Length and eigenvalue equivalence

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1 Introduction

Let M be a compact Riemannian manifold, and let $\Delta = \Delta_M$ denote the Laplace–Beltrami operator of M acting on $\mathrm{L}^2(M)$. The **eigenvalue spectrum** $\mathscr{E}(M)$ consists of the eigenvalues of Δ listed with their multiplicities. Two manifolds M_1 and M_2 are said to be **isospectral** if $\mathscr{E}(M_1) = \mathscr{E}(M_2)$. Geometric and topological constraints are forced on isospectral manifolds; for example if the manifolds are hyperbolic (complete with all sectional curvature equal to -1) then they must have the same volume [18], and so for surfaces the same genus.

Another invariant of M is the **length spectrum** $\mathscr{L}(M)$ of M; that is the set of all lengths of closed geodesics on M counted with multiplicities. Two manifolds M_1 and M_2 are said to be **iso-length spectral** if $\mathscr{L}(M_1) = \mathscr{L}(M_2)$. Under the hypothesis of negative sectional curvature the invariants $\mathscr{E}(M)$ and $\mathscr{L}(M)$ are closely related. For example, it is known that $\mathscr{E}(M)$ determines the set of lengths of closed geodesics, and in the case of closed hyperbolic surfaces, the stronger statement that $\mathscr{E}(M)$ determines $\mathscr{L}(M)$ and vice-versa holds [7, 8].

In this paper we address the issue of how much information is lost by forgetting multiplicities. More precisely, for a compact Riemannian manifold M, define the **eigenvalue set** (resp. **length set** and **primitive length set**) to be the set of eigenvalues of Δ (resp. set of lengths all closed geodesics and lengths of all primitive closed geodesics) counted without multiplicities. These sets will be denoted $E(M)$, $L(M)$ and $L_p(M)$ respectively. Two manifolds M_1 and M_2 are said to be **eigenvalue equivalent** (resp. **length equivalent** and **primitive length equivalent**) if $E(M_1) = E(M_2)$ (resp. $L(M_1) = L(M_2)$ and $L_p(M_1) = L_p(M_2)$). Although length

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spectrum and primitive length spectrum determine each other, the corresponding statement for length sets is false. Primitive length equivalent manifolds are clearly length equivalent, but we shall see that the converse is false.

We will focus mainly on hyperbolic manifolds of finite volume. Even in this setting little seems known about the existence of manifolds which are eigenvalue (resp. length or primitive length) equivalent but not isospectral or iso-length spectral. Examples of non-compact arithmetic hyperbolic 2–manifolds that are length equivalent were constructed in Theorem 2 of [24] using arithmetic methods. However, as far as the authors are aware, no examples of closed hyperbolic surfaces that are length equivalent and not iso-length spectral were known, and it would appear that no examples of eigenvalue equivalent or primitive length equivalent hyperbolic manifolds which are not isospectral or iso-length spectral were known. Our main results rectify this situation for hyperbolic surfaces and indeed for all finite volume hyperbolic m–manifolds.

Theorem 1.1. *Let M be a closed hyperbolic m–manifold. Then there exist infinitely many pairs of finite covers $\{M_j, N_j\}$ of M such that*

(a) $E(M_j) = E(N_j)$*,*

(b) $\mathrm{vol}(M_j)/\mathrm{vol}(N_j) \to \infty$.

Moreover, $E(M_j) = E(N_j)$ for any Riemannian metric on M.

The method of proof of Theorem 1.1 does not provide (primitive) length equivalent pairs of covers. However, we can prove an analogue for primitive length equivalence (and hence also length equivalence).

Theorem 1.2. *Let M be a finite volume hyperbolic m–manifold. Then there exist infinitely many pairs of finite covers $\{M_j, N_j\}$ of M such that*

(a) $L_p(M_j) = L_p(N_j)$

(b) $\mathrm{vol}(M_j)/\mathrm{vol}(N_j) \to \infty$.

Moreover, $L_p(M_j) = L_p(N_j)$ for any Riemannian metric on M.

Indeed, as we point out in §5.1, for every finite volume hyperbolic n–manifold where $n \neq 3, 4, 5$ we can produce pairs of finite sheeted covers of arbitrarily large volume ratio that are both primitive length equivalent and eigenvalue equivalent.

The methods of the paper are largely group theoretic, relying on the fundamental group rather than the geometry, and a quick way to provide lots of examples in many more situations is given by the following. Recall that a group Γ is called **large** if it contains a finite index subgroup that surjects a free non-abelian group.

Theorem 1.3. *(Theorem 3.1 is a stronger version.) Let M be a compact Riemannian manifold with large fundamental group. Then there exist infinitely many pairs of finite covers $\{M_j, N_j\}$ of M such that*

(a) $L(M_j) = L(N_j)$,

(b) $E(M_j) = E(N_j)$,

(c) $\mathrm{vol}(M_j)/\mathrm{vol}(N_j) \to \infty$.

Moreover, (a) and (b) hold for any Riemannian metric on M, and if $\pi_1(M)$ is hyperbolic, $L_p(M_j) = L_p(N_j)$ also holds for any Riemannian metric on M.

Our arguments start with Sunada's construction [28] of isospectral manifolds, which was based on a well known construction in number theory of "arithmetically equivalent" number fields (see [17]). Our length equivalence of manifolds similarly has a number theoretic counterpart called "Kronecker equivalence" of number fields, as we discovered after doing this work; see the book [10]. The results contained here can thus be viewed as providing the geometric investigation proposed in the sentence from the last paragraph of that book: *"In view of the relations between arithmetical and Kronecker equivalence, one should also study Kronecker equivalence in this geometric situation."*

In the final section we collate some remarks and questions. In particular, we note that Mark Kac's famous paper "Can one hear the shape of a drum" [9] has been a catalyst for much of the work on isospectrality, and we revisit that paper and the Gordon-Webb-Wolpert answer to his question [5] in the light of our work.

2 Equivalence

We first recall Sunada's construction.

For any finite group G and subgroups H and K of G, we say that H and K are **almost conjugate** (or "Gassmann equivalent" in the terminology of Perlis [17]) if for any g in G the following condition holds (where (g) denotes conjugacy class):

$$|H \cap (g)| = |K \cap (g)|.$$

In [28] Sunada proved the following theorem relating almost conjugate pairs with isospectral covers.

Theorem (Sunada). *Let M be a closed Riemannian manifold, G a finite group, and H and K almost conjugate subgroups of G. If $\pi_1(M)$ admits a homomorphism onto G, then the finite covers M_H and M_K associated to the pullback subgroups of H and K are isospectral. Moreover, the manifolds M_H and M_K are iso-length spectral.*

The proof of this is an easy exercise, but checking when manifolds produced by Sunada's method are non-isometric requires more work. However, for length equivalence far less is required, and the resolution of the isometry problem is built into our construction.

Length equivalence, primitive length equivalence, and eigenvalue equivalence each require a different condition on the group G. In each instance, we describe a group theoretic condition, and then explain how it is used to produce examples with the desired features.

2.1 Length and primitive length equivalence

Though it is not essential, the group G will always be finite in what follows.

Definition 2.1 (Elementwise conjugacy). *Subgroups H and K of G are said to be* ***elementwise conjugate*** *if for any g in G the following condition holds:*

$$H \cap (g) \neq \emptyset \quad \textit{if and only if} \quad K \cap (g) \neq \emptyset. \tag{1}$$

(Or, more briefly, $H^G = K^G$, where $H^G = \bigcup_{g \in G} g^{-1}Hg$.)

It is immediate from the definition that almost conjugate subgroups are elementwise conjugate.

To produce primitive length equivalent manifolds, we impose further conditions on H and K, and also on $\pi_1(M)$.

