

CALCULUS III
LIMITS AND CONTINUITY OF FUNCTIONS OF
TWO OR THREE VARIABLES

A Manual For Self-Study

prepared by

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LIMITS AND CONTINUITY

In this discussion we will introduce the notions of limit and continuity for functions of two or more variables. We will not go into great detail—our objective is to develop the basic concepts accurately and to obtain results needed in later discussions. A more extensive study of these topics is usually given in a course in advanced calculus.

LIMITS ALONG CURVES

For a function of a single variable there are two one-sided limits at a point x_0 , namely,

$$\lim_{x \rightarrow x_0^+} f(x) \quad \text{and} \quad \lim_{x \rightarrow x_0^-} f(x)$$

reflecting the fact that there are only two directions from which x can approach x_0 , the right or the left. For functions of two or three variables the situation is more complicated because there are infinitely many different curves along which one point can approach another. Our first objective in this discussion is to define the limit of $f(x, y)$ as (x, y) approaches a point (x_0, y_0) along a curve C (and similarly for functions of three variables).

If C is a smooth parametric curve in 2-space or 3-space that is represented by the equations

$$x = x(t), \quad y = y(t), \quad \text{or} \quad x = x(t), \quad y = y(t), \quad z = z(t)$$

and if $x_0 = x(t_0)$, $y_0 = y(t_0)$, and $z_0 = z(t_0)$, then the limits

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) \quad \text{and} \quad \lim_{(x,y,z) \rightarrow (x_0,y_0,z_0)} f(x, y, z)$$

are defined by

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = \lim_{t \rightarrow t_0} f(x(t), y(t)) \tag{1}$$

$$\lim_{(x,y,z) \rightarrow (x_0,y_0,z_0)} f(x, y, z) = \lim_{t \rightarrow t_0} f(x(t), y(t), z(t)) \tag{2}$$

In these formulas the limit of the function of t must be treated as a one-sided limit if (x_0, y_0) or (x_0, y_0, z_0) is an endpoint of C .

A geometric interpretation of the limit along a curve for a function of two variables. As the point $(x(t), y(t))$ moves along the curve C in the xy -plane towards (x_0, y_0) , the point $(x(t), y(t), f(x(t), y(t)))$ moves directly above it along the graph $z = f(x, y)$ with $f(x(t), y(t))$ approaching the limiting value L .

Example -1.1 Consider the function $f(x, y)$ of two variables x and y defined as

$$f(x, y) = -\frac{xy}{x^2 + y^2}.$$

Find the limit along the following curves as $(x, y) \rightarrow (0, 0)$.

(a) the x -axis (b) the y -axis the line $y = x$

(d) the line $y = -x$ (e) the parabola $y = x^2$.

Solution (a) The x -axis has parametric equations $x = t, y = 0$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{t \rightarrow 0} f(t, 0) = \lim_{t \rightarrow 0} \left(-\frac{0}{t^2} \right) = \lim_{t \rightarrow 0} 0 = 0.$$

Solution (b) The y -axis has parametric equations $x = 0, y = t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{t \rightarrow 0} f(0, t) = \lim_{t \rightarrow 0} \left(-\frac{0}{t^2} \right) = \lim_{t \rightarrow 0} 0 = 0.$$

Solution (c) The line $y = x$ has parametric equations $x = t, y = t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{t \rightarrow 0} f(t, t) = \lim_{t \rightarrow 0} \left(-\frac{t^2}{2t^2} \right) = \lim_{t \rightarrow 0} \left(-\frac{1}{2} \right) = -\frac{1}{2}.$$

Solution (d) The line $y = -x$ -axis has parametric equations $x = t, y = -t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{t \rightarrow 0} f(t, -t) = \lim_{t \rightarrow 0} \left(\frac{t^2}{2t^2} \right) = \lim_{t \rightarrow 0} \frac{1}{2} = \frac{1}{2}.$$

Solution (e) The parabola $y = x^2$ has parametric equations $x = t, y = t^2$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{t \rightarrow 0} f(t, t^2) = \lim_{t \rightarrow 0} \left(-\frac{t^3}{t^2 + t^4} \right) = \lim_{t \rightarrow 0} \left(-\frac{t}{1 + t^2} \right) = 0.$$

OPEN AND CLOSED SETS

Although limits along specific curves are useful for many purposes, they do not always tell the complete story about the limiting behavior of a function at a point; what is required is a limit concept that accounts for the behavior of the function in an *entire vicinity* of a point, not just along smooth curves passing through the point. For this purpose, we start introducing some terminology.

