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ALL KNOWN PRINCIPAL CONGRUENCE LINKS

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Abstract. This report lists the link diagrams in S^3 for all principal congruence link complements for which such a link diagram is known. Several unpublished link diagrams are included. Related to this, we also include one link diagram for an arithmetic regular tessellation link complement.

Introduction

Given a square-free positive integer d and an ideal I in the ring of integers O_d of the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$, the principal congruence manifold (or orbifold) associated with (d, I) is the quotient $\mathbb{H}^3/\Gamma(I)$ where

$$\Gamma(I) = \ker\left(\mathrm{PSL}(2, O_d) \to \mathrm{SL}(2, O_d/I)/\{\pm 1\}\right).$$

There are only finitely principal congruence manifolds that are link complements in S^3 and the complete list was given in [BGR18]. We refer the reader to [BR18] and [BR14] for an introduction to and further background on congruence manifolds.

This report serves as a repository of link diagrams for the cases (d, I) for which such a diagram is known. Since there are, in general, many links having the same complement, we show only one link diagram for a case. We plan to update this document as more principal congruence links become known. Please contact us if you have constructed a new link or know about a case not listed here. Furthermore, if you cite this article and refer to a particular link, do so by using the pair (d, I) rather than the figure number as the figure numbers might potentially change as more principal congruence links become known.

All the link diagrams are also available files [Goe18, prinCong/Links/] which can be opened and viewed with SnapPy [CDGW18], e.g.,

```
cd DIRECTORY_WITH_LINK_FILES
M = Manifold("pSL_0.5_plus_0.5_sqrt_minus_11.lnk")
M.plink()
```

Thematically related are the regular tessellation link complements defined in [Gör15] and we also include one link (Figure 16) even though it is not principal congruence.

A dot in a link diagram denotes a link component that intersects the paper plane orthogonally.

$d = 1$

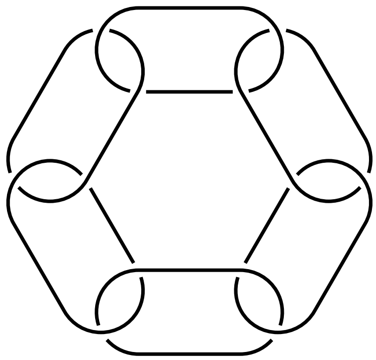

FIGURE 1. $(1, \langle\ 2\ \rangle)$.
[Bak81, Fig. I-17]
[BR14, Fig. 1a].

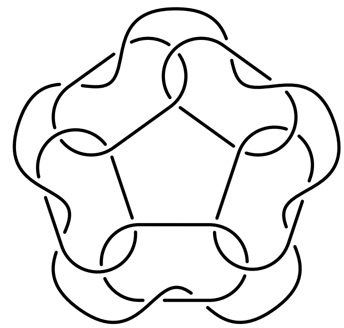

FIGURE 2. $(1, \langle\ 2+\sqrt{-1}\ \rangle)$.
By second author.

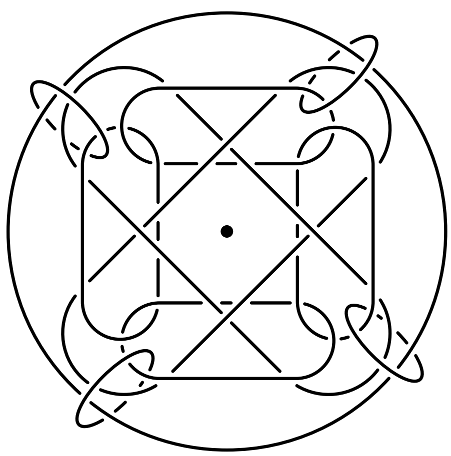

FIGURE 3. $(1, \langle\ 2+2\sqrt{-1}\ \rangle)$. By second author.

$d = 2$

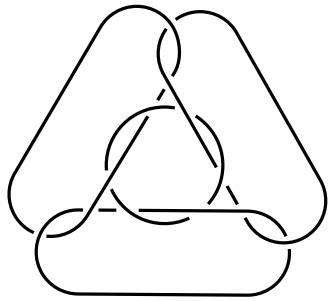

FIGURE 4. $(2, \langle\ 1 + \sqrt{-2}\ \rangle)$. [Thu79, Example 6.8.10] [BR14, Fig. 5a].

