Proc. Int. Cong. of Math. – 2018
Rio de Janeiro, Vol. 1 (1191–1214)

PROFINITE RIGIDITY

Alan W. Reid

Abstract

We survey recent work on profinite rigidity of residually finite groups.

1 Introduction

It is an old and natural idea to try to distinguish finitely presented groups via their finite quotients, and recently, there has been renewed interest, especially in the light of recent progress in 3-manifold topology, in the question of when the set of finite quotients of a finitely generated residually finite group determines the group up to isomorphism. In more sophisticated terminology, one wants to develop a complete understanding of the circumstances in which finitely generated residually finite groups have isomorphic *profinite completions*. Motivated by this, say that a residually finite group Γ is *profinitely rigid*, if whenever $\widehat{\Delta} \cong \widehat{\Gamma}$, then $\Delta \cong \Gamma$ (see Section 2.2 for definitions and background on profinite completions).

It is the purpose of this article to survey some recent work and progress on profinite rigidity, which is, in part, motivated by Remeslennikov's question (see Question 4.1) on the profinite rigidity of a free group. The perspective taken is that of a low-dimensional topologist, and takes advantage of the recent advances in our understanding of hyperbolic 3-manifolds and their fundamental groups through the pioneering work of Agol [2013] and Wise [2009].

Standing assumption: Throughout the paper all discrete groups considered will be finitely generated and residually finite.

Acknowledgments. Much of the work surveyed here is joint work of the author in several papers with multiple co-authors, and the author would like to sincerely thank his co-authors for their collaborations: M.D.E. Conder, D. B. McReynolds, R. Spitler and H. Wilton. He would like to single out M.R. Bridson for particular thanks in this regard. He would also like to thank M. Boileau, S. Friedl, D. D. Long, A. Lubotzky

The author was supported in part by an NSF grant.
MSC2010: primary 20E18; secondary 57M07.
Keywords: profinite completion, profinite rigidity, 3-manifold group.

and P. Zalesskii for many helpful and stimulating conversations on topics related to this work.

2 Preliminaries

We begin by providing some background discussion on profinite groups and profinite completions of discrete groups. We refer the reader to Ribes and Zalesskii [2000] for a more detailed account of the topics covered here.

2.1 Profinite groups. A *directed set* is a partially ordered set I such that for every $i, j \in I$ there exists $k \in I$ such that $k \geq i$ and $k \geq j$. An *inverse system* is a family of sets $\{X_i\}_{\{i \in I\}}$, where I is a directed set, and a family of maps $\phi_{ij} : X_i \to X_j$ whenever $i \geq j$, such that:

- $\phi_{ii} = id_{X_i}$;
- $\phi_{ij}\phi_{jk} = \phi_{ik}$, whenever $i \geq j \geq k$.

Denoting this system by (X_i, ϕ_{ij}, I), the *inverse limit* of the inverse system (X_i, ϕ_{ij}, I) is the set

$$\varprojlim X_i = \{(x_i) \in \prod_{i \in I} X_i | \phi_{ij}(x_i) = x_j, \text{ whenever } i \geq j\}.$$

If (X_i, ϕ_{ij}, I) is an inverse system of non-empty compact, Hausdorff, totally disconnected topological spaces (resp. topological groups) over the directed set I, then $\varprojlim X_i$ is a non-empty, compact, Hausdorff, totally disconnected topological space (resp. topological group).

In addition, if (X_i, ϕ_{ij}, I) is an inverse system, a subset $J \subset I$ is defined to be *cofinal*, if for each $i \in I$, there exists $j \in J$ with $j \geq i$. If J is cofinal we may form an inverse system (X_j, ϕ_j, J) obtained by omitting those $i \in I \setminus J$. The inverse limit $\varprojlim X_j$ can be identified with the image of $\varprojlim X_i$ under the projection map $\prod_{i \in I} X_i$ onto $\prod_{j \in J} X_j$.

2.2 Profinite completion. Let Γ be a finitely generated group (not necessarily residually finite for this discussion), and let $\mathfrak{N}$ denote the collection of all finite index normal subgroups of Γ. Note that $\mathfrak{N}$ is non-empty as $\Gamma \in \mathfrak{N}$, and we can make $\mathfrak{N}$ into directed set by declaring that

$$\text{for } M, N \in \mathfrak{N}, M \leq N \text{ whenever } M\ contains\ N.$$

In this case, there are natural epimorphisms $\phi_{NM} : \Gamma/N \to \Gamma/M$, and the inverse limit of the inverse system $(\Gamma/N, \phi_{NM}, \mathfrak{N})$ is denoted $\widehat{\Gamma}$ and defined to be to the *profinite completion* of Γ.