Definition 2.2 (Primitive). *We shall call a subgroup H of G* ***primitive in*** *G if the following holds:*

(a) All non-trivial cyclic subgroups of H have the same order p (necessarily prime).

(b) $\bigcap_{g \in G} g^{-1}Hg = \{1\}$.

Theorem 2.3. *Let M be a Riemannian manifold, G a group, and H and K elementwise conjugate subgroups of G.*

(1) If $\pi_1(M)$ admits a homomorphism onto G, then the covers M_H and M_K associated to the pullback subgroups of H and K are length equivalent.

(2) If, in addition, H and K are primitive in G and $\pi_1(M)$ has the property that any pair of distinct maximal cyclic subgroups of Γ intersect trivially, then the covers M_H and M_K associated to the pullback subgroups of H and K are primitive length equivalent.

Remark. It is well known that when M admits a metric of negative sectional curvature, then $\pi_1(M)$ satisfies the condition needed to apply Theorem 2.3.

Proof of theorem. To prove (1) it suffices to show that a closed geodesic γ on M has a lift to a closed geodesic on M_H if and only if it has a lift to a closed geodesic on M_K. Let ρ denote the homomorphism $\pi_1(M) \to G$. By standard covering space theory, γ has a closed lift to M_H if and only if $\rho([\gamma]) \in G$ is conjugate into H. By assumption this is true for H if and only if it is true for K, proving (1).

For (2) we will show the inclusion $L_p(M_H) \subseteq L_p(M_K)$; the reverse inclusion then follows by symmetry. We argue by contradiction, assuming there is a primitive γ in $\pi_1(M_H)$, every conjugate of which in $\pi_1(M_K)$ is imprimitive. Let γ_K be any conjugate of γ in $\pi_1(M_K)$ and let $\delta \in \pi_1(M_K)$ and $r > 1$ be such that $\delta^r = \gamma_K$. The arguments splits into two cases.

Case 1. $\rho(\delta) = 1$.

Since $\ker\rho < \pi_1(M_H)$, all conjugates of δ are contained in $\pi_1(M_H)$. This contradicts the primitivity of γ, as a $\pi_1(M)$–conjugate of δ powers to γ.

Case 2. $\rho(\delta) \neq 1$, so $\rho(\delta)$ has prime order p by Definition 2.2 (a). We split this into two subcases.

Case 2.1. $\rho(\gamma_K) \neq 1$.

Since $\langle\rho(\gamma_K)\rangle$ is nontrivial and contained in $\langle\rho(\delta)\rangle$ which has prime order, it is equal to $\langle\rho(\delta)\rangle$. Thus, $\rho(\mu\delta\mu^{-1}) \in \langle\rho(\gamma)\rangle$, where μ is the element of $\pi_1(M)$ conjugating γ_K to γ. Since the cyclic subgroup $\langle\rho(\gamma)\rangle$ is contained in H, $\rho(\mu\delta\mu^{-1})$ is contained in H, so $\mu\delta\mu^{-1}$ is an element of $\pi_1(M_H)$. This contradicts the primitivity of γ since $(\mu\delta\mu^{-1})^r = \gamma$.

Case 2.2. $\rho(\gamma_K) = 1$.

By Definition 2.2 (b), there exists an element g in G which conjugates $\rho(\delta)$ outside of K. For any element $\sigma \in \rho^{-1}(g)$, $\sigma\gamma_K\sigma^{-1} \in \pi_1(M_K)$ and by assumption this cannot be primitive. Therefore, there exists δ_1 in $\pi_1(M_K)$ and $s > 1$ such that $\sigma\gamma_K\sigma^{-1} = \delta_1^s$. We have the equality $(\sigma\delta\sigma^{-1})^r = \delta_1^s$. By assumption, $\left\langle\sigma\delta\sigma^{-1}\right\rangle$ and $\langle\delta_1\rangle$ are contained in a common maximal cyclic subgroup C of $\pi_1(M)$. The intersection of $\rho(C)$ with K is a cyclic subgroup which contains the image of $\langle\rho(\delta_1)\rangle$. By Definition 2.2 (a), the cyclic subgroups of K have prime order p, and so $|\rho(C) \cap K| = 1$ or p.

Assume first that the latter holds. Now $\rho(C)$ has a unique cyclic subgroup of order p, so $\rho(C) \cap K$ must equal $\left\langle\rho(\sigma\delta\sigma^{-1})\right\rangle$. Hence the element $\rho(\sigma\delta\sigma^{-1})$ is in K, which contradicts the choice of σ. Therefore, we can assume that $\rho(C) \cap K = 1$. Then $\rho(\delta_1) = 1$. Replacing γ_K by $\sigma\gamma_K\sigma^{-1}$ and δ by δ_1, Case 1 provides the desired contradiction. $\square$

2.2 Eigenvalue equivalence

To give context to our construction below of eigenvalue equivalence, we begin by recalling the following well known equivalent formulation of almost conjugacy.

Proposition 2.4. *Subgroups H and K of a finite group G are almost conjugate if and only if for every finite dimensional complex representation ρ of G,*

$$\dim \mathrm{Fix}(\rho|H) = \dim \mathrm{Fix}(\rho|K)\,,$$

where $\mathrm{Fix}(\rho|H)$ *denotes the subspace of $\rho(H)$–fixed vectors.*

Proof. A convenient reference for the character theory used here and later is [26]. The dimension of $\mathrm{Fix}(\rho|H)$ is the inner product $(\chi_1^H, \chi_{\rho|H}^H)$ of the trivial character on H and the character of $\rho|H$. By definition this is $\frac{1}{|H|}\sum_{h\in H}\chi_{\rho|H}^H(h)$, and since characters are constant on conjugacy classes, this equals $\frac{1}{|H|}\sum|(g)\cap H|\,\chi_\rho^G(g)$, where the sum is over conjugacy classes in G. Thus the equality $\dim \mathrm{Fix}(\rho|H) = \dim \mathrm{Fix}(\rho|K)$ is equivalent to

$$\frac{1}{|H|}\sum|(g)\cap H|\chi_\rho^G(g) = \frac{1}{|K|}\sum|(g)\cap K|\chi_\rho^G(g)\,. \tag{2}$$

Clearly, almost conjugacy of H and K implies this equality.

For the converse, note first that the equality $\dim \mathrm{Fix}(\rho|H) = \dim \mathrm{Fix}(\rho|K)$ applied to the regular representation of G becomes $[G:H] = [G:K]$, whence $|H| = |K|$. Since characters of irreducible representations form a basis for class functions on G, letting ρ run over all irreducible representations of G in equation (2) now implies that $|(g)\cap H| = |(g)\cap K|$ for each conjugacy class (g). $\square$

Definition 2.5. *We say subgroups H and K of a finite group G are* ***fixed point equivalent*** *if for any finite dimensional complex representation ρ of G, the restriction $\rho|_H$ has a nontrivial fixed vector if and only if $\rho|_K$ does.*

Theorem 2.6. *Let H and K be fixed point equivalent subgroups of a finite group G. If M is a compact Riemannian manifold and $\pi_1(M)$ admits a homomorphism onto G, then the covers M_H and M_K associated to the pullbacks in $\pi_1(M)$ of H and K are eigenvalue equivalent.*

Proof. Let $\widetilde{M}$ be the cover of M associated to the pullback in $\pi_1(M)$ of the trivial subgroup of G. The action of G on $\widetilde{M}$ is by isometries, and the quotients by H and K give covering maps $p_H\colon \widetilde{M} \to M_H$ and $p_K\colon \widetilde{M} \to M_K$, respectively.

