Arithmetic Hyperbolic Manifolds

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Plan for the lectures

A basic example and some preliminary material on bilinear and quadratic forms, $\mathbb{H}^n$ and $\text{Isom}(\mathbb{H}^n)$.

Arithmetic hyperbolic manifolds of simplest type.

Why you might care.

Geometric bounding

Dimensions 2 and 3 versus higher dimensions.
A basic example $\text{PSL}(2, \mathbb{Z})$

$\text{SL}_2(\mathbb{R})$ acts on the set $S$ of $2 \times 2$ real symmetric matrices.

Given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$, and $S = \begin{pmatrix} x & y \\ y & z \end{pmatrix} \in S$ we have:

$$g \cdot S \mapsto gSg^t.$$ 

Note that since $g \in \text{SL}_2(\mathbb{R})$,

$$\det(gSg^t) = \det(S) = xz - y^2.$$
$S$ is a 3-dimensional vector space, and using a basis for $S$ we get a representation $\rho : \text{SL}_2(\mathbb{R}) \to \text{GL}_3(\mathbb{R})$:

$$
\rho(g) = \begin{pmatrix}
    a^2 & 2ab & b^2 \\
    ac & bc + ad & bd \\
    c^2 & 2cd & d^2
\end{pmatrix}
$$

$\det(\rho(g)) = 1$ so $\rho(g) \in \text{SL}_3(\mathbb{R})$.

Using $\det(gSg^t) = \det(S) = xz - y^2$, it follows that $\rho(g)$ preserves the quadratic form $xz - y^2$; i.e. Set

$$
J = \begin{pmatrix}
    0 & 0 & \frac{1}{2} \\
    0 & -1 & 0 \\
    \frac{1}{2} & 0 & 0
\end{pmatrix}
$$

$$
\rho(g).J.\rho(g)^t = J.
$$
\( \ker(\rho) = \pm I \) and so this gives an isomorphism of \( \text{PSL}_2(\mathbb{R}) \) onto a subgroup of:

\[
\text{SO}(xz - y^2, \mathbb{R}) = \{ X \in \text{SL}_3(\mathbb{R}) : XJX^t = J \}.
\]

In fact \( \text{PSL}_2(\mathbb{R}) \cong \) a subgroup of index 2.

Moreover this maps \( \text{PSL}_2(\mathbb{Z}) \) onto a subgroup of

\[
\text{SO}(xz - y^2, \mathbb{Z}) = \{ X \in \text{SL}_3(\mathbb{Z}) : XJX^t = J \}.
\]
Make a change of basis: $u = (x + z)/2$ and $v = (x - z)/2$.

$$xz - y^2 = u^2 - v^2 - y^2.$$ 

PSL$_2$($\mathbb{R}$) still maps isomorphically onto a subgroup of

SO($u^2 - v^2 - y^2$, $\mathbb{R}$)

$$= \{X \in SL_3(\mathbb{R}) : X\text{diag}\{1, -1, -1\}X^t = \text{diag}\{1, -1, -1\}\}$$

but:

PSL$_2$($\mathbb{Z}$) does not map into SO($u^2 - v^2 - y^2$, $\mathbb{Z}$).
Another comment on this representation of $\text{PSL}(2, \mathbb{R})$:

Suppose $n > 1$ and let $\Gamma_0(n) < \text{PSL}(2, \mathbb{Z})$ denote the subgroup consisting of those elements congruent to $\pm \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ (mod $n$).

Note that $\tau_n = \begin{pmatrix} 0 & -1/\sqrt{n} \\ \sqrt{n} & 0 \end{pmatrix}$ normalizes $\Gamma_0(n)$.

Hence $\langle \Gamma_0(n), \tau_n \rangle \subset N_{\text{PSL}(2, \mathbb{R})}(\Gamma_0(n))$ is commensurable with $\text{PSL}(2, \mathbb{Z})$, not a subgroup of $\text{PSL}(2, \mathbb{Z})$ or even $\text{PSL}(2, \mathbb{Q})$ if $n$ is square-free.

But under the representation $\rho$ described above:

$$\rho(\tau_n) = \begin{pmatrix} 0 & 0 & 1/n \\ 0 & -1 & 0 \\ n & 0 & 0 \end{pmatrix}$$

it is rational!
What's so special about \( xz - y^2 \) or \( u^2 - v^2 - y^2 \)?

Take \( ax^2 + by^2 - cy^2 \), \( a, b, c \) integers and \( > 0 \).

Consider

\[
= \{ X \in \text{SL}_3(\mathbb{R}) : X \text{diag}\{a, b, -c\} X^t = \text{diag}\{a, b, -c\} \}
\]

and the discrete subgroup:

\[
= \{ X \in \text{SL}_3(\mathbb{Z}) : X \text{diag}\{a, b, -c\} X^t = \text{diag}\{a, b, -c\} \}
\]

What can we say about this discrete group?
They are infinite.

\( \{ X \in \text{SL}_2(\mathbb{Z}) : X \text{diag} \{ b, -c \} X^t = \text{diag} \{ b, -c \} \} \) gives an infinite cyclic subgroup.

e.g Take \( y^2 - 3z^2 \), and \( X = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \), then

\( X \text{diag} \{ 1, -3 \} X^t = \text{diag} \{ 1, -3 \} \).

If \( ax_0^2 + by_0^2 - cz_0^2 = 0 \) then can build a unipotent element \((x_0, y_0, z_0 \text{ not all 0})\).
Bilinear and Quadratic Forms

Let $V$ be a finite dimensional vector space over $k$, with characteristic of $k \neq 2$.

By a (symmetric) bilinear form $B$ on $V$, we mean a map

$$B : V \times V \longrightarrow k$$

such that

(i) $B(u, v) = B(v, u)$, for all $u, v \in V$.