Let C be a circle in 2-space that is centered at (x_0, y_0) and has positive radius δ . The set of points that are enclosed by the circle, but do not lie on the circle, is called the **open disk** of radius δ centered at (x_0, y_0) , and the set of points that lie on the circle together with those enclosed by the circle is called the **closed disk** of radius δ centered at (x_0, y_0) . Analogously, if S is a sphere in 3-space that is centered at (x_0, y_0, z_0) and has positive radius δ , then the set of points that are enclosed by the sphere, but do not lie on the sphere, is called the **open ball** of radius δ centered at (x_0, y_0, z_0) , and the set of points that lie on the sphere together with those enclosed by the sphere is called the **closed ball** of radius δ centered at (x_0, y_0, z_0) . Disks and balls are the two-dimensional and three-dimensional analogs of intervals on a line.

The notions of “open” and “closed” can be extended to more general sets in 2-space and 3-space. If D is a set of points in 2-space, then a point (x_0, y_0) is called an **interior point** of D if there is some open disk centered at (x_0, y_0) that contains only points of D , and (x_0, y_0) is called a **boundary point** of D if every open disk centered at (x_0, y_0) contains both points in D and points not in D . The same terminology applies to sets in 3-space, but in that case the definitions use balls rather than disks.

For a set D in either 2-space or 3-space, the set of all interior points is called the **interior** of D and the set of all boundary points is called the **boundary** of D . Moreover, just as for disks, we say that D is **closed** if it contains all of its boundary points and **open** if it contains none of its boundary points. The set of all points in 2-space and the set of all points in 3-space have no boundary points (why?), so by agreement they are regarded to be both open and closed.

GENERAL LIMITS OF FUNCTIONS OF TWO VARIABLES

Let me begin with the question that you wanted to ask, but were afraid to ask it:

What does it mean to say (or write) the statement

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L ?$$

By the statement, we mean to convey the idea that the values of $f(x,y)$ can be made as “close as we like” to the number L by restricting points (x,y) in the domain of f to be “sufficiently close” to (but different from) the point (x_0,y_0) . This idea has a formal expression in the following definition.

FORMAL DEFINITION OF LIMIT (n VARIABLES)

Definition: Let $f : \mathbf{D} \subseteq \mathbf{R}^n \rightarrow \mathbf{R}$ be defined for all $\mathbf{x} = (x_1, x_2, \dots, x_n)$ in some open region $\mathbf{D} \subseteq \mathbf{R}^n$ which may or may not contain a fixed n -tuple $\mathbf{x}_0 = (x_0^1, x_0^2, \dots, x_0^n)$.

The number $L \in \mathbf{R}$ is said to be the limit of $f(\mathbf{x})$ as $\mathbf{x} \in \mathbf{D}$ approaches \mathbf{x}_0 if and only if given any real number $\varepsilon > 0$, we can find a corresponding real number $\delta > 0$ (usually depends on ε) such that $f(\mathbf{x})$ satisfies

$$|f(\mathbf{x}) - L| < \varepsilon$$

whenever the distance between \mathbf{x} and \mathbf{x}_0 satisfies $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$ and we will write

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} f(\mathbf{x}) = L \text{ or } \lim_{\mathbf{x} \rightarrow \mathbf{x}_0} |f(\mathbf{x}) - L| = 0.$$

FORMAL DEFINITION OF LIMIT (TWO VARIABLES)

Definition: Let $f : \mathbf{D} \subseteq \mathbf{R}^2 \rightarrow \mathbf{R}$ be a function of two variables x and y defined for all ordered pairs (x, y) in some open disk $\mathbf{D} \subseteq \mathbf{R}^2$ centered on a fixed ordered pair (x_0, y_0) , except possibly at (x_0, y_0) .

