RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - JANUARY 2020

This is a 4 hour, closed book, closed notes exam. There are six problems; complete all of them. Justify all of your work, as much as time allows. Write and sign the Rice honor pledge at the end of the exam.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

You can assume all spaces are path-connected.

1. Let $T = S^1 \times S^1$ and let $f : T \to T$ be defined by
$$f(x, y) = (2x + y, x + y).$$
Here we are viewing S^1 as $\mathbb{R}/\mathbb{Z}$. Let $X = (T \times [0, 1])/\sim$ be the 3-manifold obtained by identifying $(x, y) \times \{0\}$ with $f(x, y) \times \{1\}$. Compute $\pi_1(X)$.

2. Let $W = S^1 \vee S^1$ be the wedge of 2 circles. Describe four distinct connected 3-fold covering spaces of W including at least one irregular cover. In each case, give the group of covering transformations (with proof), say whether or not the covering is regular and give the corresponding subgroup of $\pi_1(W)$. Make sure to justify everything you claim, e.g. why the coverings are distinct, why regular or irregular, why the subgroup of $\pi_1(W)$ corresponding to the cover is what you say it is.

3. Let $f : H \to \mathbb{R}^3$ be a topological embedding where H is a solid handlebody of genus 2 (i.e. a thickened wedge of circles) as in Figure 1. Let $X = \text{int}(f(H))$ be the image of f in $\mathbb{R}^3$. Compute $H_p(\mathbb{R}^3 - X)$ for all p.

   Hint: $\mathbb{R}^3 = (\mathbb{R}^3 - X) \cup X$.

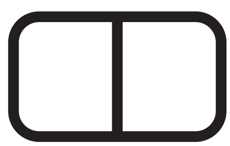

FIGURE 1. H, a solid handlebody of genus 2

4. Let X be the space obtained from a solid octagon by identifying sides as shown Figure 2 below.

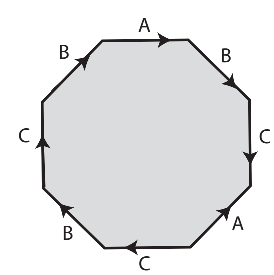

FIGURE 2

   (a) Give a CW-structure for X (be careful with the vertices) and describe the cellular chain complex.

(b) Give a presentation for $\pi_1(X)$.
(c) Calculate $H_n(X; \mathbb{Z}_3)$ and $H^n(X; \mathbb{Q})$ for all $n \geq 0$.

5. Prove that any continuous map $f : S^4 \to S^2 \times S^2$ has degree 0.

6. Let $f : X \to Y$ be a map between smooth manifolds which is transverse to a submanifold Z in Y. Then $W = f^{-1}(Z)$ is a submanifold of X. Prove that $T_x(W)$ is the preimage of $T_{f(x)}(Z)$ under the linear map $df_x : Tx(X) \to T_{f(x)}(Y)$.