(ii) $B(u + u', v) = B(u, v) + B(u', v)$. for all $u, u', v \in V$.

(iii) $B(\alpha u, v) = \alpha B(u, v)$, for all $\alpha \in k$ and $u, v \in V$. 
Definition With $V$ and $B$ as above, we call $(V, B)$ a bilinear space.

Associated to $B$ is a quadratic map

$$q : V \longrightarrow k$$

defined by

$$q(v) = B(v, v).$$

We see that $q$ satisfies

(i)

$$q(\alpha v) = \alpha^2 q(v),$$

for all $\alpha \in k$ and $v \in V$.

(ii)

$$q(u + v) - q(u) - q(v) = 2B(u, v),$$

for all $u, v \in V$. 
By specifying a basis for $V$, $\mathcal{B} = \{e_i\}$, one can write $B$ and $q$ as follows:

Associated to $B$ is the symmetric matrix

\[
\begin{pmatrix}
B(e_i, e_j)
\end{pmatrix}
\]

and

\[
q = q_B(x) = x^T \begin{pmatrix} B(e_i, e_j) \end{pmatrix} x.
\]

is the associated quadratic form for the basis $\mathcal{B}$.

All bilinear forms (or quadratic forms) will be non-degenerate (i.e. $B(x, y) = 0$ for all $y \in V$ implies $x = 0$)
Example:

Let $V = \mathbb{R}^{n+1}$ with the standard basis $\mathcal{B} = \{e_i\}$.

Define $B = \langle \cdot , \cdot \rangle$ by

$$\langle x, y \rangle = x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1},$$

where

$$x = (x_1, \ldots, x_{n+1}) \quad y = (y_1, \ldots, y_{n+1}).$$

and the quadratic form

$$q(x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2.$$
Let $V_1$ and $V_2$ be $n$-dimensional vector spaces over $k$ equipped with quadratic forms $q_1$ and $q_2$ (call the associated symmetric matrices $Q_1$ and $Q_2$).

Say $(V_1, q_1)$ is **equivalent over $k$** to $(V_2, q_2)$ if there exists $T \in \text{GL}_n(k)$ so that:

$$T^t Q_1 T = Q_2$$

Write $q_1 \simeq_k q_2$. 
Example Take $V_1 = V_2 = \mathbb{R}^3$ and

\[
q_1 = x_1^2 + x_2^2 - x_3^2, \quad q_2 = x_1^2 + x_2^2 - 3x_3^2
\]

\[
q_3 = x_1^2 + x_2^2 - 4x_3^2, \quad q_4 = x_1x_2 + x_3^2
\]

$q_1 \cong_\mathbb{R} q_2$

$q_1 \cong_\mathbb{Q} q_3$

Is $q_1 \cong_\mathbb{Q} q_2$?

$q_1 \cong_\mathbb{Q} q_4$. 
Equivalence over $\mathbb{R}$

Let $V = \mathbb{R}^n$, $B$ and $q$ be bilinear and quadratic forms.

**Sylvester’s Law:** There exists a basis $\{v_1, \ldots, v_n\}$ of $V$ such that $q$ has the description

$$Q = \left(B(v_i, v_j)\right) = \begin{cases} 0, & i \neq j \\ 1, & 1 \leq i \leq p \\ -1, & p < i \leq n \end{cases},$$

for some $p$.

So

$$Q = \text{diag}(1, 1, \ldots, 1, -1, -1, \ldots, -1),$$

with $p$ 1’s and $s = (n - p) - 1$’s.

$(p, s)$ is called the **signature** of the form.
If $Q_1$ and $Q_2$ are symmetric, invertible matrices over $\mathbb{R}$, then

$$Q_1 \sim_{\mathbb{R}} Q_2,$$

if and only if the signature of $Q_1$ and $Q_2$ are the same.

**Example**

The forms

$$q = x_1^2 + \ldots + x_n^2 - x_{n+1}^2, \quad q' = x_1^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}^2$$

have signature $(n, 1)$ and so are equivalent over $\mathbb{R}$.

The forms

$$q_1 = x_1^2 + \ldots + x_n^2 + x_{n+1}^2, \quad q'_1 = x_1^2 + \ldots + x_n^2 + \sqrt{2}x_{n+1}^2$$

have signature $(n + 1, 0)$ and so are equivalent over $\mathbb{R}$. 
Hyperboloid Model

$< \cdot , \cdot >$ will denote the bilinear form on $\mathbb{R}^{n+1}$ described earlier.

Let

$$\mathbb{H}^n = \{(x_1, \ldots , x_{n+1}) \in \mathbb{R}^{n+1} : < x , x > = -1 , x_{n+1} > 0\}.$$ 

We shall define a metric $d$ on $\mathbb{H}^n$ and $(\mathbb{H}^n , d)$ will be the hyperboloid model of hyperbolic $n$-space.
Proposition

Let

\[ d : \mathbb{H}^n \times \mathbb{H}^n \rightarrow \mathbb{R} \]

be the function that assigns to each pair \((x, y) \in \mathbb{H}^n \times \mathbb{H}^n\) the unique number \(d(x, y) \geq 0\) such that

\[ \cosh d(x, y) = -<x, y>. \]

Then \(d\) is a metric on \(\mathbb{H}^n\).
Remarks: $\cosh d(x, y) = -\langle x, y \rangle$ is well-defined since

$$\langle x, y \rangle \leq -1 \text{ for all } x, y \in \mathbb{H}^n$$

(Cauchy Schwartz)

Equality holds iff $x = y$ since $\langle x, y \rangle = -1$ iff $x = y$.

Symmetry follows from $\langle x, y \rangle = \langle y, x \rangle$.

