Math 542: Homework 5

1. (a) Prove that the cross-ratio $[z_1, z_2, z_3, z_4]$ of four points in $\mathbf{C} \cup \infty$ is $\mathrm{PSL}(2, \mathbf{C})$-invariant.

(b) Prove that the cross-ratio of four points is real if and only the points lie on a circle in $\mathbf{C} \cup \infty$.

2. Let Γ be a non-elementary subgroup of $\mathrm{PSL}(2, \mathbf{C})$ with a finite generating set

$$\{\gamma_1, \gamma_2, \ldots, \gamma_n\}, \text{ with } \mathrm{tr}(\gamma_\mathrm{i}) \neq 0 \text{ for } \mathrm{i} = 1, \ldots, \mathrm{n}.$$

Define Γ^{sq} (with respect to these generators) to be the subgroup generated by

$$< \gamma_1^2, \gamma_2^2, \ldots, \gamma_n^2 > .$$

Show that $k\Gamma = \mathbf{Q}(\mathrm{tr}\ \Gamma^{\mathrm{sq}})$.

3. Referring to Question 2, give an example of a finitely generated Kleinian group for which Γ^{sq} has infinite index in Γ.

4. Let $\mathcal{L}$ denote the Lie algebra of $\mathrm{SL}(2, \mathbf{C})$, and let $\mathrm{Ad} : \mathrm{SL}(2, \mathbf{C}) \to \mathrm{GL}(\mathcal{L})$ denote the adjoint representation. Let Γ be a Kleinian group of finite co-volume. Show that $k\Gamma = \mathbf{Q}(\mathrm{trAd}(\gamma) : \gamma \in \Gamma)$.

5. Let k be a number field with ring of integers R_k. The *ideal class group of* k is the quotient $H_k = I_k/P_k$ where I_k is the group of fractional ideals of k and P_k the subgroup of non-zero principal fractional ideals. The *class number of* k is the order of H_k; i.e. it is a measure of how far from being a principal ideal domain R_k is.

(a) Prove that for $k = \mathbf{Q}(\sqrt{-2})$, $h_k = 1$.

(b) Prove that for $k = \mathbf{Q}(\sqrt{-5})$, $h_k \neq 1$.

(c) Prove that the orbifold $\mathbf{H}^3/\mathrm{PSL}(2, \mathrm{O_d})$ has h_d cusps (where h_d is the class number of $\mathbf{Q}(\sqrt{-d})$).

6. Suppose $S^3 \setminus K = \mathbf{H}^3/\Gamma$ is a hyperbolic knot complement. Conjugate $\Gamma < \mathrm{PSL}(2, \mathrm{k}\Gamma)$ (recall lectures) and assume that every element of $k\Gamma \cup \infty$ is a fixed point of a parabolic element of Γ. Prove that $k\Gamma$ has class number 1.

7. (a) By drawing a diagram, prove that the figure-eight knot complement contains an immersed (totally geodesic) twice punctured disc.

(b) Prove that a knot complement in S^3 can never contain an embedded incompressible twice punctured disc (**Hint:** Think homology!).

8. Prove that for $d \neq 1, 3$, the "standard copy" of $\mathbf{H}^2/\mathrm{PSL}(2, \mathbf{Z})$ embeds in $\mathbf{H}^3/\mathrm{PSL}(2, \mathrm{O_d})$. In addition, prove that the image is non-separating.

9. Referring to the proof of Theorem 4.14 in lectures, let K be a number field with $k\Gamma \subset K \subset \mathbf{Q}(\mathrm{tr}\ \Gamma)$ and let $\Gamma_K = \{\gamma \in \Gamma : \mathrm{tr}(\gamma) \in \mathrm{K}\}$. Prove that $\Gamma^{(2)}\Gamma_K < \Gamma_K$: i.e. if $A \in \Gamma^{(2)}$ and $B \in \Gamma_K$ then $AB \in \Gamma_K$ (you cannot use that Γ_K is a subgroup as this claim is used in the proof of that statement!).

10. Let A be a quaternion algebra over the field k (characteristic $\neq 2$). Let A_0 be the subset of pure quaternions. Prove that $x \in A_0$ if and only if $x^2 \in k$ but $x \notin k$.