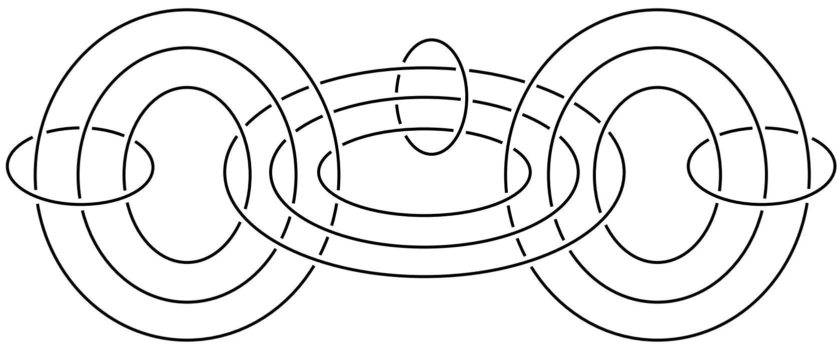

FIGURE 5. $(2, \langle\ 2\ \rangle)$. [Bak81, Fig. I-13] [BR14, Fig. 1b].

$d = 3$

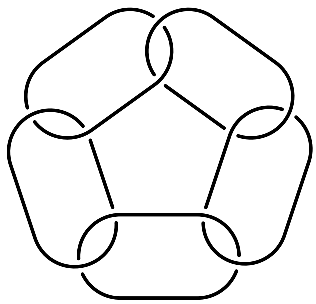

FIGURE 6. $(3, \langle\ 2\ \rangle)$.
[Bak81, Fig. I-9]
[DT03, Fig. 2].

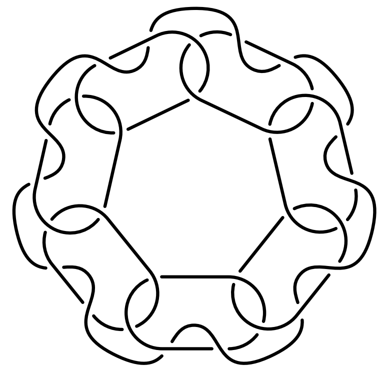

FIGURE 7. $(3, \langle\ \frac{5+\sqrt{-3}}{2}\ \rangle)$.
[Thu98].

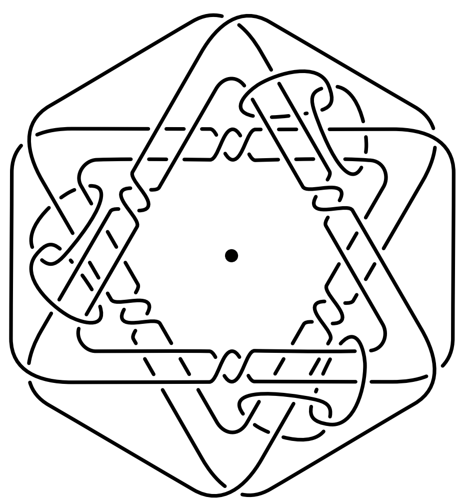

FIGURE 8. $(3, \langle\ 3\ \rangle)$. [Goe11, Fig. 1.7] .

