Math 542: Homework 1

1. (a) Let ρ be defined on $\mathbf{H}^2$ as in class. Prove that it is a metric.

(b) Prove that the topology induced on $\mathbf{H}^2$ by ρ is the same as the Euclidean topology.

2. (a) Prove that the hyperbolic metric ρ can be expressed as follows: Let $z, w \in \mathbf{H}^2$, then:

$$\cosh(\rho(z,w)) = 1 + \frac{|z-w|^2}{2\mathrm{Im(z)Im(w)}}.$$

(b) Use (a) to show that $||\gamma||^2 = 2\cosh(\rho(i, \gamma i))$.

3. (a) Prove that $\mathrm{PSL}(2, \mathbf{R})$ acts transitively on $\mathbf{H}^2$.

(b) Prove that $\mathrm{PSL}(2, \mathbf{C})$ acts triply transitively on $\mathbf{C} \cup \infty$.

4. Prove that the Möbius transformation action of $\mathrm{PSL}(2, \mathbf{Z}[\mathrm{i}])$ on $\mathbf{C} \cup \infty$ *is not* a discontinuous action.

5. Let $A, B \in \mathrm{PSL}(2, \mathbf{C})$. Prove that:

(a) A and B have a common fixed-point in $\mathbf{C} \cup \infty$ if and only if $\mathrm{tr[A,B]} = 2$.

(b) If A is a hyperbolic element and A and B have *exactly* one fixed point in common, then $< A, B >$ is not discrete.

6. Let Γ be a Kleinian group and $P < \Gamma$ with $P \cong \mathbf{Z} \oplus \mathbf{Z}$. Show that all non-trivial elements of P are parabolic elements.

7. Let Γ be a Kleinian group and $N(\Gamma) = \{g \in \mathrm{PSL}(2, \mathbf{C}) : \mathrm{g\Gamma g^{-1}} = \Gamma\}$ denote the normalizer of Γ in $\mathrm{PSL}(2, \mathbf{C})$. Prove that $N(\Gamma)$ is a Kleinian group (i.e. it is a discrete group).

8. (a) Prove that the homomorphism $\phi_n : \mathrm{PSL}(2, \mathbf{Z}) \to \mathrm{PSL}(2, \mathbf{Z}/\mathrm{n}\mathbf{Z})$ from class is onto.

(b) Prove that $\Gamma(2)$ is torsion-free, and identify the quotient surface $\mathbf{H}^2/\Gamma(2)$.

9. Prove that $T \in \mathrm{PSL}(2, \mathbf{R})$ preserves area; i.e. if $E \subset \mathbf{H}^2$ then $\mathrm{Area(T(E))} = \mathrm{Area(E)}$.