The covering projection induces an embedding $p_H^*\colon \mathrm{L}^2(M_H) \to \mathrm{L}^2(\widetilde{M})$ whose image is the H–fixed subspace $\mathrm{L}^2(\widetilde{M})^H$ of $\mathrm{L}^2(\widetilde{M})$. Since $\Delta_{\widetilde{M}} \circ p_H^* = p_H^* \circ \Delta_{M_H}$, the action of H on $\mathrm{L}^2(\widetilde{M})$ restricts to an action on the λ–eigenspace $\mathrm{L}^2(\widetilde{M})_\lambda$ and p_H^* identifies $\mathrm{L}^2(M_H)_\lambda$ with $(\mathrm{L}^2(\widetilde{M})_\lambda)^H$. Thus λ is an eigenvalue for M_H if and only if $(\mathrm{L}^2(\widetilde{M})_\lambda)^H$ has positive dimension.

Since G is finite, the representation of G on $\mathrm{L}^2(\widetilde{M})_\lambda$ decomposes as a direct sum of finite dimensional representations (in fact, compactness of M implies $\mathrm{L}^2(\widetilde{M})_\lambda$ is finite dimensional, but we do not need this). Hence, if H and K are fixed point equivalent, $(\mathrm{L}^2(\widetilde{M})_\lambda)^H$ will be non-trivial if and only if $(\mathrm{L}^2(\widetilde{M})_\lambda)^K$ is non-trivial. $\square$

Remark. 1. The compactness assumption on M is not necessary. If M is non-compact our argument extends easily to show that under the conditions of the theorem both the discrete and non-discrete spectra of M_H and M_K agree when viewed as sets.

2. What makes the Sunada construction work for both the length and eigenvalue spectra is the equivalence of almost conjugacy with the condition of Proposition 2.4. Our weakening of almost conjugacy to elementwise conjugacy on the one hand, and, via Proposition 2.4, to fixed point equivalence on the other, go in dual directions. They therefore cannot be expected to be equivalent, and it is a little surprising that in the examples we know, the two weaker conditions still tend to have significant overlap.

2.3 Examples

An elementary example of elementwise conjugacy is the following. Let G be the alternating group $\mathrm{Alt}(4)$, and set $a=(12)(34)$ and $b=(14)(23)$. Then the subgroups $H=\{1,a\}$ and the Klein 4–group $K=\{1,a,b,ab\}$ are elementwise conjugate. However note that K is not primitive since it is a normal subgroup. In addition H and K are not fixed point equivalent since K has no fixed vector under the irreducible 3–dimensional representation of G while H has a fixed vector. On the other hand, it is not hard to check that H is fixed point equivalent to the trivial subgroup $\{1\}$.

We now generalize this example.

Let $\mathbb{F}_p$ be the prime field of order p, and let $n\geq 2$ be a positive integer. The **n–dimensional special $\mathbb{F}_p$–affine group** is the semidirect product $\mathbb{F}_p^n \rtimes \mathrm{SL}(n;\mathbb{F}_p)$ defined by the standard action of $\mathrm{SL}(n;\mathbb{F}_p)$ on $\mathbb{F}_p^n$. We call any $\mathbb{F}_p$–vector subspace V of $\mathbb{F}_p^n$ a **translational subgroup** of $\mathbb{F}_p^n \rtimes \mathrm{SL}(n;\mathbb{F}_p)$.

Theorem 2.7. *Let V and W be translational subgroups of $G=\mathbb{F}_p^n \rtimes \mathrm{SL}(n;\mathbb{F}_p)$. Then,*

(i) if V and W are both non-trivial then they are elementwise conjugate in G, and they are moreover primitive if they are proper subgroups of $\mathbb{F}_p^n$;

(ii) if V and W are both proper subgroups of $\mathbb{F}_p^n$ then they are fixed point equivalent in G.

Proof. (i). Since $\mathrm{SL}(n;\mathbb{F}_p)$ acts transitively on non-trivial elements of $\mathbb{F}_p^n$, the elementwise conjugacy is immediate. Moreover, conditions (a) and (b) of Definition 2.2 clearly hold for V if V is a proper subgroup of $\mathbb{F}_p^n$.

(ii). It suffices to show that any proper translational subgroup V is fixed point equivalent to the trivial subgroup. So we must show that for any m-dimensional representation ρ of $\mathbb{F}_p^n \rtimes \mathrm{SL}(n;\mathbb{F}_p)$ with $m > 0$, the restriction $\rho|_V$ has a nontrivial fixed subspace when restricted to V. To this end, let χ be the character of ρ. The dimension of the fixed space of $\rho|_V$ is $\dim(\mathrm{Fix}(\rho|_V)) = \frac{1}{|V|}\sum_{v\in V}\chi(v)$. Since $\chi(1) = m$ and any two nontrivial elements of V are conjugate in $\mathbb{F}_p^n \rtimes \mathrm{SL}(n;\mathbb{F}_p)$, we can rewrite this:

$$\dim(\mathrm{Fix}(\rho|_V)) = \frac{1}{|V|}\left(m + (|V|-1)\chi(x)\right),$$

where $x \in V - \{0\}$. Similarly, the dimension of the fixed space for the full translation subgroup $\mathbb{F}_p^n$ is

$$\dim(\mathrm{Fix}(\rho|_{\mathbb{F}_p^n})) = \frac{1}{\left|\mathbb{F}_p^n\right|}\left(m + (\left|\mathbb{F}_p^n\right|-1)\chi(x)\right).$$

Thus $m + (\left|\mathbb{F}_p^n\right|-1)\chi(x) \geq 0$, so $\chi(x) \geq \frac{-m}{\left|\mathbb{F}_p^n\right|-1}$. Hence,

$$\dim(\mathrm{Fix}(\rho|_V)) = \tfrac{1}{|V|}\left(m + (|V|-1)\chi(x)\right) \geq \tfrac{1}{|V|}\left(m - m\frac{|V|-1}{\left|\mathbb{F}_p^n\right|-1}\right) > 0. \qquad \square$$

3 Proofs of main results

The following is a stronger version of Theorem 1.3:

Theorem 3.1. *Let M be a compact Riemannian manifold whose fundamental group is large. For every integer $n \geq 2$ and every odd prime p, there exists a finite tower of covers of M*

$$M_0 \longrightarrow M_1 \longrightarrow \ldots \longrightarrow M_{n-1} \longrightarrow M_n \longrightarrow M,$$

with each $M_i \to M_{i+1}$ of degree p, such that:

(a) $L(M_j) = L(M_k)$ *for* $0 \leq j,k \leq n-1$*;*

(b) $E(M_j) = E(M_k)$ *for* $1 \leq j,k \leq n$*;*

Moreover, (a),(b) hold for any Riemannian metric on M. Finally, if $\pi_1(M)$ is hyperbolic then for any Riemannian metric on M,

(c) $L_p(M_j) = L_p(M_k)$ *for* $1 \leq j,k \leq n-1$.

Proof. Since M is large we can find finite index subgroups which surject any finitely generated free group, so there is a finite cover X of M with

$$\pi_1(X) \twoheadrightarrow \mathbb{F}_p^n \rtimes \mathrm{SL}(n;\mathbb{F}_p) \ .$$

Consider any complete $\mathbb{F}_p$–flag

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V_n = \mathbb{F}_p^n$$

in $\mathbb{F}_p^n$. Pulling these subgroups back to $\pi_1(X) \subset \pi_1(M)$ we obtain a tower

$$M_0 \longrightarrow M_1 \longrightarrow \ldots \longrightarrow M_{n-1} \longrightarrow M_n \longrightarrow M$$

of corresponding covers of M. The theorem then follows from Theorem 2.7 combined with Theorems 2.3 and 2.6. $\square$

This theorem implies Theorems 1.1, 1.2, and 1.3 in the case of closed hyperbolic surfaces. In addition, it is well-known that closed and finite volume hyperbolic manifolds whose fundamental groups are large exist in all dimensions (see e.g., [12]). This provides examples of hyperbolic manifolds in all dimensions satisfying the conclusions of Theorems 1.1, 1.2, and 1.3. To prove that **any** closed or finite volume hyperbolic manifold has finite sheeted covers with these properties requires additional work.