Triangle inequality requires work — uses the Hyperbolic Law of Cosines:

Let $A$, $B$, and $C$ be distinct points in $\mathbb{H}^n$. Form the triangle with these points as the vertices: Let $\gamma$ be the angle at the vertex of $C$: With $a = d(B, C)$, $b = d(A, C)$, and $c = d(A, B)$,

Then

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma.$$
Isometries

\[ O(n, 1) = \{ S \in \text{GL}_{n+1}(\mathbb{R}) \mid S^T J_n S = J_n \}, \]

where

\[ J_n = \text{diag}(1, 1, \ldots, 1, -1) \]

is called the Orthogonal group of the quadratic form

\[ x_1^2 + \cdots + x_n^2 - x_{n+1}^2. \]

This is equivalent to

\[ O(n, 1) = \{ S \in \text{GL}_{n+1}(\mathbb{R}) \mid < Su, Sv > = < u, v >, \ u, v \in \mathbb{R}^{n+1} \}, \]

where \(< \cdot, \cdot >\) is as before.
Note that the matrix $\text{diag}(-1, 1, \ldots, 1) \in \text{O}(n, 1)$ has determinant $-1$.

Define: $\text{SO}(n, 1) = \text{O}(n, 1) \cap \text{SL}_{n+1}(\mathbb{R})$, called the **Special Orthogonal Group**.

This has index 2 in $\text{O}(n, 1)$.

Define

$$\text{O}_0(n, 1) = \{S \in \text{O}(n, 1) : S \text{ preserves } \mathbb{H}^n\},$$

and

$$\text{SO}_0(n, 1) = \text{SO}(n, 1) \cap \text{O}_0(n, 1).$$

$\text{diag}(1, 1, \ldots, 1, -1) \in \text{O}(n, 1)$ flips $x_{n+1}$ to $-x_{n+1}$ and so

$$[\text{O}(n, 1) : \text{O}_0(n, 1)] = 2.$$
By construction of the metric $d$, $O_0(n, 1)$ preserves $d$. Thus

$$O_0(n, 1) \subset \text{Isom}(\mathbb{H}^n).$$

**Theorem 1**

$$O_0(n, 1) = \text{Isom}(\mathbb{H}^n) \text{ and } SO_0(n, 1) = \text{Isom}^+(\mathbb{H}^n).$$

**Definition**

By a hyperbolic $n$-manifold we mean a manifold (resp. orbifold) $M^n = \mathbb{H}^n/\Gamma$ where $\Gamma < O_0(n, 1)$ is torsion-free (otherwise).

If $\Gamma < SO_0(n, 1)$, $M^n$ is orientable.

$M^n = \mathbb{H}^n/\Gamma$ has **finite volume** if $\Gamma$ admits a fundamental polyhedron of finite volume (say $\Gamma$ has finite co-volume).

Say $\Gamma$ is **cocompact** if $M^n$ is closed.
How do we construct examples closed or finite volume hyperbolic $n$-manifolds?

Note: $O(n,1,\mathbb{Z}) = O(n,1) \cap GL_{n+1}(\mathbb{Z})$ is a discrete subgroup of $O(n,1)$

**Example:** $O_0(2,1,\mathbb{Z}) = (2,4,\infty)$ reflection triangle group. In particular $O_0(2,1,\mathbb{Z})$ has finite co-volume and is non-cocompact.
Reflection generators:

$$\tau_{e_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(reflection in the plane $\langle z, e_2 \rangle = 0$)

$$\tau_v = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(reflection in the plane $\langle z, (-1/\sqrt{2}, 1/\sqrt{2}, 0) \rangle = 0$)

$$\tau_u = \begin{pmatrix} -1 & -2 & 2 \\ -2 & -1 & 2 \\ -2 & -2 & 3 \end{pmatrix}$$

(reflection in the plane $\langle z, (1, 1, 1) \rangle = 0$)
Arithmetic groups of simplest type

Let $k$ be a totally real number field of degree $d$ over $\mathbb{Q}$ equipped with a fixed embedding into $\mathbb{R}$ which we refer to as the identity embedding, and denote the ring of integers of $k$ by $R_k$.

Let $V$ be an $(n + 1)$-dimensional vector space over $k$ equipped with a quadratic form $f$ (with associated symmetric matrix $F$) defined over $k$ which has signature $(n, 1)$ at the identity embedding, and signature $(n + 1, 0)$ at the remaining $d - 1$ embeddings.

Call such quadratic forms admissible.

Define the linear algebraic groups defined over $k$:

$$O(f) = \{X \in \text{GL}_{n+1}(\mathbb{C}) : X^tFX = F\}$$ and

$$SO(f) = \{X \in \text{SL}_{n+1}(\mathbb{C}) : X^tFX = F\}.$$
For a subring $L \subset \mathbb{C}$, we denote the $L$-points of $O(f)$ (resp. $SO(f)$) by $O(f, L)$ (resp. $SO(f, L)$).

An arithmetic lattice in $O(f)$ (resp. $SO(f)$) is a subgroup $\Gamma < O(f)$ commensurable with $O(f, R_k)$ (resp. $SO(f, R_k)$).

Note that an arithmetic subgroup of $SO(f)$ is an arithmetic subgroup of $O(f)$, and an arithmetic subgroup $\Gamma < O(f)$ determines an arithmetic subgroup $\Gamma \cap SO(f)$ in $SO(f)$.
Examples:

1. Let \( q = a_1 x_1^2 + \cdots + a_n x_n^2 - a_{n+1} x_{n+1}^2 \), where \( a_i > 0 \) and \( a_i \in \mathbb{Z} \). This has signature \((n, 1)\).

2. \( f = x_1^2 + \cdots + x_n^2 - \sqrt{2} x_{n+1}^2 \).