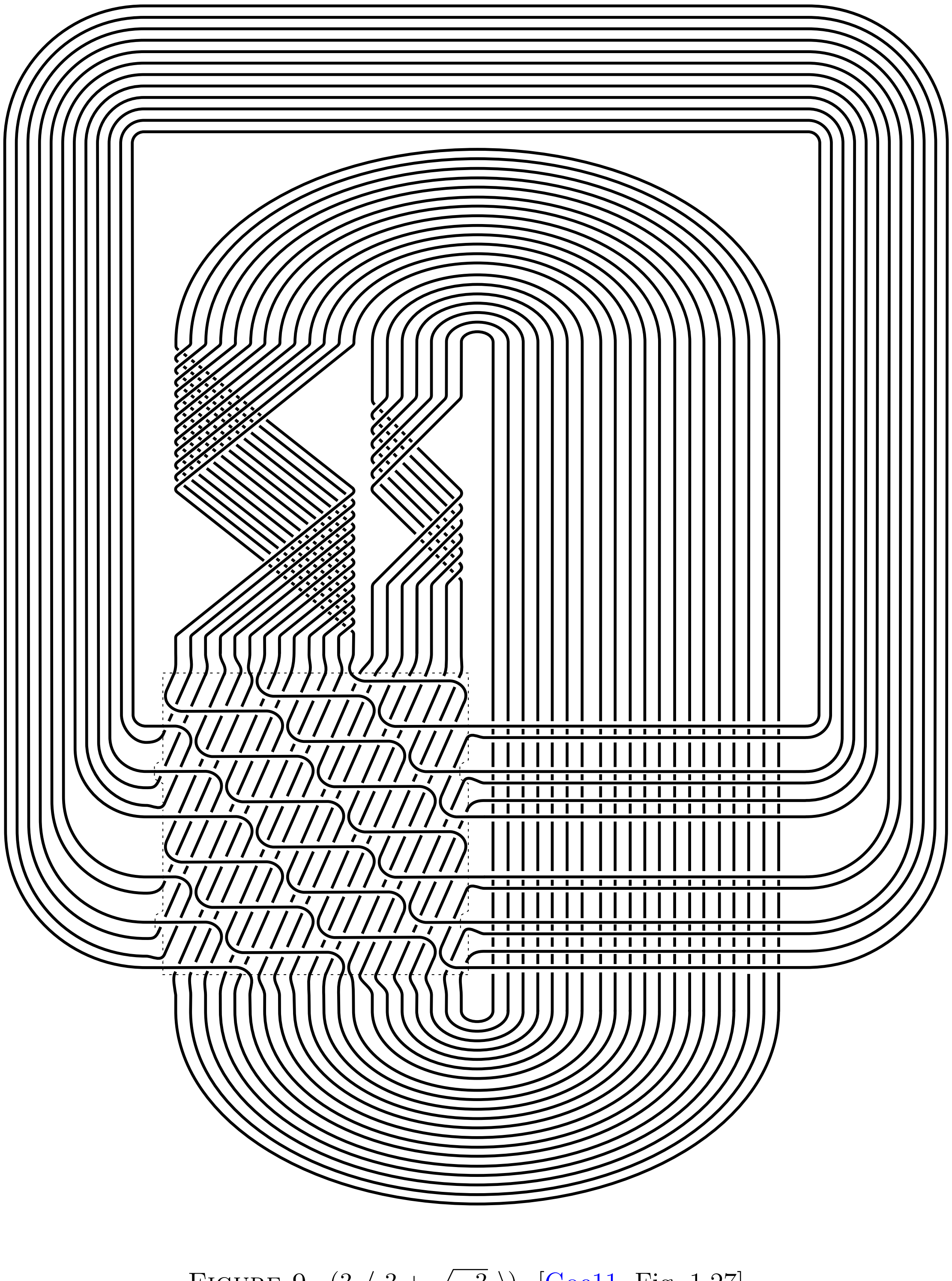

FIGURE 9. $(3, \langle\ 3+\sqrt{-3}\ \rangle)$. [Goe11, Fig. 1.27] .

$d = 7$

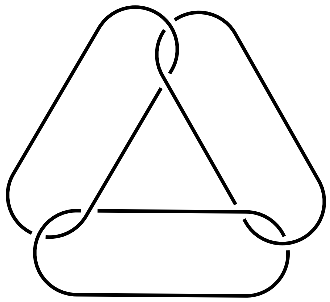

FIGURE 10. $(7, \langle \frac{1+\sqrt{-7}}{2} \rangle)$.
[GS93, Fig 1(a)]
[BR14, Fig. 3].

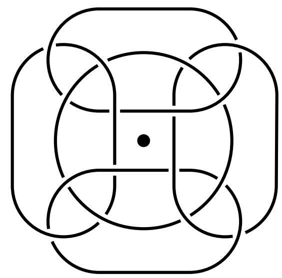

FIGURE 11. $(7, \langle \frac{3+\sqrt{-7}}{2} \rangle)$.
By second author.

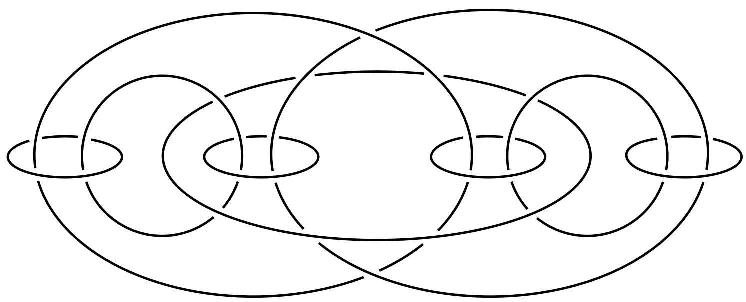

FIGURE 12. $(7, \langle 2 \rangle)$. [Bak81, Fig. I-5] [BR14, Fig. 1d].