We mention in passing that Theorem 3.1 (b) applied to surfaces produces arbitrarily long towers of abelian covers

$$M_1 \longrightarrow \ldots \longrightarrow M_{n-1} \longrightarrow M_n$$

whose first nontrivial eigenvalue remains constant. On the other hand, it is well known that any infinite tower of abelian covers of a fixed hyperbolic surface has λ_1 tending to zero (see [1] and [28]).

3.1 More families

Theorem 3.2. *(i) Let $p > 3$ be a prime. Then $\mathrm{PSL}(2;\mathbb{Z}/p^2\mathbb{Z})$ contains subgroups $K < H$ with $[H : K] = p$ which are fixed point equivalent.*

(ii) Let $k \neq \mathbb{Q}$ be a number field with ring of integers $\mathscr{O}_k$. Let $\mathscr{P}$ be the set of non-dyadic prime ideals $\mathfrak{p}$ of $\mathscr{O}_k$ for which $\mathscr{O}_k/\mathfrak{p} = \mathbb{F}_q$ is a non-prime field (this set is infinite by the Cebotarev Density Theorem). Then for $\mathfrak{p}$ in $\mathscr{P}$ the group $\mathrm{PSL}(2;\mathscr{O}_k/\mathfrak{p}^2)$ contains subgroups $K < H$ with $[H : K] = p$ which are primitive and elementwise conjugate in $\mathrm{PSL}(2;\mathscr{O}_k/\mathfrak{p}^2)$.

The proof of Theorem 3.2 will be deferred until §4. Assuming this we complete the proofs of Theorems 1.1 and 1.2 in the next subsection.

3.2 Completion of proofs

We shall need the following special case of the Strong Approximation Theorem (see [30] and [16]; see also [11] for a discussion of the proof in the particular case of hyperbolic manifolds). Suppose M^m is a finite volume hyperbolic manifold with $m \geq 3$. We shall identify $\mathrm{Isom}(\mathbf{H}^m)$ with $\mathrm{PO}_0(m,1)$ so $\pi_1(M) < \mathrm{PO}_0(m,1)$. We can assume there is a number field k such that $\pi_1(M) < \mathrm{PO}_0(m,1;S)$ for a finite extension ring S of $\mathscr{O}_k$ with k the field of fractions of S (see [22] for the details). We choose k minimal.

Theorem 3.3 (Strong Approximation). *For all but finitely many primes $\mathfrak{p}$ of S the image of $\pi_1(M)$ under the reduction homomorphism*

$$r_{\mathfrak{p}^j}:\ \mathrm{PO}_0(m,1;S) \longrightarrow \mathrm{PO}(m,1;S/\mathfrak{p}^j)$$

contains the commutator subgroup $\Omega(m,1;S/\mathfrak{p}^j)$ of $\mathrm{PO}(m,1;S/\mathfrak{p}^j)$ for all $j \geq 1$. □

Proof of Theorem 1.1. Theorem 1.1 is shown for hyperbolic surfaces in the comment following the proof of Theorem 3.1, so we can assume that M is a closed hyperbolic manifold of dimension $m \geq 3$. We will produce surjections of $\pi_1(M)$ onto finite groups containing $\mathrm{PSL}(2;\mathbb{Z}/p^2\mathbb{Z})$ for infinitely many p.

Let S be as in the Strong Approximation Theorem above. For all but a finite number of primes $\mathfrak{p}$ of S the image of $\pi_1(M) < \mathrm{PO}_0(m,1;S)$ under the restriction homomorphism $r_{\mathfrak{p}^j}$ contains $\Omega(m,1;S/\mathfrak{p}^j)$ for all $j \geq 1$. If p is the integer prime that $\mathfrak{p}$ divides then $r_{\mathfrak{p}^j}(\pi_1(M))$ therefore contains the subgroup $\Omega(m,1;\mathbb{Z}/p^j\mathbb{Z})$, and therefore also the subgroup $\Omega(2,1;\mathbb{Z}/p^j\mathbb{Z})$ of $\Omega(m,1;\mathbb{Z}/p^j\mathbb{Z})$.

We claim the finite groups $\Omega(2,1;\mathbb{Z}/p^j\mathbb{Z})$ and $\mathrm{PSL}(2;\mathbb{Z}/p^j\mathbb{Z})$ are isomorphic. To see this, first recall that the p–adic Lie groups $\mathrm{PSL}(2;\mathbb{Q}_p)$ and $\Omega(2,1;\mathbb{Q}_p)$ are isomorphic (as p–adic Lie groups). The groups $\mathrm{PSL}(2;\mathbb{Z}_p)$ and $\Omega(2,1;\mathbb{Z}_p)$ are the respective maximal compact subgroups of $\mathrm{PSL}(2;\mathbb{Q}_p)$ and $\Omega(2,1;\mathbb{Q}_p)$, and are unique up to isomorphism (see [19, Ch 3.4]). Hence the groups $\mathrm{PSL}(2;\mathbb{Z}_p)$ and

$\Omega(2,1;\mathbb{Z}_p)$ are isomorphic as p–adic Lie groups. Reducing modulo the ideal generated by the jth power of the uniformizer π of $\mathbb{Z}_p$ yields the asserted isomorphism between $\Omega(2,1;\mathbb{Z}/p^j\mathbb{Z})$ and $\mathrm{PSL}(2;\mathbb{Z}/p^j\mathbb{Z})$.

Restricting now to $j=2$ we have shown $r_{\mathfrak{p}^2}(\pi_1(M))$ contains a subgroup isomorphic to $\mathrm{PSL}(2;\mathbb{Z}/p^2\mathbb{Z})$. So by passage to a subgroup of finite index in $\pi_1(M)$, we can arrange a finite cover X of M with a surjection $\pi_1(X)\to\mathrm{PSL}(2;\mathbb{Z}/p^2\mathbb{Z})$. The existence of pairs $\{M_j,N_j\}$ as stated in Theorem 1.1 now follows from Theorem 2.6, Theorem 3.2(i), and the infinitude of $\mathscr{P}$. $\square$

Proof of Theorem 1.2. We have already shown this for hyperbolic surfaces in the comments following Theorem 3.1. We next consider hyperbolic 3–manifolds. Let M be a hyperbolic 3–manifold with holonomy representation $\pi_1(M)<\mathrm{PSL}(2;S)$ (again S chosen minimally). The field of fractions of S, a finite ring extension of $\mathbb{Z}$, is necessarily a proper extension of $\mathbb{Q}$, see [14]. In particular, by the Cebotarev Density Theorem, there exist infinitely many prime ideals $\mathfrak{p}$ of S such that $S/\mathfrak{p}$ is a nontrivial extension of $\mathbb{F}_p$. The Strong Approximation Theorem applies here to see that for all but finitely many among this infinite set of prime ideals of S the reduction maps

$$r_{\mathfrak{p}^2}\colon\, \pi_1(M)\longrightarrow \mathrm{PSL}(2;S/\mathfrak{p}^2),$$

are onto. By Theorem 3.2 (ii) and Theorem 2.3 (ii), there exists a pair of covers

$$N_{\mathfrak{p}}\longrightarrow M_{\mathfrak{p}}\longrightarrow M$$

with $L_p(N_{\mathfrak{p}})=L_p(M_{\mathfrak{p}})$ and $\mathrm{vol}(N_{\mathfrak{p}})/\mathrm{vol}(M_{\mathfrak{p}})=p$.