Then for the two Galois embeddings

\[
\sigma_1 : \sqrt{2} \mapsto \sqrt{2} \\
\sigma_2 : \sqrt{2} \mapsto -\sqrt{2}.
\]

we see that \( f \) has signature \((n, 1)\) and \( f_2^\sigma = x_1^2 + \cdots + x_n^2 + \sqrt{2} x_{n+1}^2 \), so has signature \((n + 1, 0)\).
With $q$ as in Example 1, $\text{SO}(q, \mathbb{Z})$ is clearly discrete.

What about Example 2?

What about co-compact or finite covolume?
Will discuss this below.
The form \( q \) may or may not represent 0 over \( \mathbb{Q} \) (equivalently \( \mathbb{Z} \)).

e.g. \( x_1^2 + x_2^2 - x_3^2 \) does, \( x_1^2 + x_2^2 - 3x_3^2 \) does not.

\( x_1^2 + x_2^2 + x_3^2 - 3x_3^2 \) does, \( x_1^2 + x_2^2 + x_3^2 - 7x_3^2 \) does not.

Let \( d \) be any positive integer then \( x_1^2 + x_2^2 + x_3^2 + x_4^2 - dx_5^2 \) represents 0 (\( d \) is a sum of 4 squares).

More generally **Meyer’s Theorem** shows that whenever \( n \geq 4 \), \( q \) as in Example 1 always represent 0 non-trivially.

In the second example there is no solution over \( \mathbb{Q}(\sqrt{2}) \) to the equation \( f(x) = 0 \).
Recap from Lecture 1

Arithmetic groups of simplest type

Let \( k \) be a totally real number field of degree \( d \) over \( \mathbb{Q} \) equipped with a fixed embedding into \( \mathbb{R} \) which we refer to as the identity embedding, and denote the ring of integers of \( k \) by \( R_k \).

Let \( V \) be an \((n + 1)\)-dimensional vector space over \( k \) equipped with a quadratic form \( f \) (with associated symmetric matrix \( F \)) defined over \( k \) which has signature \((n, 1)\) at the identity embedding, and signature \((n + 1, 0)\) at the remaining \( d - 1 \) embeddings.

Call such quadratic forms admissible.
Define the linear algebraic groups defined over $k$:

$$O(f) = \{ X \in \text{GL}_{n+1}(\mathbb{C}) : X^tFX = F \}$$

$$SO(f) = \{ X \in \text{SL}_{n+1}(\mathbb{C}) : X^tFX = F \}.$$ 

For a subring $L \subset \mathbb{C}$, we denote the $L$-points of $O(f)$ (resp. $SO(f)$) by:

$$O(f, L) = O(f) \cap \text{GL}_{n+1}(L), \quad SO(f, L) = SO(f) \cap O(f, L)$$

An arithmetic lattice in $O(f)$ (resp. $SO(f)$) is a subgroup $\Gamma < O(f)$ commensurable with $O(f, \mathbb{R}_k)$ (resp. $SO(f, \mathbb{R}_k)$).
Two examples to keep in mind:

1. Let \( q = a_1x_1^2 + \cdots + a_nx_n^2 - a_{n+1}x_{n+1}^2 \), where \( a_i > 0 \) and \( a_i \in \mathbb{Z} \). This has signature \((n, 1)\).

Meyer's Theorem says this represents 0 non-trivially over \( \mathbb{Z} \) whenever \( n \geq 4 \).

2. \( f = x_1^2 + \cdots + x_n^2 - \sqrt{2}x_{n+1}^2 \).

\( f \) has signature \((n, 1)\) and \( f^2 = x_1^2 + \cdots + x_n^2 + \sqrt{2}x_{n+1}^2 \), so has signature \((n + 1, 0)\).
Theorem 2

Let $q$ be an admissible quadratic form over the totally real field $k$. Then $\text{SO}(q, \mathbb{R}_k)$ is a discrete subgroup of finite covolume in $\text{SO}(q, \mathbb{R})$. Moreover it is cocompact if and only if $q$ is anisotropic.

Suppose that $q$ is a quadratic form of signature $(n, 1)$.

By Sylvester’s Theorem, there exists $T \in \text{GL}_{n+1}(\mathbb{R})$ such that $T^tQT = J_n$.

This effects a conjugation:

$$T^{-1}O(Q, \mathbb{R})T = O(J_n, \mathbb{R}) = O(n, 1).$$
A subgroup \( \Gamma < O_0(n, 1) \) is called *arithmetic of simplest type* if \( \Gamma \) is commensurable with the image in \( O_0(n, 1) \) of an arithmetic subgroup of \( O(f) \) (under the conjugation map described above).

An arithmetic hyperbolic \( n \)-manifold \( M = \mathbb{H}^n / \Gamma \) is called *arithmetic of simplest type* if \( \Gamma \) is.

The same set-up using special orthogonal groups constructs orientation-preserving arithmetic groups of simplest type (and orientable arithmetic hyperbolic \( n \)-manifolds of simplest type).
Discreteness

1. Assume that there is a sequence \( \{A_m = (a_{ij}^m)\} \subset O(q, R_k) \) such that \( A_m \to I \).

For sufficiently large \( m \), we have \( |a_{ij}^m| < 2 \).

2. If \( \sigma \) is a non-identity embedding \( q^\sigma \) is equivalent over \( \mathbb{R} \) to \( x_1^2 + x_2^2 + \ldots + x_{n+1}^2 \).

Hence \( O(q^\sigma, \mathbb{R}) \) is conjugate to \( O(n + 1) \) and so is a compact group.

This implies that \( |\sigma(a_{ij}^m)| < K_\sigma \) for some \( K_\sigma \in \mathbb{R} \).

3. There are only finitely many algebraic integers \( x \) of bounded degree, such that \( x \) and all of its Galois conjugates are bounded.
Finite co-volume of these arithmetic groups follows from general results of Borel and Harish-Chandra.