We will say that the number $L \in \mathbf{R}$ is the limit of $f(x, y)$ as $(x, y) \in \mathbf{D}$ approaches (x_0, y_0) if and only if given any real number $\varepsilon > 0$, we can find a real number $\delta > 0$ (usually depends on ε) such that $f(x, y)$ satisfies

$$|f(x, y) - L| < \varepsilon$$

whenever the distance between (x, y) and (x_0, y_0) satisfies $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$ and we will write

$$\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = L \quad \text{or} \quad \lim_{(x, y) \rightarrow (x_0, y_0)} |f(x, y) - L| = 0.$$

FORMAL DEFINITION OF LIMIT (THREE VARIABLES)

Definition: Let $f : \mathbf{E} \subseteq \mathbf{R}^3 \rightarrow \mathbf{R}$ be a function of three variables x , y , and z defined for all ordered triples (x, y, z) in some open sphere $\mathbf{E} \subseteq \mathbf{R}^3$ centered on a fixed ordered triple (x_0, y_0, z_0) , except possibly at (x_0, y_0, z_0) .

We will say that the number $L \in \mathbf{R}$ is the limit of $f(x, y, z)$ as $(x, y, z) \in \mathbf{E}$ approaches (x_0, y_0, z_0) if and only if given any real number $\varepsilon > 0$, we can find a real number $\delta > 0$ (usually depends on ε) such that $f(x, y, z)$ satisfies

$$|f(x, y, z) - L| < \varepsilon$$

whenever the distance between (x, y, z) and (x_0, y_0, z_0) satisfies

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta \quad \text{and we will write}$$

$$\lim_{(x, y, z) \rightarrow (x_0, y_0, z_0)} f(x, y, z) = L \quad \text{or} \quad \lim_{(x, y, z) \rightarrow (x_0, y_0, z_0)} |f(x, y, z) - L| = 0.$$

IMPORTANT: In using the definitions above, you **must** have some idea of what the number L is before hand. Once you have some idea of L then the definitions above can be used to verify that L is the limit.

Example -1.2 Let $f : \mathbf{D} \subseteq \mathbf{R}^2 \rightarrow \mathbf{R}$ be the function $f(x, y) = \frac{4x^2y}{x^2+y^2}$ of two variables x and y defined on the region $\mathbf{D} = \{(x, y) \in \mathbf{R}^2 : (x, y) \neq (0, 0)\} = \mathbf{R}^2/\{(0, 0)\}$. Use the definition of limits to show that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{4xy^2}{x^2 + y^2} = 0.$$

Solution

Preliminary discussion: According to the definition of limits given above for two variables, given any real number $\varepsilon > 0$ whatsoever, we must “find a corresponding real number $\delta > 0$ (usually depends on ε and also on the point $(0, 0)$)” such that $f(x, y)$ satisfies the inequality $|f(x, y) - 0| < \varepsilon$ whenever $(x, y) \in \mathbf{D}$ and the distance between (x, y) and $(0, 0)$ satisfies the inequality $0 < \sqrt{x^2 + y^2} < \delta$:

So here is what we do to find the corresponding real number δ (in terms of ε): Suppose you are given $\varepsilon > 0$ arbitrarily. You **must to find a corresponding real number** $\delta > 0$ (but you don't know how to you are going to choose it yet!) such that $f(x, y) = \frac{4x^2y}{x^2+y^2}$ satisfies the inequality

$$\left| \frac{4x^2y}{x^2 + y^2} - 0 \right| < \varepsilon$$

whenever the distance between $(x, y) \in \mathbf{D}$ and $(0, 0)$ satisfies the inequality $0 < \sqrt{x^2 + y^2} < \delta$ or equivalently,