We extend this to all hyperbolic m–manifolds with $m>3$ as follows. Let $S=\mathbb{Z}[i]$ and let $\mathscr{P}$ be the set of prime ideals defined in Theorem 3.2 (specifically, these are the ideals $p\mathbb{Z}[i]$ with $p\equiv 3 \bmod 4$). For $m>3$ and $\mathfrak{p}\in\mathscr{P}$ we first claim we have an injection of $\mathrm{PSL}(2;\mathbb{Z}[i]/\mathfrak{p}^j)$ into $\Omega(m,1;\mathbb{Z}/p^j\mathbb{Z})$. For this, we argue as follows. First, there exists a quadratic form B_4 defined over $\mathbb{Q}$ of signature $(3,1)$ and an injection

$$\mathrm{PSL}(2;\mathbb{Z}[i])\longrightarrow \mathrm{PSO}_0(B_4;\mathbb{Z}).$$

For each prime $\mathfrak{p}=p\mathbb{Z}[i]$, this induces isomorphisms

$$\mathrm{PSL}(2;\mathbb{Z}[i]/\mathfrak{p}^j)\longrightarrow \Omega(B_4;\mathbb{Z}/p^j\mathbb{Z}).$$

For $j=1$, this can be found in [29]. For $j>1$, this is established by an argument similar to that used in the proof of the equivalence of $\mathrm{PSL}(2;\mathbb{Z}/p^j\mathbb{Z})$ and $\Omega(2,1;\mathbb{Z}/p^j\mathbb{Z})$ in proving Theorem 1.1. Extending the form B_4 from $\mathbb{Q}^4$ to $\mathbb{Q}^{m+1}$ for $m>3$ by the identity produces injections

$$\Omega(B_4;\mathbb{Z}/p^j\mathbb{Z})\longrightarrow \Omega(m,1;\mathbb{Z}/p^j\mathbb{Z}).$$

In particular, we can view

$$\mathrm{PSL}(2;\mathbb{Z}[i]/\mathfrak{p}^j) < \Omega(m,1;\mathbb{Z}/p^j\mathbb{Z})$$

for all $m > 3$, all j, and all $\mathfrak{p} \in \mathscr{P}$ as claimed. Since we have already shown in the proof of Theorem 1.1 that $\pi_1(M)$ surjects finite groups containing $\Omega(m,1;\mathbb{Z}/p^j\mathbb{Z})$ for all but finitely many primes, Theorem 3.2 (ii) with $S = \mathbb{Z}[i]$ and Theorem 2.3 (ii) now complete the proof. $\square$

4 Proof of Theorem 3.2

Throughout this section p will be an odd prime. For any ring R let $M(2;R)$ be the algebra of 2×2 matrices over R. The Lie algebra $\mathfrak{sl}(2;\mathbb{F}_p)$ of $\mathrm{SL}(2;\mathbb{F}_p)$ consists of traceless matrices: $\mathfrak{sl}(2;\mathbb{F}_p) = \{X \in M(2;\mathbb{F}_p) \mid X_{11} = -X_{22}\}$. The adjoint action of $\mathrm{SL}(2;\mathbb{F}_p)$ on $\mathfrak{sl}(2;\mathbb{F}_p)$ is the action by conjugation. As a vector space $\mathfrak{sl}(2;\mathbb{F}_p)$ has a natural $\mathrm{SL}(2;\mathbb{F}_p)$–invariant bilinear form, the Killing form B defined by $B(X,Y) = \mathrm{Tr}(XY)$. The associated quadratic form Q_B (defined by $B(X,X) = 2Q_B(X)$) is thus also invariant. Explicitly, for $X,Y \in \mathfrak{sl}(2;\mathbb{F}_p)$:

$$B(X,Y) = 2X_{11}Y_{11} + X_{12}Y_{21} + X_{21}Y_{12}\,, \qquad Q_B(X) = X_{11}^2 + X_{12}X_{21}\,.$$

Lemma 4.1. *There is a short exact sequence*

$$1 \longrightarrow \mathfrak{sl}(2;\mathbb{F}_p) \longrightarrow \mathrm{SL}(2;\mathbb{Z}/p^2\mathbb{Z}) \longrightarrow \mathrm{SL}(2;\mathbb{F}_p) \longrightarrow 1.$$

The conjugation action of $\mathrm{SL}(2;\mathbb{F}_p)$ *on* $\mathfrak{sl}(2;\mathbb{F}_p)$ *induced by this sequence is the adjoint action.*

Proof. The inclusion $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z}$ is given by $a \mapsto pa$. It induces an inclusion $M(2;\mathbb{Z}/p\mathbb{Z}) \to M(2;\mathbb{Z}/p^2\mathbb{Z})$ given by $X \mapsto pX$.

Reduction modulo p induces the surjection $\pi\colon \mathrm{SL}(2;\mathbb{Z}/p^2\mathbb{Z}) \to \mathrm{SL}(2;\mathbb{F}_p)$ whose kernel is clearly

$$\ker(\pi) = \{I + pX \in M(2;\mathbb{Z}/p^2\mathbb{Z}) \mid \det(I+pX) = 1\}\,.$$

Now $\det(I+pX) = 1 + p\,\mathrm{Tr}(X) + p^2\det(X) = 1 + p\,\mathrm{Tr}(X)$ since we are in $\mathbb{Z}/p^2\mathbb{Z}$, so we can rewrite:

$$\ker(\pi) = \{I + pX \mid X \in \mathfrak{sl}(2;\mathbb{F}_p)\}\,.$$

The equation $(I+pX)(I+pY) = I + pX + pY$ now shows that the map $X \to I + pX$ is an isomorphism of the additive group $\mathfrak{sl}(2;\mathbb{F}_p)$ to $\ker(\pi)$. The final sentence of the lemma is clear. $\square$

Lemma 4.2. *The number of* $\mathrm{SL}(2;\mathbb{Z}/p^2\mathbb{Z})$*–conjugacy classes in* $\mathfrak{sl}(2;\mathbb{F}_p)$ *(ie, orbits of the adjoint action of* $\mathrm{SL}(2;\mathbb{F}_p)$*) is exactly* $(p+2)$*, as listed in the following table. In the table n represents a fixed quadratic non-residue in* $\mathbb{F}_p$ *and "qr" is short for quadratic residue (i.e., a square). Each of rows 2 and 3 represents* $(p-1)/2$ *conjugacy classes, as* $Q = Q_B(X)$ *runs respectively through the quadratic residues and non-residues in* $\mathbb{F}_p - \{0\}$*.*

description	size	# classes	representative
trivial	1	1	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
anisotropic qr	$p(p+1)$	$(p-1)/2$	$\begin{pmatrix} 0 & 1 \\ Q & 0 \end{pmatrix}$
anisotropic non-qr	$p(p-1)$	$(p-1)/2$	$\begin{pmatrix} 0 & 1 \\ Q & 0 \end{pmatrix}$
isotropic qr	$(p^2-1)/2$	1	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
isotropic non-qr	$(p^2-1)/2$	1	$\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$

Proof. We will prove this in several steps.

Step 1. *Any* $\begin{pmatrix} x & y \\ z & -x \end{pmatrix} \in \mathfrak{sl}(2;\mathbb{F}_p)$ *is* $\mathrm{SL}(2;\mathbb{F}_p)$*–equivalent to a matrix of the form* $\begin{pmatrix} 0 & y' \\ z' & 0 \end{pmatrix}$.

To see this, note first that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} (1+2bc)x+bdz-acy & * \\ * & * \end{pmatrix},$$

so we want to solve the equations $ad-bc=1$ and $(1+2bc)x+bdz-acy=0$ for a,b,c,d.

- If $y \neq 0$ choose $b=0$, $a=d=1$ and solve $x-cy=0$ for c.
- If $y=0$ and $z \neq 0$ choose $a=0$, $b=-c=1$ and solve for d.
- If $y=z=0$ choose $2bc=-1$, $a=1$ and solve for d.

