Math 542: Homework 2

1. As in class let $I(\gamma)$ denote the isometric circle of $\gamma \in \mathrm{PSL}(2, \mathbf{C})$. Prove

(a) $\gamma : I(\gamma) \to I(\gamma^{-1})$.

(b) γ maps the outside of $I(\gamma)$ to the inside of $I(\gamma^{-1})$.

(c) $\gamma(\infty)$ is the centre of $I(\gamma^{-1})$

2. (a) Prove that the product of two inversions or reflections is an element of $\mathrm{PSL}(2, \mathbf{C})$.

(b) Let $\gamma \in \mathrm{PSL}(2, \mathbf{C})$, with $\gamma(\infty) \neq \infty$. Describe the action of γ on $\mathbf{C} \cup \infty$ in terms of inversions in isometric circles associated to γ and γ^{-1}, a reflection, and in the case of a hyperbolic a rotation.

3. (a) Prove that the Ford fundamental polyhedron is indeed a fundamental polyhedron.

(b) Compute a Ford fundamental polyhedron for $\mathrm{PSL}(2, \mathbf{Z}[\sqrt{-2}])$.

4. Prove that the Dirichlet fundamental polyhedron at a point p is a fundamental polyhedron.

(c) Complete the proof that Ford fundamental polygon for $\mathrm{PSL}(2, \mathbf{Z})$ is also a Dirichlet fundamental polygon.

5. Let Γ be a Kleinian group containing the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Prove that for any other element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, either $c = 0$ or $|c| \geq 1$.

6. Prove that if Γ is a Kleinian group of finite covolume, the center of Γ is trivial.

7. Prove that if $K = K_1 \# K_2$ is the connect sum of two non-trivial knots, the swallow-follow torus T (as constructed in class) is incompressible, i.e.:

$$\pi_1(T) \hookrightarrow \pi_1(S^3 \setminus (K_1 \# K_2)) \text{ is injective.}$$

8. A Kleinian group Γ is called *geometrically finite* if it has a finite sided fundamental polyhedron. Otherwise the group is called *geometrically infinite*.

(a) Prove that the following group is geometrically finite. Here n is a positive integer.

$$\Gamma_n = < \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

(b) Let Γ be a Kleinian group and $H < \Gamma$ of finite index. Then Γ is geometrically finite if and only if H is geometrically finite.

9. Prove that a non-elementary subgroup of $\mathrm{PSL}(2, \mathbf{C})$ contains infinitely many hyperbolic elements.