$$4|x| \cdot \frac{y^2}{x^2 + y^2} < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Let us pretend for a brief moment that we have found the corresponding real number δ , then since we know that $y^2 \leq x^2 + y^2$ for all $(x, y) \neq (0, 0)$ implies that $\frac{y^2}{x^2+y^2} \leq 1$ for all $(x, y) \neq (0, 0)$ we can write the following statement (very important): *This is where you get to discover how the corresponding real number $\delta > 0$ should be chosen (recall that you only pretended to have chosen it, so now you get to choose it for real).*

$$|f(x, y) - 0| = \left| \frac{4x^2y}{x^2 + y^2} - 0 \right| = 4|x| \cdot \frac{y^2}{x^2 + y^2} \leq 4|x| \cdot 1 = 4\sqrt{x^2} \leq 4\sqrt{x^2 + y^2} < 4\delta.$$

So if (x, y) is in \mathbf{D} and $0 < \sqrt{x^2 + y^2} < \delta$, we see that you should choose the corresponding positive real number δ to be $\varepsilon/4$ (or smaller) and we get that

$$\left| \frac{4x^2y}{x^2 + y^2} - 0 \right| \leq 4\sqrt{x^2 + y^2} < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon$$

or equivalently, that

$$|f(x, y) - 0| = \left| \frac{4x^2y}{x^2 + y^2} - 0 \right| < 4\delta = \varepsilon.$$

Now that you have found $\delta = \frac{\varepsilon}{4} > 0$, you can now give your answer to the question as follows:

The answer: The number 0 is the limit of the function $f(x, y) = \frac{4x^2y}{x^2 + y^2}$ as (x, y) in \mathbf{D} approaches $(0, 0)$ because for any given number $\varepsilon > 0$, we have shown that we can produce a corresponding number $\delta = \varepsilon/4 > 0$ so that $f(x, y) = \frac{4x^2y}{x^2 + y^2}$ satisfies the inequality

$$\left| \frac{4x^2y}{x^2 + y^2} - 0 \right| < \varepsilon$$

whenever the distance between (x, y) and $(0, 0)$ satisfies $0 < \sqrt{x^2 + y^2} < \varepsilon/4 = \delta$. So we can write

$$\lim_{(x,y) \rightarrow (0,0)} \frac{4x^2y}{x^2 + y^2} = 0.$$

Notice: The answer to limit questions such as above are usually very short compared to the amount of work required to discover your answer.

Example -1.3 Show that the function

$$f(x, y) = \frac{2xy}{x^2 + y^2}$$

has no limit at the origin $(0, 0)$ as $(x, y) \rightarrow (0, 0)$.

Solution Let us suppose that $f(x, y)$ has a limit L , as $(x, y) \rightarrow (0, 0)$. If $(x, y) \rightarrow (0, 0)$ along a vertical path, e.g., if $x = 0$ and $y \downarrow 0$, then $L = 0$ (because $f(0, y) = 0$ for all $y \neq 0$). If $(x, y) \rightarrow (0, 0)$ along a “diagonal” path, e.g., if $y = x$ and $x \downarrow 0$, then $L = 1$ (because $f(x, x) = 1$ for all $x \neq 0$). Since $L = 0 \neq 1 = L$ (a function cannot have two different limits at the same point), $f(x, y)$ has no limit at $(0, 0)$.

The Example above and example 1.1 provides us with a way to show that a function of two variables has no limit at a particular point. We shall call this procedure the “Two-Path Test”:

THE TWO-PATH TEST (*used for showing that the limit does not exist at a particular point*)

- (a) If $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = L$, then $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = L$ along any smooth curve or path.
- (b) If the limit of $f(x, y)$ fails to exist as $(x, y) \rightarrow (x_0, y_0)$ along some smooth curve or path, or if $f(x, y)$ has different limits as $(x, y) \rightarrow (x_0, y_0)$ along two different smooth curves or path, then the limit of $f(x, y)$ does not exist as $(x, y) \rightarrow (x_0, y_0)$.

