Math 542: Homework 7

1. Let k be a number field with ring of integers R_k. Prove that $M(2, R_k)$ is a maximal order of $M(2, k)$.

2. Let k be a number field with class number one. Assuming that every maximal ideal of $M(2, k)$ is conjugate to one of the form $\mathcal{O}(I) \subset M_2(k)$ given by $\begin{pmatrix} R_k & I \\ I^{-1} & R_k \end{pmatrix}$ for some integral ideal I (recall HW 6). Prove that all maximal orders are conjugate to $M(2, R_k)$.

3. Let $A = \left(\frac{-1,3}{\mathbf{Q}}\right)$ and $\mathcal{O} = \mathbf{Z}[1, i, j, ij]$. Prove that A is a division algebra and that $\mathcal{O}$ is not a maximal order.

4. Let A be a quaternion algebra over the number field k. Let $I \subset A$ be a finitely generated R_k-module that contains a k-basis of A (an ideal in the language of HW 6 Qn 10). Set $I^{-1} = \{a \in A : IaI \subset I\}$ show that I^{-1} is also an ideal.

5. Let A be a quaternion algebra over the number field k. Define the *discriminant* of an order $\mathcal{O} \subset A$ (denoted $d(\mathcal{O})$) as the ideal in R_k generated by the set $\{\mathrm{det}(\mathrm{tr}(\mathrm{x_i x_j})) : 1 \leq \mathrm{i}, \mathrm{j} \leq 4, \mathrm{x_i} \in \mathcal{O}\}$.

(i) Prove that the discriminant is always a non-zero ideal.

(ii) Prove that if $\mathcal{O}$ has a free R_k-basis $\{e_1, e_2, e_3, e_4\}$, then $d(\mathcal{O})$ is the principal ideal $(\mathrm{det}(\mathrm{tr}(\mathrm{e_i e_j})))\mathrm{R_k}$.

6. Referring to Q5, compute the discriminant of the order $\mathcal{O}$ in Q3.

7. Let $A = \left(\frac{a,b}{k}\right)$. Complete the proof from class of the following equivalences:

(a) A has zero divisors.

(b) $n_{A/k}$ is isotropic.

(c) $n_{A/k}$ restricted to A_0 is isotropic.

(d) There is a solution to $ax^2 + by^2 = 1$ in $k \times k$.

(e) If $E = k(\sqrt{b})$, then $a \in N_{E/k}(E)$ (i.e. a is in the image of the norm map from $E \to k$).

8. Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $c \neq 0$. Let $\mathcal{H}$ be a horosphere tangent to ∞ at height h. Prove that $T(\mathcal{H})$ is a horosphere tangent to a/c of diameter $\frac{1}{h|c|^2}$.

9. Let p, q, r be odd primes and $L(p)$, $L(q)$ and $L(r)$ Lens spaces with fundamental groups of order p, q, r respectively. Let $M = L(p)\#L(q)\#L(r)$. Prove that for p, q and r sufficiently large, M does not contain a knot K with $M \setminus K \to \mathbf{H}^3/\mathrm{PSL}(2, \mathrm{O_d})$.

10. Prove that the (orientation-preserving) triangle group $\Delta(2,3,8)$ is arithmetic. Compute the Hilbert Symbol of the invariant quaternion algebra.