1. Suppose \( f: Z \to Z \) is a self-map of a path connected space \( Z \), and \( M_f \) is the mapping torus of \( f \). Construct a map to the circle \( \phi: M_f \to S^1 \) such that \( \phi_*: \pi_1(M_f) \to \pi_1(S^1) \) is surjective.

Recall: \( M_f \) is the quotient space of \( Z \times [0, 1] \) identifying \((z, 0)\) to \((f(z), 1)\), for all \( z \in Z \).

2. Suppose \( f: \Sigma_2 \to \Sigma_3 \) is a map from a closed, orientable surface of genus 2 to a closed, orientable surface of genus 3. Prove that \( f_*: \pi_1(\Sigma_2) \to \pi_1(\Sigma_3) \) has infinite index in \( \pi_1(\Sigma_3) \).

3. Let \( X \) be the space obtained by attaching 2 disks to the wedge of circles as shown below, so that the boundaries of the disks trace out the loops \( abab^{-1} \) and \( ab^2a^{-1}b^{-1} \).

(a) Prove there is a unique connected, 2-sheeted covering space \( \tilde{X} \to X \), up to isomorphism.

(b) Compute the homology groups of the covering space from part (a), \( H_p(\tilde{X}) \) for all \( p \geq 0 \).

4. Let \( W \) be the quotient space of the solid torus \( S^1 \times D^2 \) obtained by identifying all points in the boundary \( S^1 \times \partial D^2 \) to a point (that is, \( W = (S^1 \times D^2)/(S^1 \times \partial D^2) \)). Compute the reduced homology groups \( \tilde{H}_p(W) \), for all \( p \geq 0 \).

5. Let \( N \) be a closed, connected 5–manifold with \( \pi_1(N) \cong \mathbb{Z}_{15} \) and \( H_2(N) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2 \).

(a) Prove that \( N \) is orientable.

(b) Compute \( H_p(N) \) and \( H^p(N; \mathbb{Z}) \), for all \( p \geq 0 \).

6. Prove that there is no degree 1 map \( f: S^2 \times S^2 \to \mathbb{C}P^2 \).

7. (a) Prove that \( F([x : y : z]) = [x^2 + y^2 : y^2 + z^2] \) well-defines a smooth map \( F: \mathbb{R}P^2 \to \mathbb{R}P^1 \).

(b) Prove that \( F^{-1}([1 : 2]) \) is a smooth submanifold, and determine its dimension.

8. (a) Prove that if \( \theta \) is a smooth \((n-1)\)–form on a closed, smooth \( n \)–manifold, then \( d\theta \) is zero at some point.

(b) Find an example of a smooth \((n-1)\)–form \( \theta \) on a compact, smooth \( n \)–manifold-with-boundary for which \( d\theta \) is nowhere zero.

9. Let \( \Delta \) be the distribution on \( \mathbb{R}^3 \) defined by the form \( \omega = dz - ydx \); that is, \( \omega \) generates the ideal of forms vanishing on \( \Delta \).

(a) Use \( \omega \) to decide whether or not \( \Delta \) is integrable.

(b) Find vector fields generating \( \Delta \) and use them to give an alternate proof of (a).