Example -1.4 Determine whether the function $f : \mathbf{D} \subseteq \mathbf{R}^2 \rightarrow \mathbf{R}$ of two variables x and y defined for all $(x, y) \neq (0, 0)$ given as

$$f(x, y) = \frac{xy^2}{x^2 + y^4}$$

has a limit as $(x, y) \rightarrow (0, 0)$.

Solution The vertical path $x = 0$ gives $f(0, y) = 0 = L$ even before we take the limit as $y \rightarrow 0$. On the other hand, the parabolic path $x = y^2$ gives

$$f(y^2, y) = \frac{y^4}{2y^4} = L = \frac{1}{2} \neq 0.$$

Therefore, the two-path test tells us that $f(x, y)$ cannot have a limit at $(0, 0)$ as $(x, y) \rightarrow (0, 0)$.

THE “SQUEEZE” THEOREM or THE “SANDWICH” THEOREM

Suppose that $g(x, y)$, $f(x, y)$ and $h(x, y)$ are functions of two variables all defined for all $(x, y) \neq (x_0, y_0)$ in some open disk \mathbf{D} centered on (x_0, y_0) . If $g(x, y) \leq f(x, y) \leq h(x, y)$ for all $(x, y) \neq (x_0, y_0)$ and that $L = \lim_{(x,y) \rightarrow (x_0,y_0)} g(x, y) = \lim_{(x,y) \rightarrow (x_0,y_0)} h(x, y)$, then

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = L.$$

COROLLARY

Suppose that $|f(x, y) - L| \leq g(x, y)$ for all (x, y) in the interior of some disk centered at (x_0, y_0) , except possibly at (x_0, y_0) . If $\lim_{(x,y) \rightarrow (x_0,y_0)} g(x, y) = 0$, then $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = L$.

Example -1.5 (From your Textbook) Evaluate

$$\lim_{(x,y) \rightarrow (1,0)} \frac{(x-1)^2 \ln(x)}{(x-1)^2 + y^2}.$$

Solution Along the smooth curve $x = 1$, we have

$$\lim_{(1,y) \rightarrow (1,0)} \frac{0}{y^2} = 0.$$

Along the path or curve $y = 0$, we have

$$\lim_{(x,0) \rightarrow (1,0)} \frac{(x-1)^2 \ln(x)}{(x-1)^2} = \lim_{x \rightarrow 1} \ln(x) = 0.$$

A third path through $(1, 0)$ is the line $y = x - 1$ (note that in this case, we must have $y \rightarrow 0$ as $x \rightarrow 1$). We have

$$\lim_{(x,x-1) \rightarrow (1,0)} \frac{(x-1)^2 \ln(x)}{(x-1)^2 + (x-1)^2} = \lim_{x \rightarrow 1} \frac{(x-1)^2 \ln(x)}{2(x-1)^2} = \lim_{x \rightarrow 1} \frac{\ln(x)}{2} = 0.$$

It seems like 0 could be the limit, but we are not sure about it, or we could try to find a path through $(1, 0)$ where the limit does not exist or the limit exists and is different from 0. To show this, we consider

$$|f(x, y) - L| = \left| \frac{(x-1)^2 \ln(x)}{(x-1)^2 + y^2} \right|.$$

Notice that if the y^2 term were not present in the denominator, then we could cancel the $(x-1)^2$ terms. We have

$$|f(x, y) - L| = \left| \frac{(x-1)^2 \ln(x)}{(x-1)^2 + y^2} \right| \leq \left| \frac{(x-1)^2 \ln(x)}{(x-1)^2} \right| = |\ln(x)|.$$

Since $\lim_{(x,y) \rightarrow (1,0)} |\ln(x)| = 0$, it follows by the squeeze theorem that

$$\lim_{(x,y) \rightarrow (1,0)} \frac{(x-1)^2 \ln(x)}{(x-1)^2 + y^2} = 0.$$

CHANGING TO POLAR COORDINATES (for limits at the origin)