**Cocompactness**

$f$ be a diagonal, anisotropic (over $\mathbb{Q}$) quadratic form of signature $(n, 1)$ and $\mathbb{Z}$-coefficients. Then $\Lambda = \text{SO}(f, \mathbb{Z})$ is cocompact in $G = \text{SO}(f, \mathbb{R})$.

**Caution** Meyer’s theorem implies that this is only possible for $n \leq 3$. 
We have a map

\[ \pi : \text{SL}_{n+1}(\mathbb{R}) \to \text{SL}_{n+1}(\mathbb{R})/\text{SL}_{n+1}(\mathbb{Z}) \]

**Key Claim:** Using \( \pi \), we can define a map

\[ \phi : G/\Lambda \to \text{SL}_{n+1}(\mathbb{R})/\text{SL}_{n+1}(\mathbb{Z}). \]

The image of \( \phi \) is compact.

This can be established using the Mahler Compactness Criterion.
Theorem 3 (Mahler’s Compactness Criterion)

Let $C \subset SL_m(\mathbb{R})$, then the image of $C$ in $SL_m(\mathbb{R})/SL_m(\mathbb{Z})$ is precompact (i.e. compact closure) if and only if 0 is not an accumulation point of

$$C\mathbb{Z}^m = \{ c \cdot v : c \in C, \, v \in \mathbb{Z}^m \}.$$ 

Use this to prove: The image of $G$ in $SL_{n+1}(\mathbb{R})/SL_{n+1}(\mathbb{Z})$ is precompact.

Uses anisotropic and defined over $\mathbb{Z}$ to ensure you stay away from 0.
Associated to $f$ is the bilinear form $B$, with

$$B(x, y) = \frac{1}{2}(f(x + y) - f(x) - f(y)).$$

This has $\mathbb{Z}$-coefficients ($f$ is diagonal with $\mathbb{Z}$-coefficients). Note that

(i) $B(\mathbb{Z}^{n+1}, \mathbb{Z}^{n+1}) \in \mathbb{Z}$.

(ii) $|B(v_m, v_m)| \geq 1$. (anisotropic)

You obtain a contradiction from a sequence of $g_m \in G$ and $v_m \in \mathbb{Z}^{n+1}$ such that

$$g_m v_m \to 0.$$

One can show the image is actually closed and this finishes the proof.
Comments on non-compact case

Theorem 4

TFAE

- $\text{SO}(f, R_k)$ is non-compact and finite volume.
- $f$ is defined over $\mathbb{Q}$ and is isotropic.
- $\text{SO}(f, R_k)$ contains a unipotent element.
Finding unipotent elements

Main idea

Let $a, b, c \in \mathbb{Z}$ and $f = ax^2 + by^2 + cz^2$ represent 0 non-trivially over $\mathbb{Z}$; e.g. $ax_0^2 + by_0^2 + cz_0^2 = 0$. Assume $x_0 \neq 0$.

This form is equivalent over $\mathbb{Q}$ to the form $t(xz - y^2)$ for some $t \in \mathbb{Q}$. Note $SO(t(xz - y^2), \mathbb{Q}) = SO(xz - y^2, \mathbb{Q})$.

If $(x_0, y_0, z_0)$ is as above, consider the matrix:

$$T = \begin{pmatrix} bcx_0 & 0 & 4x_0 \\ bcy_0 & -4cz_0 & -4y_0 \\ bcz_0 & 4by_0 & -4z_0 \end{pmatrix}.$$

Then $T'fT = \begin{pmatrix} 0 & 0 & t/2 \\ 0 & -t & -0 \\ t/2 & 0 & -0 \end{pmatrix}$ where $t = 16abcx_0^2$. 
Already seen $SO(xz - y^2, \mathbb{Z})$ contains unipotent elements. This group up to finite index is the image of $PSL(2, \mathbb{Z})$. As we now discuss: Use the equivalence above to construct unipotents in $SO(f, \mathbb{Z})$ via commensurability.
Commensurability

Suppose $q_1$ and $q_2$ are admissable quadratic forms over $k$ and $q_1 \simeq_k q_2$.
So there exists $T \in \text{GL}_{n+1}(k)$ such that $T^t Q_1 T = Q_2$.

Claim:

$$T^{-1} O(Q_1, k) T = O(Q_2, k)$$

The converse also holds.

However

$$T^{-1} O(Q_1, R_k) T \subset O(Q_2, k).$$

But need not preserve $R_k$-points
However, $T^{-1}O(Q_1, R_k)T$ is commensurable with $O(Q_2, R_k)$.

**Idea** By considering the entries of $T$ and $T^{-1}$ can choose a congruence subgroup $\Gamma < O(Q_1, R_k)$ such that $T^{-1}\Gamma T < O(Q_2, R_k)$.

**Example** $f = xz - y^2$ and $f_n = nxz - y^2$.

These are equivalent over $\mathbb{Q}$: Let $Y = \begin{pmatrix} n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$Y^t \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 0 \end{pmatrix} Y = \begin{pmatrix} 0 & 0 & n/2 \\ 0 & -1 & 0 \\ n/2 & 0 & 0 \end{pmatrix}$$
Then if $X = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in O(f, \mathbb{Z})$,

$$Y^{-1}XY = \begin{pmatrix} a_1 & a_2/n & a_3/n \\ nb_1 & b_2 & b_3 \\ nc_1 & c_2 & c_3 \end{pmatrix}$$

which will lie in $O(f_n, \mathbb{Z})$ if we choose $a_2$ and $a_3$ divisible by $n$; so take $\Gamma$ to be the principal congruence subgroup of level $n$ in $O(f, \mathbb{Z})$. 
Examples

1. The \((2, 4, 6)\) triangle group arises from the form \(x^2 + y^2 - 3z^2\).
2. The \((2, 3, 7)\) triangle group arises from a form defined over \(\mathbb{Q}(\cos \pi/7)\).

