Math 542: Homework 4

1. Let Γ_1 and Γ_2 be commensurable non-cocompact Kleinian groups of finite co-volume. Let F_i denote the field generated by the entries of the parabolic elements of Γ_i, $i = 1, 2$. Show that $F_1 = F_2$.
2. (a) Prove that the subgroup $< a, b > \subset \mathrm{SL}(2, \mathbf{C})$ is reducible if and only if $\mathrm{tr[a, b]} = 2$.

(b) Let $< g, h >$ be an irreducible subgroup of $\mathrm{SL}(2, \mathbf{C})$. Prove that $\{1, g, h, gh\}$ is a basis of $M(2, \mathbf{C})$.

3. Let V be an irreducible algebraic set defined over $\mathbf{Q}$. Prove that if $\dim(V) = 0$ then V consists of a single point whose co-ordinates are algebraic numbers.
4. Let Γ be a finitely generated Kleinian group and $\Delta < \Gamma$ a finitely generated normal subgroup, prove that $k\Gamma = k\Delta$ and $A\Gamma = A\Delta$.
5. Let A be a quaternion algebra over field K (characteristic $\neq 2$) with Hilbert Symbol $\left(\frac{a,b}{K}\right)$. Recall that the reduced norm $n_{A/K}$ determines a quadratic form (called the *norm form*) $x_0^2 - ax_1^2 - bx_2^2 + abx_3^2$ with $x_i \in K$.

(a) Show that if the norm form represents 0 non-trivially, then A is not a division algebra (and so is isomorphic to $M(2, K)$).

(b) Let A be the quaternion algebra over $\mathbf{Q}$ with Hilbert Symbol $\left(\frac{-2,3}{\mathbf{Q}}\right)$. Show that A is not a division algebra.

(c) Let A be the quaternion algebra over $\mathbf{Q}$ with Hilbert Symbol $\left(\frac{-1,3}{\mathbf{Q}}\right)$. Does $n_{A/K}$ represent 0 non-trivially?

6. (a) Let $M = \mathbf{H}^3/\Gamma$ be a finite volume hyperbolic 3-manifold admitting an orientation-reversing involution. Prove that the invariant trace-field $k\Gamma$ contains a real subfield of index 2.

(b) Prove that the complements of the knots 5_2 and 8_{20} from Rolfsen's tables do not admit an orientation-reversing involution.

7. Compute invariant trace-fields for $(4, 0)$-surgery on the figure eight-knot complement and $(3, 0)$-surgery on 5_2.
8. Let A be a quaternion algebra over field K (characteristic $\neq 2$) with Hilbert Symbol $\left(\frac{a,b}{K}\right)$. Prove directly that A is simple as follows:

(i) If I is a proper 2-sided ideal and $\alpha = t + xi + yj + zk \in I$, show that $t + xi \in I$ by considering $\alpha i + i\alpha$.

(ii) Arguing symmetrically and taking a linear combination, show that $t \in I$. Conclude that $t = 0$, and then $x = y = z = 0$.

9. Let K be a field so that K^*/K^{*2} is finite. Show that there are only finitely many isomorphism classes of quaternion algebras over K.

10. Suppose that A is a quaternion algebra over a field $K \subset \mathbf{C}$, and assume that A is embedded in $M(2, \mathbf{C})$. Prove that the $n_{A/K}$ and $\mathrm{tr}_{\mathrm{A/K}}$ coincide with matrix determinant and trace respectively.