Step 2. *If* $Q = Q\begin{pmatrix} x & y \\ z & -x \end{pmatrix} \neq 0$ *then* $\begin{pmatrix} x & y \\ z & -x \end{pmatrix}$ *is* $\mathrm{SL}(2;\mathbb{F}_p)$*–equivalent to* $\begin{pmatrix} 0 & 1 \\ Q & 0 \end{pmatrix}$.

We have shown we can assume $x=0$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} bdz-acy & a^2y-b^2z \\ d^2z-c^2y & acy-bdz \end{pmatrix}. \tag{3}$$

Since $Q=yz\neq 0$ we have $y,z\neq 0$ so $a^2y-b^2z=1$ can be solved for a,b. Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ bz & ay \end{pmatrix}$ does what is required.

Step 3. *Excluding the zero-element, if* $Q\begin{pmatrix} x & y \\ z & -x \end{pmatrix} = 0$ *then* $\begin{pmatrix} x & y \\ z & -x \end{pmatrix}$ *is* $\mathrm{SL}(2;\mathbb{F}_p)$*–equivalent to exactly one of* $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ *or* $\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$*, where n is a fixed quadratic non-residue.*

We can assume $x = 0$. If $z = 0$ we conjugate by an element with $a = 0$ to get $y = 0$. Thus we can assume $x = y = 0$ and $z \neq 0$. Now looking at equation (3), one sees that if $x = y = 0$ then z can be changed only by squares.

Step 4. *It remains to verify the sizes of the conjugacy classes.*

For each class in row 2 or 3 we must simply count the number of elements $\begin{pmatrix} x & y \\ z & -x \end{pmatrix}$ with $x^2 + yz = Q$. Here $Q \neq 0$. If Q is a quadratic non-residue then we must have $yz \neq 0$, so for each of p choices of x and each of $p-1$ choices of $y \neq 0$ we get a unique z. There are therefore $p(p-1)$ elements in the class. A similar count gives $p(p+1)$ elements if Q is a residue.

If $Q = 0$ it is easier to work out the isotropy group of a representative of the class. For an element in our normal form $x = y = 0$ the isotropy group consists of all $\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix}$ with $d^2 = 1$. This clearly has size $2p$ so the class has size $|\mathrm{SL}(2;\mathbb{F}_p)|/2p = (p^2-1)/2$. □

We now investigate the $\mathrm{SL}(2;\mathbb{F}_p)$–classes of proper non-trivial subgroups in $\mathfrak{sl}(2;\mathbb{F}_p)$. The group $\mathfrak{sl}(2;\mathbb{F}_p)$ itself has order p^3.

We first consider the subgroups of order p. Using Lemma 4.2 it is clear there are three classes. Namely

I. *Isotropic lines*. Each isotropic line has $(p-1)/2$ isotropic qr elements and $(p-1)/2$ isotropic non-qr elements. There are $p+1$ such lines in this class. A representative is the line $\left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \mid y \in \mathbb{F}_p \right\}$.

R. *Anisotropic qr lines*. Each such line has exactly two elements in each anisotropic qr conjugacy class. There are $p(p+1)/2$ such lines in this class. A representative is the line $\left\{ \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix} \mid y \in \mathbb{F}_p \right\}$.

N. *Anisotropic non-qr lines*. Each such line has exactly two elements in each anisotropic non-qr conjugacy class. There are $p(p-1)/2$ such lines in this class. A representative is the line $\left\{ \begin{pmatrix} 0 & y \\ ny & 0 \end{pmatrix} \mid y \in \mathbb{F}_p \right\}$.

Next, we determine the conjugacy classes of subgroups of order p^2, i.e., planes. Since the Killing form B is nonsingular, the orthogonal complement of such a plane with respect to B will be a line, and vice versa, so we can classify planes up to conjugacy by the conjugacy classes of their orthogonal complements. There are therefore three classes of planes:

$I^\perp$. *Orthogonal complements of isotropic lines*. A representative such plane is

$$I^\perp = \left\{ \begin{pmatrix} x & 0 \\ y & -x \end{pmatrix} \mid x,y \in \mathbb{F}_p \right\}.$$

The Killing form is degenerate on this plane, with nullspace I. This nullspace contains all isotropic elements of the plane and the remaining elements consist of $2p$ elements from each anisotropic qr conjugacy class. The plane has no anisotropic non-qr elements. There are $p+1$ of these planes.

$R^\perp$. *Orthogonal complements of anisotropic qr lines*. A representative such plane is

$$R^\perp = \left\{ \begin{pmatrix} x & y \\ -y & -x \end{pmatrix} \mid x,y \in \mathbb{F}_p \right\}.$$

Such a plane has exactly $2p-2$ isotropic elements, which, together with 0, form two isotropic lines (in $R^\perp$ the lines $x=y$ and $x=-y$). For any $Q \neq 0$ there are exactly $p-1$ elements $X \in R^\perp$ with $Q_B(X) = Q$. Thus such a plane intersects every conjugacy class in $\mathfrak{sl}(2;\mathbb{F}_p)$. There are $p(p+1)/2$ of these planes.

$N^\perp$. *Orthogonal complements of anisotropic non-qr lines*. A representative such plane is

$$N^\perp = \left\{ \begin{pmatrix} x & y \\ -ny & -x \end{pmatrix} \mid x,y \in \mathbb{F}_p \right\}.$$

Such a plane has no isotropic elements and for any $Q \neq 0$ it has $p+1$ elements with $Q_B(X) = Q$. There are $p(p-1)/2$ of these planes.

We note for future reference

Lemma 4.3. *Any plane of type $R^\perp$ is elementwise conjugate in* $\mathrm{SL}(2;\mathbb{Z}/p^2\mathbb{Z})$ *to* $\mathfrak{sl}(2;\mathbb{F}_p)$ □

Proof of Theorem 3.2 (i). We will show that the trivial subgroup is fixed point equivalent to any anisotropic qr line R. It suffices to show that the only finite dimensional representation of $\mathrm{SL}(2;\mathbb{Z}/p^2\mathbb{Z})$ without an R–fixed vector is the trivial representation. Given such a representation, each subgroup H containing R will also have no fixed vector. We will use this information for the subgroups H of type R, $I^\perp$, $R^\perp$, $N^\perp$, and $\mathfrak{sl}(2;\mathbb{F}_p)$ to show the representation must be trivial.

To begin, the sum of the character χ of a representation over the non-zero elements of a line in $\mathfrak{sl}(2;\mathbb{F}_p)$ will only depend on the conjugacy class of the line, and thus give numbers that we shall call $X_I(\chi)$, $X_R(\chi)$, $X_N(\chi)$, depending on whether the line is isotropic, anisotropic qr, or anisotropic non-qr. Also, let $X_0(\chi)$ be the dimension of the representation; this is χ evaluated on the trivial element in $\mathfrak{sl}(2;\mathbb{F}_p)$. If H is a subgroup of $\mathfrak{sl}(2;\mathbb{F}_p)$, then the sum of χ over the elements of H gives $|H|$ times the dimension of the fixed space of the representation restricted to H, hence zero under our assumption that H has no non-trivial fixed points. Since H is a union of lines that are disjoint except at 0, this then gives an equation of the form

$$X_0(\chi) + I_H X_I(\chi) + R_H X_R(\chi) + N_H X_N(\chi) = 0\,.$$

Here the coefficients I_H, R_H, N_H are the number of lines of each type in H. By our discussion above, these numbers for the subgroups of interest to us are:

	I_H	R_H	N_H
$H = R$	0	1	0
$H = I^\perp$	1	p	0
$H = R^\perp$	2	$(p-1)/2$	$(p-1)/2$
$H = N^\perp$	0	$(p+1)/2$	$(p+1)/2$
$H = \mathfrak{sl}(2;\mathbb{F}_p)$	$p+1$	$(p^2+p)/2$	$(p^2-p)/2$

These five different types of subgroups containing R yield five linear equations in the four unknown quantities $X_0(\chi), X_I(\chi), X_R(\chi), X_N(\chi)$. Since already the coefficient matrix of the first four equations,

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & p & 0 \\ 1 & 2 & (p-1)/2 & (p-1)/2 \\ 1 & 0 & (p+1)/2 & (p+1)/2 \end{pmatrix},$$

has nonzero determinant (namely $-p^2$), the equations have only the trivial solution. This implies $X_0(\chi) = 0$, proving the representation is trivial, as desired. $\square$

Remark. By computing the character table of $\mathrm{SL}(2;\mathbb{Z}/p^2\mathbb{Z})$ one can show that there is no other fixed point equivalence in $\mathrm{SL}(2;\mathbb{Z}/p^2\mathbb{Z})$ between non-conjugate subgroups of $\mathfrak{sl}(2;\mathbb{F}_p)$.