Of course most Fuchsian groups are not arithmetic.

3. The Bianchi groups \(\text{PSL}(2, \mathcal{O}_d)\) represent the totality of commensurability classes of non-cocompact arithmetic Kleinian groups.

These arise from the quadratic forms \(dx_1^2 + x_2^2 + x_3^2 - x_4^2\).

4. The minimal volume hyperbolic 3-orbifold arises from the quadratic form \(x_1^2 + x_2^2 + x_3^2 + (3 - 2\sqrt{5})x_4^2\).

Most arithmetic hyperbolic 3-manifolds are not of simplest type.
5. The groups generated by reflections in the compact 120 cell in $\mathbb{H}^4$ and the ideal 24-cell in $\mathbb{H}^4$ are arithmetic of simplest type. The quadratic forms are:

\[
x_1^2 + x_2^2 + x_3^2 + x_4^2 - \left(\frac{1 + \sqrt{5}}{2}\right)x_5^2
\]

\[
x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2
\]
Some other remarks

There are volume formula for certain arithmetic groups of simplest type; e.g. maximal groups.

If $n$ is even and $f$ is an admissible quadratic form defined over $k$, then every arithmetic subgroup in $\text{SO}(f)$ is contained in $\text{SO}(f, k)$ (Borel). This is not true when $n$ is odd.
 Totally Geodesic Submanifolds

Consider the form $f = x_1^2 + x_2^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}$ and write as $f = x_1^2 + g$ where

$$g = x_2^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}$$

Hence the group $\text{SO}(f, \mathbb{Z}[\sqrt{2}])$ is cocompact and contains a subgroup $\text{SO}(g, \mathbb{R}_k)$ which is a cocompact subgroup of $\text{SO}(g, \mathbb{R})$.

This allows us to construct arithmetic hyperbolic $(n + 1)$-manifolds that contains a hyperbolic $n$-manifold.

Indeed to construct arithmetic hyperbolic $(n + 1)$-manifolds that contains an immersed hyperbolic $m$-submanifold for all $1 \leq m \leq n$. 
But we can get many many more....

The reason is this: If $T \in O(f, k)$ then $TO(f, R_k)T^{-1}$ is commensurable with $O(f, R_k)$.

Given this: we have a subgroup $U = TO(f, R_k)T^{-1} \cap O(f, R_k)$ of finite index in both $TO(f, R_k)T^{-1}$ and $O(f, R_k)$. We can therefore build another co-dimension 1 totally geodesic submanifold by considering $H = TSO(g, R_k)T^{-1} \cap U < O(f, R_k)$.

Another way to say this is: Let $W \subset V$ be the subspace spanned by $\{e_2, \ldots e_{n+1}\}$ so that $W$ equipped with $g$ gives a copy of $\mathbb{H}^{n-1}$. Let $T \in O(f, k)$ as above, then $T(W)$ is invariant by a cocompact subgroup $H$. 
Theorem 5

Let $M$ be an arithmetic hyperbolic $n$-manifold of simplest type. Then $M$ contains infinitely many immersed totally geodesic co-dimension 1 submanifolds.
And more....

Again take \( f = x_1^2 + x_2^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1} \).

If we can find an admissible \( n \)-dimensional quadratic form \( q \) defined over \( \mathbb{Q}(\sqrt{2}) \) and an element \( a > 0 \in \mathbb{Q}(\sqrt{2}) \) such that:

\[
 f \sim_{\mathbb{Q}(\sqrt{2})} ax_1^2 + q = q'
\]

then commensurability of \( O(f, \mathbb{Z}[\sqrt{2}]) \) and \( T^{-1}O(q', \mathbb{Z}[\sqrt{2}])T \) can be used to built a subgroup of finite index in \( T^{-1}O(q, \mathbb{Z}[\sqrt{2}])T \) in \( O(f, \mathbb{Z}[\sqrt{2}]) \).
Recap from Lectures 1 and 2

Given an admissible quadratic form $f$ over a totally real field $k$ of signature $(n, 1)$ we have an algebraic group $O(f)$ and an equivalence over $\mathbb{R}$ of $f$ with $J_n$ so that.

$$T^tFT = J_n \text{ implies a conjugation } T^{-1}O(f, \mathbb{R})T = O(n, 1).$$

Then $T^{-1}O(f, R_k)T \cap O_0(n, 1)$ determines a commensurability class of arithmetic hyperbolic $n$-manifolds of simplest type.

Note if $f \simeq_k q$ then this provides further commensurabilities.
One striking fact about these arithmetic hyperbolic manifolds of simplest type is:

**Theorem 6**

*Let $M$ be an arithmetic hyperbolic $n$-manifold of simplest type. Then $M$ contains infinitely many immersed totally geodesic co-dimension 1 submanifolds.*

Does this characterize arithmeticity?
Embedding and Bounding

Theorem 7 (Millson, Bergeron-Haglund-Wise)

Let $M$ be an arithmetic hyperbolic $n$-manifold of simplest type and $N$ an immersed co-dimension 1 totally geodesic submanifold. Then $N$ embeds in a finite sheeted cover of $M$.

Need to promote immersed to embedded.
LERF

Definition ($H$-separable)

Let $G$ be a group, $H < G$. We say $G$ is $H$-separable if for each $g \in G \setminus H$, there exists a subgroup $K < G$ of finite index such that $H \subset K$ and $g \notin K$.

Example Let $H = 1$. Then $G$ is $H$-separable is equivalent to saying $G$ is Residually Finite.

