Part A: All Students

1.4 (a) Let $V$ be a simple $R$-module. Then $V \neq 0$, so let $v \in V$ be a nonzero element. Consider $\phi: R \to V$ by sending $r \mapsto r \cdot v$. Then $\phi$ is surjective since $V$ is simple. Let $M = \ker(\phi)$. So $R/M \cong V$. We show that $M$ is a maximal ideal, or equivalently that $R/M$ is a field. Let $r \in R \setminus M$ so that $r$ is a nonzero element of $R/M$. Then the submodule generated by $rv$ is a nonzero submodule of $V$ so must be equal to $V$. Hence there exists an $r' \in R$ such that $rr'v = v$ or equivalently $rr' - 1 \in M$ which shows $r$ is invertible in $R/M$. Thus $R/M$ is a field.

(b) Let $\phi: S \to S'$ be a homomorphism of simple modules. By (a), we know $S, S'$ can be viewed as fields and $\phi$ a homomorphism of fields. Hence either $\phi$ is injective or is the zero map. If $\phi$ is injective, then the image is a nonzero submodule of $S'$ which shows that $\phi$ must also be surjective. Hence $\phi$ is an isomorphism.

2.1 It’s clear that $M$ is spanned by \{x, y\}. However it is not a basis since $yx + (-x)y = 0$ is a nontrivial dependence relation.

Suppose $M$ is a free module over $\mathbb{C}[x, y]$, then $M$ has basis \{g_1, \ldots, g_k\} since $M$ is finitely generated. Let $x = \sum f_i g_i$ and $y = \sum h_i g_i$. Then $yx - xy = \sum (yf_i - xh_i)g_i = 0$ which implies $yf_i = xh_i$ for each $i$. It follows $f_i = xq_i$ for some polynomial $q_i$ so that $x = \sum xq_i g_i$ or $\sum q_i g_i = 1$. But then $1 \in M$ which is a contradiction.

2.2 If $R = 0$, every finitely generated $R$-module is 0, so they are all free. Suppose $R \neq 0$ and let $I \subseteq R$ be an ideal. Since $R/I$ is generated by the image of 1 under the quotient map, it is free by assumption. Let $a \in R/I$ be a basis element for $R/I$. Then for any $x \in I$ we have $xa \in I$, hence $xa = 0$ in $R/I$. Since $a$ is a basis element, either $a = 0$ and $R/I = 0$ (hence $I = R$), or $x = 0$, hence $I = (0)$, since $x$ was chosen arbitrarily. Therefore the only ideals of $R$ are $(0)$ and $R$, so $R$ is a field.

1. (a) Tor($M$) is nonempty since $0 \in$ Tor($M$). Closed under addition: Let $m, m' \in$ Tor($M$). Then there exist nonzero $r, r' \in R$ such that $rm = r'm' = 0$. So $rr' \neq 0$ and $rr'(m + m') = r'(rm) + r(r'm') = 0$. Hence $m + m' \in$ Tor($M$). Closed under scalar multiplication: If $m \in$ Tor($M$) and $rm = 0$, then for any $r' \in R$, we have $rr'm = r'rm = 0$ so $r'm \in$ Tor($M$) as well.

(b) Let $R = \mathbb{Z}/6\mathbb{Z}$ and $M = R$. Then Tor($M$) = \{0, 2, 3, 4\}. However $2 + 3 = 5 \notin$ Tor($M$), so it is not closed under addition.

(c) Let $p, q \in R$ such that $p, q \neq 0$ but $pq = 0$. Let $m \in M$ be a nonzero element. If $qm = 0$ then $m \in$ Tor($M$). Otherwise, $p(qm) = 0$ so that $qm \in$ Tor($M$) is nonzero.

2. Closed under addition: Given $r, r' \in R$ such that $rN = r'N = 0$ we have that $(r + r')N = rN + r'N = 0$ so that $r + r'$ is also an annihilator of $N$. Closed under scalar multiplication: Let $r \in R$. Then for any $r' \in R$, we have $r'rN = r'(rN) = 0$. Hence $r'r$ is an annihilator or $N$.

3. Closed under addition: If $m, m' \in M$ are annihilators of $N$, then $I(m + m') = Im + Im' = 0$ so $m + m'$ also annihilates $I$. Closed under scalar multiplication: If $r \in R$, then $I(rm) = r(Im) = 0$ so $rm$ also annihilates $I$. 

2.7 Let $\chi$ be the character of the representation with zero for all $s \neq 1$. Let $\chi'$ be any degree 1 representation. Then

$$(\chi \mid \chi') = 1/g \sum_{t \in G} \chi(t)\chi'(t)^* = 1/g \chi(1).$$

This number must be an integer, so we have $g \mid \chi(1)$. Since the character of the regular representation is $r_G(s) = 0$ for $s \neq 1$ and $r_G(1) = g$, we have that $\chi$ is an integral multiple of $r_G$. 