$d = 11$

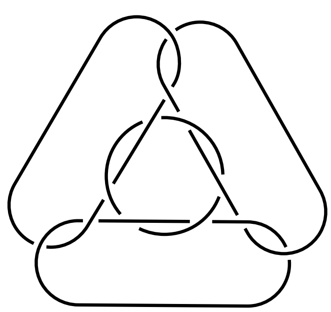

FIGURE 13. $(11, \langle \frac{1+\sqrt{-11}}{2} \rangle)$. [Hat83, Fig. 9] [BR14, Fig. 5b].

$d = 15$

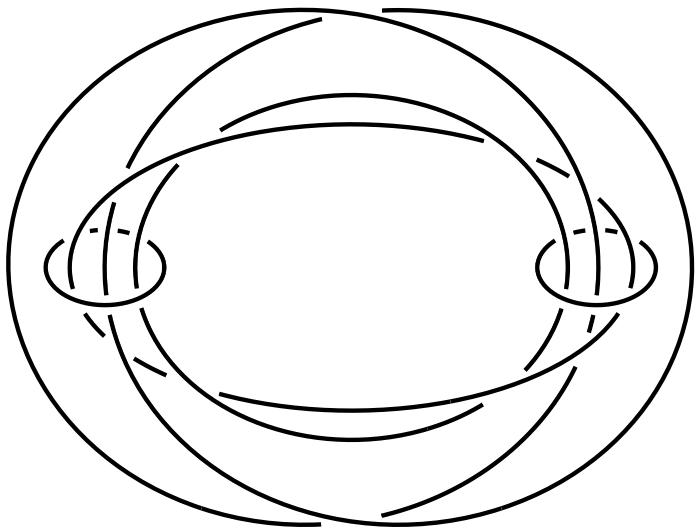

FIGURE 14. $(15, \langle\, 2, \frac{1+\sqrt{-15}}{2}\,\rangle)$. [Bak92, Fig. 1] [BR14, Fig. 4a].

$d = 23$

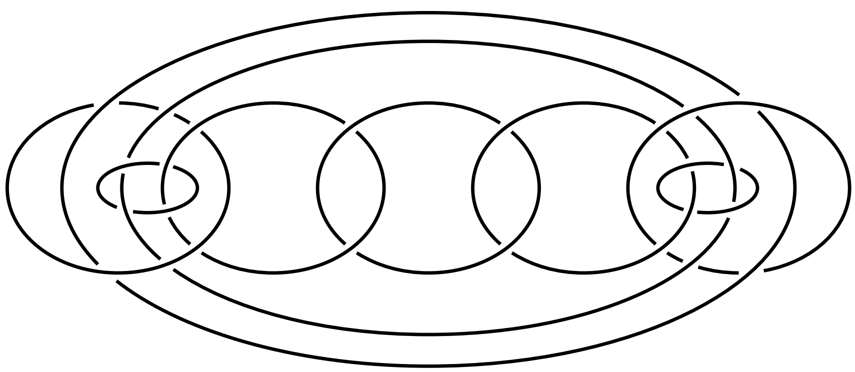

FIGURE 15. $(23, \langle\, 2, \frac{1+\sqrt{-23}}{2}\,\rangle)$. [Bak92, Fig. 2] [BR14, Fig. 4b].

Cubical

The link in Figure 16 is a regular tessellation link (as defined in [Gör15]) but not a principal congruence link. Similar to a principal congruence manifold, a regular tessellation manifold is a regular covering space of an orbifold. In case of a regular tessellation manifold, this orbifold is the quotient of $\mathbb{H}^3$ by a Coxeter group instead of a Bianchi group $\mathrm{PSL}(2, O_d)$. In the case of the link shown in Figure 16, the complement admits a tessellation into 6 regular ideal cubes such that the symmetries of the complement act transitively on flags. The complement is also arithmetic as it is also a non-regular cover of the Bianchi orbifold $Q_3 = \mathbb{H}^3/\mathrm{PSL}(2, O_3)$. We refer the reader to [Gör15] for further details.

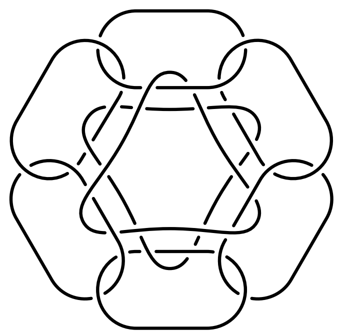

Figure 16. $\mathcal{U}^{\{4,3,6\}}_{\frac{3+\sqrt{-3}}{2}}$. By second author.

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