Definition

Let $G$ be a group. We say that $G$ is subgroup separable or LERF if $G$ is $H$-separable for all finitely generated subgroups $H$.

Thanks to work of Peter Scott this is what is needed to promote immersed to embedded!
Example

Take \( f = x_1^2 + x_2^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1} \), and write \( f = x_1^2 + g \) with \( g = x_2^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}^2 \).

We can inject \( O(g) \hookrightarrow O(f) \) as follows:

\[
A \in O(g) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in O(f)
\]

Let the image of \( O(g, \mathbb{Z}[\sqrt{2}]) \) under this map be denoted by \( H \).
Claim: $O(f, \mathbb{Z}[\sqrt{2}])$ is $H$-separable.

Let $\gamma \in O(f, \mathbb{Z}[\sqrt{2}]) \setminus H$.

Two cases to focus on: (1) $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & A \end{pmatrix}$, $A \in O(g, \mathbb{Z}[\sqrt{2}])$ and (2) $\gamma$ has a non-zero entry $x$ in the first column or row (and not in the $(1, 1)$-entry).

In either case choose a prime ideal $P \subset \mathbb{Z}[\sqrt{2}]$ so that $P \neq < \sqrt{2} >$ and $x \not\in P$. 
Now let $\Gamma$ be a torsion free subgroup of finite index in $O(f, \mathbb{Z}[\sqrt{2}])$, then $\Gamma$ is $\Gamma \cap H$-separable.

This gives hyperbolic $n$-manifolds containing embedded totally geodesic submanifolds. they may be non-orientable.

However, can always arrange (perhaps on a passage to a further finite sheeted cover) that everything is orientable.

So we can construct an orientable hyperbolic $n$-manifold $M$ containing an embedded co-dimension 1 totally geodesic submanifold.

It either is separating or non-separating.

Latter case gives a map $\pi_1(M) \to \mathbb{Z}$, the former $N$ bounds a compact hyperbolic $n$-manifold.
Theorem 8 (Long, Bergeron)

$M$ an arithmetic hyperbolic $n$-manifold of simplest type, and $N$ an immersed co-dimension one totally geodesic submanifold for which $N$ does not factor through a cover. Then $\pi_1(M)$ is $\pi_1(N)$-separable.

Embedded vs Bounding
Geometrically bounding

Definition

A closed connected orientable, hyperbolic $n$-manifold $M$ bounds geometrically if $M$ is realized as the totally geodesic boundary of a compact orientable, hyperbolic $(n + 1)$-manifold $W$.

Definition

A closed connected flat $n$-manifold $M$ bounds geometrically if $M$ is realized as the cusp cross-section of a finite volume 1-cusped hyperbolic $(n + 1)$-manifold.

Can also make sense of just saying that complete orientable finite volume hyperbolic $n$-manifold bounds geometrically.
Theorem 9 (Long-R)

1. There are closed hyperbolic 3-manifolds that do not bound geometrically.
2. There are flat 3-manifolds that do not bound geometrically.
3. For every $n$ there are closed orientable hyperbolic $n$-manifolds which are arithmetic of simplest type that bound geometrically.
Other results:

**Theorem 10 (Kolpakov-Martelli-Tschantz)**

*There exist infinitely many closed hyperbolic 3-manifolds (whose volumes are known) which bound geometrically a compact hyperbolic 4-manifold (whose volumes are known).*

This uses the tessellation of $\mathbb{H}^4$ coming from the 120-cell.

**Theorem 11 (Slavich)**

*The figure-eight knot complement bounds geometrically.*
Recent work:

**Theorem 12 (Kolpakov-R-Slavich)**

Let $M = \mathbb{H}^n / \Gamma$ ($n \geq 2$ and even) be an orientable arithmetic hyperbolic $n$-manifold of simplest type.
Then $M$ embeds as a totally geodesic submanifold of an orientable arithmetic hyperbolic $(n + 1)$-manifold $W$.
If $M$ double covers a non-orientable hyperbolic manifold, then $M$ bounds geometrically.

Moreover, when $M$ is not defined over $\mathbb{Q}$ (and is therefore closed), the manifold $W$ can be taken to be closed.
Weaker statement can be made in odd dimensions.
Obstructions to bounding

**Theorem 13 (Long-R)**

If \( M \) is a closed hyperbolic 3-manifold or a flat 3-manifold that bounds geometrically then \( \eta(M) \in \mathbb{Z} \).

Meyerhoff-Neumann: *For surgeries on a hyperbolic knot in \( S^3 \) the \( \eta \) invariant takes on a dense set of values in \( \mathbb{R} \).*

This allows us to build hyperbolic examples that don’t bound geometrically.

\( \eta(M) \in \mathbb{Z} \) is rare: from the SnapPy census of approx. 11,000 closed hyperbolic 3-manifolds 41 have this property.
For a flat 3-manifold $M$, it is known that $\eta(M)$ depends only on the topology of $M$ and is independent of the flat metric.

Take the unique orientable flat 3-manifold with base for the Seifert fibration $S^2$ and Seifert invariants $(2, 1), (3, -1), (6, -1)$. Then $\eta(M) = -4/3$

**Theorem 14 (Long-R, McReynolds)**

*Every flat n-manifold arises as the cusp cross-section of some cusp of a multi-cusped non-compact arithmetic hyperbolic $(n + 1)$-manifold.*
**WARNING** At the time when we proved these results it was unknown as to whether there were any 1-cusped finite volume hyperbolic 4-manifolds.

**Theorem 15 (Kolpakov-Martelli)**

There exist 1-cusped arithmetic hyperbolic 4-manifolds of simplest type.

These are built using the 24-cell, and are plentiful.

Still none known in any other dimension $> 4$.