Sometimes if you cannot make any headway with

$$\lim_{(x,y) \rightarrow (0,0)} f(x,y)$$

in rectangular coordinates, you could try changing domain of f to polar coordinates instead. Substitute $x = r \cos \theta$, $y = r \sin \theta$ and investigate the limit of the resulting expression as $r \rightarrow 0$. In other words, try to decide whether there exists a number L satisfying the following criterion:

Given any positive real number ε whatsoever, we can find a corresponding positive real number δ (usually depending on ε) such that for all r and θ , $F(r, \theta) = f(r \cos \theta, r \sin \theta)$ satisfies

$$|F(r, \theta) - L| < \varepsilon$$

whenever the distance between r and 0 satisfies $0 < |r| < \delta$.

If such an L exists, then

$$\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{r \rightarrow 0} F(r, \theta) = L.$$

Example -1.6 Determine whether the function given by

$$f(x,y) = \frac{x^3}{x^2 + y^2}$$

has limit as (x,y) approaches $(0,0)$.

Solution After evaluating the limit to be 0 on several paths through the origin $(0,0)$, we believe that the limit might be 0 (but we can never be sure! It requires verification).

Let $F(r, \theta) = f(r \cos \theta, r \sin \theta) = \frac{r^3 \cos^3 \theta}{r^2} = r \cos^3 \theta$.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^3 \cos^3 \theta}{r^2} = \lim_{r \rightarrow 0} r \cos^3 \theta = 0.$$

We can verify this using the definition of limits:

Suppose that $\varepsilon > 0$ is given arbitrarily. Choose $\delta = \varepsilon$ so that for all r and θ , $F(r, \theta) = r \cos^3 \theta$ satisfies

$$|F(r, \theta) - L| = |r \cos^3 \theta - 0| = |r| |\cos \theta|^3 \leq |r| < \delta = \varepsilon$$

whenever the distance between r and 0 satisfies $0 < |r| < \delta$.

CONTINUITY

Stated informally, a function of a single variable is continuous if its graph is an unbroken curve without jumps or holes. To extend this idea to functions of two variables, imagine that the graph of $z = f(x, y)$ is molded from a thin sheet of clay that has been hollowed or pinched into peaks and valleys. We will regard f as being continuous if the clay surface has no tears or holes.

The precise definition of continuity at a point for functions of two variables is similar to that for functions of one variable—we require the limit of the function and the value of the function to be the same at the point.

FORMAL DEFINITION OF CONTINUITY (TWO VARIABLES)

Let $f : \mathbf{D} \subseteq \mathbf{R}^2 \rightarrow \mathbf{R}$ be defined for all (x, y) in some open region $\mathbf{D} \subseteq \mathbf{R}^2$ which contains a fixed ordered pair (x_0, y_0) .

We will say that the function $f(x, y)$ is continuous at the point (x_0, y_0) if and only if given any real number $\varepsilon > 0$, there is a real number $\delta > 0$ (usually depends on ε) such that

$$|f(x, y) - f(x_0, y_0)| < \varepsilon \text{ whenever } (x, y) \in \mathbf{D} \text{ and } \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

and we will write

$$\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0)$$

to express the fact that $f(x_0, y_0)$ is the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0) and the $f(x, y)$ is continuous at (x_0, y_0) . In addition, if a function is continuous at each point in its domain \mathbf{D} , then we say that the function is continuous on \mathbf{D} , and if f is continuous at every point in the xy -plane, then we say that f is **continuous everywhere**.

FORMAL DEFINITION OF CONTINUITY (n VARIABLES)

Let $f : \mathbf{D} \subseteq \mathbf{R}^n \rightarrow \mathbf{R}$ be defined for all $\mathbf{x} = (x_1, x_2, \dots, x_n)$ in some open region $\mathbf{D} \subseteq \mathbf{R}^n$ which contains a fixed ordered pair $\mathbf{x}_0 = (x_0^1, x_0^2, \dots, x_0^n)$.

