4.1 The kernel of \( \varphi \) is \((x - 1)\). The corresponding theorem gives a correspondence between ideals of \( \mathbb{Z}[x] \) containing \((x - 1)\) and ideals of \( \mathbb{Z} \). Since ideals of \( \mathbb{Z} \) are of the form \((n)\) for some integer \(n\), we get that the ideals of \( \mathbb{Z}[x] \) containing \((x - 1)\) have the form \((x - 1, n)\).

4.3 (c) \( \mathbb{Z}[x]/(6, 2x - 1) \cong \mathbb{Z}[x]/(3, x + 1) \cong \mathbb{Z}/3\mathbb{Z} \).

(e) \( \mathbb{Z}[x]/(x^2 + 3, 5) \cong \mathbb{F}_5[x]/(x^2 + 3) \). Since \(x^2 + 3\) is irreducible, this is a field with 25 elements. Hence \( \mathbb{F}_5[x]/(x^2 + 3) \cong \mathbb{F}_{25} \).

4.4 Suppose \( \phi: \mathbb{Z}[x]/(2x^2 + 7) \to \mathbb{Z}[x]/(x^2 + 7) \) is an isomorphism. Let \( \phi(x) = ax + b \) for some \(a, b \in \mathbb{Z}\). Then \(0 = \phi(2x^2 + 7) = 2(ax + b)^2 + 7 = 4abx + (-14a^2 + 2b^2 + 7)\). This has no solutions for \(a, b \in \mathbb{Z}\) so we get a contradiction.

5.1 Let \( g(x) = (x^3 + x^2 + x)(x^5 + 1) = x^8 + x^7 + x^6 + x^3 + x^2 + x \). Applying the division algorithm on \( g \) by \( f \) gives remainder \(2x^3 + 2x^2 + 2x\). Hence \( g(\alpha) = 2\alpha^3 + 2\alpha^2 + 2\alpha\).

5.3 Adjoining the inverse of 2 is the same as \( \mathbb{Z}[x]/(12, 2x - 1) \). In this ring, \(0 = 6(2x - 1) = 12x - 6 = -6\), so \(6 = 0\). Then \(0 = 3(2x - 1) = -3\) so \(3 = 0\) also. Finally \(x + 1 = 3x - (2x - 1) = 0\), so we get \( \mathbb{Z}[x]/(12, 2x - 1) \cong \mathbb{Z}/3\mathbb{Z} \).

5.4 (a) \( 2\alpha = 6 \) implies \(6\alpha = 18\) so \(18 = 15\) gives \(3 = 0\). The relations then become just \(2\alpha = 0\) or equivalently \(\alpha = 0\). So we get \( R' \cong \mathbb{Z}/3\mathbb{Z} \) (= \( \mathbb{F}_3 \)).

(c) Note \(\alpha^3 + \alpha^2 + 1 = \alpha(\alpha^2 + \alpha) + 1 = 1\) so we get \(0 = 1\). Hence \( R' \) is the zero ring.

5.5 Yes, suppose \( F = \mathbb{Z}/2\mathbb{Z} \). Then \(x^2 - 1 = (x - 1)^2\). Hence the substitution \(y = x - 1\) shows that \( F[x]/(x^2 - 1) = F[y]/(y^2) \) which is clearly isomorphic to \( F[x]/(x^2) \).

Part B: 563 students only

1 (a) \( V \otimes V \) is a 4-dimensional vector space with basis \( \{1 \otimes 1, x \otimes 1, 1 \otimes y, x \otimes y\} \). Hence we can define an isomorphism to \( \mathbb{Q}[x, y]/(x^2 + 3, y^2 + 3) \), also a 4-dimensional vector space, by sending generators to \(1, x, y, xy\) respectively.

(b) \( \text{Sym}^2(V) \) consists of elements which are symmetric with respect to \(x\) and \(y\), so it has basis \(\{1, x + y, xy\}\). \( \text{Alt}^2(V) \) are antisymmetric elements, so it has basis \(\{x - y\}\).
\( \rho(0) \otimes \rho(0) = \text{Id} \)

\[
\rho(1) \otimes \rho(0) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\rho(0) \otimes \rho(1) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
\end{bmatrix}
\]

\[
\rho(1) \otimes \rho(1) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{bmatrix}
\]