Proof of Theorem 3.2 (ii). Let $\mathfrak{p}$ be a prime ideal of $\mathscr{O}_k$ such that $\mathscr{O}_k/\mathfrak{p} = \mathbb{F}_q$ is a proper extension of $\mathbb{F}_p$ and $p > 3$; that such a prime exists follows from the Cebotarev Density Theorem. Consider the following inclusion of short exact sequences:

$$\begin{tikzcd} 1 \arrow[r] & V_p \arrow[r] \arrow[d] & \mathrm{PSL}(2;\mathbb{Z}/p^2\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{PSL}(2;\mathbb{F}_p) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & V_{\mathfrak{p}} \arrow[r] & \mathrm{PSL}(2;\mathscr{O}_k/\mathfrak{p}^2) \arrow[r] & \mathrm{PSL}(2;\mathbb{F}_q) \arrow[r] & 1 \end{tikzcd}$$

By Lemma 4.1 we already know the kernel V_p in the first sequence is $\mathfrak{sl}(2;\mathbb{F}_p)$ (the transition from SL to PSL just factors by $\{\pm I\}$ and does not affect the kernel).

Although we do not need it, we note that $V_\mathfrak{p} = \mathfrak{sl}(2;\mathbb{F}_q)$. If $\mathfrak{p}$ is principal, $\mathfrak{p} = (\pi)$, say, then we could argue as in the proof of Lemma 4.1. In general we can replace k by its localization at $\mathfrak{p}$ without changing the second exact sequence and then $\mathfrak{p}$ becomes principal, so the argument applies.

We claim that in $\mathrm{PSL}(2;\mathscr{O}_k/\mathfrak{p}^2)$ any element of V_p can be conjugated out of V_p. We only need show this for the representatives of conjugacy classes given in Lemma 4.2 and the claim is then a simple calculation using equation (3) with $b \in \mathbb{F}_q - \mathbb{F}_p$, $a = d = 1$, and $c = 0$.

The proof is now complete, since Lemma 4.3 gives elementwise conjugate subgroups in V_p and we have just shown they are primitive in $\mathrm{PSL}(2;\mathscr{O}_k/\mathfrak{p}^2)$. $\square$

5 Locally symmetric manifolds and other generalities

5.1 $\mathbb{R}$–rank 1 geometries

We shall denote by $\mathbf{H}^n_Y$ the n–dimensional hyperbolic spaces modelled on $Y \in \{\mathbb{C},\mathbb{H},\mathbb{O}\}$ (where $n = 2$ when $Y = \mathbb{O}$). The methods used to produce eigenvalue, length, and primitive length equivalent manifolds extend with little fuss to complex, quaternionic, and Cayley hyperbolic manifolds. We give the version for primitive length.

Theorem 5.1. *Let Γ be a torsion-free lattice in $\mathrm{Isom}(\mathbf{H}^n_Y)$. Then there exist infinitely many pairs of finite covers of $M = \mathbf{H}^n_Y/\Gamma$, $\{M_j, N_j\}$ such that*

(a) $L_p(M_j) = L_p(N_j)$,

(b) $\mathrm{vol}(M_j)/\mathrm{vol}(N_j) \to \infty$.

Moreover, (a) and (b) hold for any finite volume Riemannian metric on M.

Proof. The argument we give breaks into a few cases. First, in most cases we have the inclusion

$$\mathrm{PO}_0(B_4;\mathbb{Z}) < \mathbf{G}_{Y,n}(\mathbb{Z})$$

where B_4 is the form from the proof of Theorem 1.2, $\mathbf{G}_{Y,n}$ is $\mathbb{Q}$–algebraic, and $\mathbf{G}_{Y,n}(\mathbb{R})$ with the analytic topology is Lie isomorphic to $\mathrm{Isom}(\mathbf{H}^n_Y)$. For $Y = \mathbb{C}$, this fails only when $n = 1,2$. For $Y = \mathbb{H}$, when $n \geq 3$, this is clear. The remaining cases of $n = 1,2$ follows from the exceptional isometry between $\mathbf{H}^1_\mathbb{H}$ and $\mathbf{H}^4$ together with the isometric inclusion of $\mathbf{H}^1_\mathbb{H}$ into $\mathbf{H}^2_\mathbb{H}$. Finally, for $Y = \mathbb{O}$, this follows from the isometric inclusion of $\mathbf{H}^2_\mathbb{H}$ into $\mathbf{H}^2_\mathbb{O}$. For all these cases, as in the proof of Theorem 1.2, an application of the Strong Approximation Theorem (cf [30], [16]) in combination with the Cebotarev Density Theorem provides infinitely many primes $\mathfrak{p}$ such that Γ surjects onto certain finite groups $\mathbf{G}(S/\mathfrak{p}^2)$ of Lie type which contain $\mathrm{PSL}(2;\mathbb{Z}[i]/p^2\mathbb{Z}[i])$. The proof is completed just as it was in the proof of Theorem 1.2.

It remains to deal with $Y = \mathbb{C}$ and $n = 1,2$. The case of n=1 is simply the case of hyperbolic surfaces. Case 2 cannot be handled indirectly, and we must use primitive pairs in the finite groups $\mathrm{PU}(2,1;\mathscr{O}_k/\mathfrak{p}^2)$, where $k/\mathbb{Q}$ is an imaginary quadratic extension of $\mathbb{Q}$ and $\mathfrak{p}$ is a prime ideal of $\mathscr{O}_k$. Selecting $\mathfrak{p}$ such that $\mathscr{O}_k/\mathfrak{p}$ is a quadratic extension of $\mathbb{F}_p$, we have the short exact sequence

$$1 \longrightarrow \mathfrak{su}(2,1;\mathscr{O}_k/\mathfrak{p}) \longrightarrow \mathrm{SU}(2,1;\mathscr{O}_k/\mathfrak{p}^2) \longrightarrow \mathrm{SU}(2,1;\mathscr{O}_k/\mathfrak{p}) \longrightarrow 1,$$

where $\mathfrak{su}(2,1;\mathscr{O}_k/\mathfrak{p})$ is the Lie algebra of $\mathrm{SU}(2,1)$ over the field $\mathscr{O}_k/\mathfrak{p}$. With the inclusions

$$\mathfrak{sl}(2;\mathbb{F}_p) < \mathfrak{su}(2,1;\mathscr{O}_k/\mathfrak{p}), \quad \Omega(2,1;\mathbb{F}_p) < \mathrm{SU}(2,1;\mathscr{O}_k/\mathfrak{p}),$$

The subgroups $\mathfrak{sl}(2;\mathbb{F}_p)$ and $R^\perp$ are elementwise conjugate in $\mathrm{SU}(2,1;\mathscr{O}_k/\mathfrak{p}^2)$ where $R^\perp$ is a 2–plane from Lemma 4.3. It is straightforward to verify that the pair satisfies the additional requirements needed for the primitive case. $\square$

Our methods also produce eigenvalue equivalent covers for all of these groups as well. In addition, for sufficiently large n, we can produce covers which are both primitive length and eigenvalue equivalent; here $n \geq 5$ and Y can be $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. To do this, by [13] Proposition 4 Window 2 for $n \geq 5$ we can arrange for the simple groups of orthogonal type to contain a copy of $(\mathrm{P})\,\mathrm{SL}(3;\mathbb{F}_p)$ which contains a group of the type given in Theorem 2.7.