We will say that the function $f(\mathbf{x})$ is continuous at the point \mathbf{x}_0 if and only if given any real number $\varepsilon > 0$, there is a real number $\delta > 0$ (usually depends on ε) such that

$$|f(\mathbf{x}) - f(\mathbf{x}_0)| < \varepsilon \text{ whenever } \mathbf{x} \in \mathbf{D} \text{ and } \|\mathbf{x} - \mathbf{x}_0\| < \delta.$$

and we will write

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$$

to express the fact that $f(\mathbf{x}_0)$ is the limit of $f(\mathbf{x})$ as \mathbf{x} approaches \mathbf{x}_0 and the $f(\mathbf{x})$ is continuous at the point \mathbf{x}_0 . In addition, if a function $f(\mathbf{x})$ is continuous at each point \mathbf{x}_0 in its domain \mathbf{D} , then we say that the function is continuous or is continuous on \mathbf{D} .

Example -1.7 In the previous example, you showed that the function

$$f(x, y) = \frac{4x^2y}{x^2 + y^2}$$

has limit 0 at the origin $(0, 0)$. You also noticed that $f(0, 0)$ is undefined (that is, $(0, 0)$ is not in the domain of the function). We now ask this question: Is $f(x, y)$ continuous at $(0, 0)$? If not, how can we make it into a continuous function at $(0, 0)$?

Solution: The definition of continuity says that the limit 0 of $f(x, y)$ as (x, y) approaches the origin $(0, 0)$ and the value of the function, $f(0, 0)$, at $(0, 0)$ must be the same. Since $f(0, 0)$ is undefined the function cannot be continuous at $(0, 0)$. However, the function as limit at the origin given by $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$ and so we can define $f(x, y)$ to be continuous at $(0, 0)$ as:

$$f(x, y) = \begin{cases} \frac{4x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

The following theorem, which we state without proof, illustrates some of the ways in which continuous functions can be combined to produce new continuous functions.

THEOREM

- (a) If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0 , then $f(x, y) = g(x)h(y)$ is continuous at (x_0, y_0) .
- (b) If $h(x, y)$ is continuous at (x_0, y_0) and $g(u)$ is continuous at $u = h(x_0, y_0)$, then the composition $f(x, y) = g(h(x, y))$ is continuous at (x_0, y_0) .
- (c) If $f(x, y)$ is continuous at (x_0, y_0) , and if $x(t)$ and $y(t)$ are continuous at t_0 with $x(t_0) = x_0$ and $y(t_0) = y_0$, then the composition $f(x(t), y(t))$ is continuous at t_0 .

Problem #1

Find the limit of f as $(x, y) \rightarrow (0, 0)$ or show that the limit does not exist.

$$(a) \quad f(x, y) = \frac{x^3 - xy^2}{x^2 + y^2}$$

$$(b) \quad f(x, y) = \cos\left(\frac{x^3 - y^3}{x^2 + y^2}\right)$$

$$(c) \quad f(x, y) = \tan^{-1}\left(\frac{|x| + |y|}{x^2 + y^2}\right)$$

You may use any of the methods discussed above to verify our answer.

SECTION 1

Problem #2

Define $f(0,0)$ in such a way that “extends” f to be continuous at the origin.

$$(a) \quad f(x, y) = \ln \left(\frac{3x^2 - x^2y^2 + 3y^2}{x^2 + y^2} \right)$$

$$(b) \quad f(x, y) = \frac{2xy^2}{x^2 + y^2}$$

$$(c) \quad f(x, y) = xy \frac{x^2 - y^2}{x^2 + y^2}$$

SECTION 2

Problem #3

Does knowing that

$$1 - \frac{x^2 y^2}{3} < \frac{\tan^{-1} xy}{xy} < 1 \quad \text{and} \quad -1 \leq \cos\left(\frac{1}{y}\right) \leq 1$$

tell you anything about

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\tan^{-1} xy}{xy} \quad \text{and} \quad \lim_{(x,y) \rightarrow (0,0)} x \cos \frac{1}{y} ?$$

Give reasons for your answer.