**Theorem 16 (Stover)**

There are no 1-cusped arithmetic hyperbolic $n$-orbifolds whenever $n \geq 30$. 
Non-arithmetic hyperbolic n-manifolds—after Gromov and Piatetski-Shapiro

Cut-and-paste arithmetic hyperbolic manifolds of simplest type along a common co-dimension 1 totally geodesic submanifold (not necessarily connected)

How to do this:

Take $f = x_1^2 + x_2^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}$, and write $f = x_1^2 + g$ with $g = x_2^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}^2$.

Plan: Find a form $q = ax_1^2 + g$ such that $q$ is not $\simeq_{\mathbb{Q}(\sqrt{2})}$ to $f$.

If we can do this its a win: we cut and paste and glue. This produces a group $\Lambda$ built from the pieces that can’t be arithmetic.
Lemma 17

Let $\Gamma_1$ and $\Gamma_2$ be arithmetic lattices in $O_0(n, 1)$ such that $\Gamma_1 \cap \Gamma_2$ is Zariski dense in $O_0(n, 1)$. Then $\Gamma_1$ and $\Gamma_2$ are commensurable.

Take $\Gamma_1 = O(f, \mathbb{Z}[\sqrt{2}])$ and $\Gamma_2 = \Lambda$, and apply the lemma.

Do the same with $\Gamma_1 = O(q, \mathbb{Z}[\sqrt{2}])$ and $\Gamma_2 = \Lambda$ and apply the lemma.

The orthogonal groups are not commensurable.

$a$ can be constructed using the theory of quadratic forms

Are there non-arithmetic hyperbolic n-manifolds $n \geq 4$ that are ”not built from arithmetic pieces”?
Theorem 18 (Bergeron-Haglund-Wise)

Let $M$ be an arithmetic hyperbolic $n$-manifold of simplest type. Then $\pi_1(M)$ is virtually $C$-special; i.e. $\pi_1(M)$ contains a finite index subgroup contained in an abstract right angled Coxeter group. In particular $\pi_1(M)$ is separable on geometrically finite subgroups—it has the virtual retract property over geometrically finite subgroups.

Remark: The theorem with Kolpakov-Slavich uses this as a key ingredient.
Starting point: Co-dimension 1 totally geodesic submanifolds are abundant in the following sense:

Given $u, v \in \partial \mathbb{H}^n$ there exists codimension 1 geodesic submanifold $H$ whose boundary $\partial H \subset \partial \mathbb{H}^n$ separates $u$ and $v$, and there exists $\Gamma < \pi_1(M)$ leaving $H$ invariant.

Wont say any more about this, but discuss an earlier special case of the Bergeron-Haglund-Wise result.
Theorem 19 (Agol-Long-R)

The Bianchi groups are virtually $C$-special.
Recent work of M. Chu improves the argument and controls the "virtual part":

**Theorem 20 (Chu)**

Let $R_d$ be the subgroup of the Bianchi group $\text{PSL}_2(O_d)$ given by

$$R_d = \left\{ \gamma \in \text{PSL}_2(\mathbb{Z}[\sqrt{-d}]) : \gamma \equiv \text{Id} \mod 2 \right\}.$$  

Then $R_d$ embeds in a RACG and has index $[\text{PSL}_2(O_d) : R_d] =$

1. $48$ if $d \equiv 1, 2 \mod (4)$
2. $288$ if $d \equiv 7 \mod (8)$
3. $480$ if $d \equiv 3 \mod (8)$
Ideas in the proof of ALR

As mentioned in Lecture 2 $\text{PSL}(2, O_d)$ can be realized up to commensurability as $\text{SO}(p_d, \mathbb{Z})$ where $p_d = dx_1^2 + x_2^2 + x_3^2 - x_4^2$.

The form $du^2 + dv^2 + dw^2 + p_d$ is equivalent over $\mathbb{Q}$ to the form $J_6$ (uses the 4-squares theorem).

$O(6, 1, \mathbb{Z})$ contains an all right Coxeter group of finite index.

Hence $\text{PSL}(2, O_d)$ is virtually $C$-special.
Contrast with dimensions 2 and 3

\( M \) a hyperbolic 2,3-manifold. \( \pi_1(M) \) is LERF.

**Theorem 21 (Hongbin Sun)**

*Let \( M \) be an arithmetic hyperbolic \( n \)-manifold of simplest type with \( n \geq 4 \). Then \( \pi_1(M) \) is not LERF.*

Idea in the proof when \( M \) is non-compact

Exploit non-LERF non-geometric closed 3-manifolds

Comment: LERF for \( \pi_1(\text{closed 3-manifold}) \) completely understood now.

**Theorem 22 (Hongbin Sun)**

*\( M \) a closed 3-manifold. \( \pi_1(M) \) is LERF if and only \( M \) is geometric.*
In fact following work of Yi Liu, Hongbin proves:

**Theorem 23**

Let $M$ be a mixed non-geometric closed 3-manifold. Then $\pi_1(M)$ contains a non-separable surface group.

How to exploit this?

Here is the basic idea now given Hongbin’s previous result.

**Theorem 24 (Long-R)**

Let $M$ denote the exterior of the figure-eight knot complement, and $DM$ its double. Then $\pi_1(DM)$ admits a faithful representation (with geometrically finite image) into an arithmetic group of simplest type commensurable with $SO_0(4, 1, \mathbb{Z})$. 
Putting these together:

**Corollary 25**

\[ \text{SO}_0(4, 1, \mathbb{Z}) \text{ is not LERF.} \]

Indeed one gets more:

**Corollary 26**

\[ \text{SO}_0(n, 1, \mathbb{Z}) \text{ is not LERF.} \]

The general non-co-compact case is a generalization of this.

The closed case uses amalgams of closed hyperbolic 3-manifold groups along infinite cyclic groups \((n = 4 \text{ needs a different argument}).\)

THE END