5.2 Locally symmetric manifolds

Length and eigenvalue equivalent covers As is clear from this discussion (and the generality of the Strong Approximation Theorem in [30] and [16]) our methods also apply to lattices in every non-compact higher rank simple Lie group. The discussion given at the end of §5.1 also applies in this setting to arrange for the finite groups of Lie type occurring in Strong Approximation to contain a group a copy of $(\mathrm{P})\,\mathrm{SL}(3;\mathbb{F}_p)$.

Primitive length equivalent covers Construction of primitive length equivalent covers over a fixed locally symmetric manifold is more subtle since in many settings the associated fundamental group fails to have the needed condition on maximal cyclic subgroups. It seems interesting to try to weaken the condition on maximal cyclic subgroups to produce examples in this setting.

6 Final Remarks

6.1 Relations among length, primitive length, and eigenvalue equivalence

Example A. Let M be a closed surface of genus at least 2 equipped with a hyperbolic metric. Let G be the alternating group $\mathrm{Alt}(4)$ and H and K the elementwise conjugate pair described in §2.3. Then, given a surjection $\pi_1(M) \to G$, let $\gamma \in \pi_1(M)$ map to $a \in G$ and correspond to a primitive closed geodesic in M (there are infinitely many primitive elements mapping to any element of G). The non-primitive geodesic of M corresponding to γ has four lifts to M_H, two primitive and two not, and it has three lifts to M_K, all non-primitive. Of course, there might accidentally be some unrelated primitive geodesic in M_K of the right length, but for a generic hyperbolic metric, and a homomorphism to G that factors through a free group this does not happen and M_H and M_K are not primitive length equivalent. Indeed, assume γ is the shortest closed geodesic on M and every other closed geodesic has much larger length. Then if γ maps to a, one can see that M_H and M_K are not primitive length equivalent.

Example B. Eigenvalue equivalent surfaces obtained from Theorem 3.1 using the trivial subspace $\{0\}$ and any proper subspace of $\mathbb{F}_p^n$ generically produce examples which are not length equivalent. In particular, eigenvalue equivalence need not imply length or primitive length equivalence.

It seems plausible that length equivalent hyperbolic examples constructed from Theorem 3.1 using $\mathbb{F}_p^n$ and any nontrivial subspace of $\mathbb{F}_p^n$ will generically fail to be eigenvalue equivalent, but this is more subtle. Using the results of Zelditch [31] it is easy to see that for a hyperbolic manifold M^m of sufficiently high dimension this approach *will* give length equivalent but not eigenvalue equivalent examples for generic (not necessarily hyperbolic) deformations of the metric on M. Using $G = \mathrm{Alt}(4)$ this allows one to find such examples in dimensions $m \geq 3$.

All of our examples of primitive length equivalence are also examples of eigenvalue equivalence.

Question 6.1. Are two primitive length equivalent hyperbolic manifolds necessarily eigenvalue equivalent?

6.2 Complex lengths

All our results for equal length sets actually produce manifolds which have the same **complex length sets**. Recall that the complex length of a closed geodesic γ in a Riemannian m–manifold is a pair $(\ell(\gamma), V)$ where $\ell(g)$ is the length of γ and

$V \in \mathrm{O}(m-1)$ is determined by the holonomy of γ. The complex length spectrum is the collection of such complex lengths with multiplicities, and the complex length set forgets multiplicities as before. The point is that Theorem 2.3 gives manifolds with the same complex length sets, just as Sunada's theorem gives equal complex length spectra. See [23] for more on the complex length spectrum.

6.3 Commensurability

The known methods of producing isospectral or iso-length spectral hyperbolic manifolds result in commensurable manifolds and it is an open question as to whether this is always the case. By construction, the eigenvalue and (primitive) length equivalent hyperbolic manifolds constructed here are also commensurable.

Question 6.2. Let M_1 and M_2 be eigenvalue (resp. length or primitive length) equivalent closed hyperbolic manifolds. Are they commensurable?

There has been some recent activity on this question. It is shown that Question 6.2 has an affirmative answer in the length equivalent setting if the manifolds M_1 and M_2 are arithmetic hyperbolic 3–manifolds ([3]), or if the manifolds are even dimensional arithmetic hyperbolic manifolds ([21]). Indeed, the results of [21] apply to more general locally symmetric spaces. In contrast, [21] also exhibts arbitrarily large collections of incommensurable hyperbolic 5–manifolds which are length commensurable. The commensurability classes of these manifolds seem to be the best candidates for producing a negative answer Question 6.2.

6.4 Infinite sets of examples

Our constructions show that there can be no uniform bound on the number of pairwise eigenvalue (resp. length or primitive length) equivalent, non-isometric manifolds. Thus a natural question is.

Question 6.3. Are there infinite sets of pairwise eigenvalue (resp. length or primitive length) equivalent, closed hyperbolic m–manifolds?

In the context of length equivalence a positive answer would follow if one can find infinitely many mutually elementwise conjugate subgroups of finite index in a finitely generated free group. C. Praeger pointed out to us that a slightly stronger version of this question is listed as an open problem (Problem 11.71) in the Kourovka Notebook [15]. It was asked there in the parallel context of Kronecker equivalence of number fields. It seems likely that the answer to this question is "no", but the limited partial answers that are known involved considerable effort, see [20].

6.5 Can one hear the size of a drum?

Mark Kac's famous paper "Can one hear the shape of a drum" [9] is quoted in many papers on isospectrality. Of course, the "drums" of his title were not closed hyperbolic manifolds, but rather flat plane domains. The first pair of different "drums" with the same sound (i.e., non-isometric isospectral plane domains) were found in the 1990's by Gordon, Webb, and Wolpert [5].

However, one might question whether the sounds of their drums D_1 and D_2 are really indistinguishable. They comment: "... to produce the same sound (i.e., the same frequencies with the same amplitudes) as would result from striking D_1 at a given point with a given (unit) intensity ... one must strike D_2 simultaneously at seven points with appropriate intensities". A more obvious example of this issue is a pretty example of S. Chapman [2]. Chapman reinterprets earlier discussion of the Gordon–Webb–Wolpert examples in terms of paper folding and cutting, as is familiar from making paper dolls. Of course, by cutting too much one can create disconnected objects, and by this means Chapman derives from the Gordon–Webb–Wolpert example the following simple example: D_1 is the disjoint union of a unit square and an isosceles right triangle with legs of length 2, and D_2 is the disjoint union of a 1×2 rectangle and an isosceles right triangle with legs of length $\sqrt{2}$. This pair of domains is isospectral, but one can ask to what extent they really sound the same.

A more honest example of equal sound might be the following: purchase three identical drums and let D_1 consists of one of them and D_2 consist of the disjoint union of the other two. It would be hard to distinguish D_1 from D_2 on hearing a drummer strike either one once. This example suggests that eigenvalue equivalence may have as much right as isospectrality to be interpreted as "same sound."

In his paper Kac gave a proof that drums that sound the same have equal area, but this was based on isospectrality. Revisiting this in the context of eigenvalue equivalence we ask:

Question 6.4. Do there exist connected eigenvalue equivalent plane domains of unequal area?

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