Note that there is a natural map $\iota : \Gamma \to \widehat{\Gamma}$ defined by

$$g \mapsto (gN) \in \varprojlim \Gamma/N,$$

and it is easy to see that ι is injective if and only if Γ is residually finite.

An alternative, perhaps more concrete way of viewing the profinite completion is as follows. If, for each $N \in \mathfrak{N}$, we equip each Γ/N with the discrete topology, then $\prod\{\Gamma/N : N \in \mathfrak{N}\}$ is a compact space and $\widehat{\Gamma}$ can be identified with $\overline{j(\Gamma)}$ where $j : \Gamma \to \prod\{\Gamma/N : N \in \mathfrak{N}\}$ is the map $g \mapsto (gN)$.

2.3 Profinite Topology. It will also be convenient to recall the profinite topology on a discrete group Γ, its subgroups and the correspondence between the subgroup structure of Γ and $\widehat{\Gamma}$.

The profinite topology on Γ is the topology on Γ in which a base for the open sets is the set of all cosets of normal subgroups of finite index in Γ.

Now given a tower $\mathfrak{T}$ of finite index normal subgroups of Γ:

$$\Gamma > N_1 > N_2 > \ldots\ldots. > N_k > \ldots$$

with $\cap N_k = 1$, this can be used to define an inverse system and thereby determines a completion of $\widehat{\Gamma}_{\mathfrak{T}}$ (in which Γ will inject). If the inverse system determined by $\mathfrak{T}$ is cofinal (recall Section 2.1) then the natural homomorphism $\widehat{\Gamma} \to \widehat{\Gamma}_{\mathfrak{T}}$ is an isomorphism. That is to say $\mathfrak{T}$ determines the full profinite topology of Γ.

The following is important in connecting the discrete and profinite worlds (see Ribes and Zalesskii [ibid., p. 3.2.2], where here we use Nikolov and Segal [2007] to replace "open" by "finite index").

Notation. Given a subset X of a profinite group G, we write $\overline{X}$ to denote the closure of X in G.

Proposition 2.1. *If Γ is a finitely generated residually finite group, then there is a one-to-one correspondence between the set $\mathfrak{X}$ of subgroups of Γ that are open in the profinite topology on Γ, and the set $\mathfrak{Y}$ of all finite index subgroups of $\widehat{\Gamma}$.*

Identifying Γ with its image in the completion, this correspondence is given by:

- *For $H \in \mathfrak{X}$, $H \mapsto \overline{H}$.*
- *For $Y \in \mathfrak{Y}$, $Y \mapsto Y \cap \Gamma$.*

If $H, K \in \mathfrak{X}$ and $K < H$ then $[H : K] = [\overline{H} : \overline{K}]$. Moreover, $K \triangleleft H$ if and only if $\overline{K} \triangleleft \overline{H}$, and $\overline{H}/\overline{K} \cong H/K$.

Thus Γ and $\widehat{\Gamma}$ have the same finite quotients. The key result to formalize the precise connection between the collection of finite quotients of Γ and those of $\widehat{\Gamma}$ is the following. This is basically proved in Dixon, Formanek, Poland, and Ribes [1982] (see also Ribes and Zalesskii [2000, pp. 88-89]), the mild difference here, is that we employ Nikolov and Segal [2007] to replace *topological isomorphism* with *isomorphism*. To state this we introduce the following notation:

$$\mathcal{C}(\Gamma) = \{Q : Q \text{ is a finite quotient of } \Gamma\}$$

Theorem 2.2. *Suppose that Γ_1 and Γ_2 are finitely generated abstract groups. Then $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$ are isomorphic if and only if $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$.*

Given this, we make the following definition—this definition is taken, by analogy with the theory of quadratic forms over $\mathbb{Z}$, where two integral quadratic forms can be locally equivalent (i.e. at all places of $\mathbb{Q}$), but not globally equivalent over $\mathbb{Z}$.

Definition 2.3. The genus of a finitely generated residually finite group Γ is: $\mathcal{G}(\Gamma) = \{\Delta : \widehat{\Delta} \cong \widehat{\Gamma}\}$.

In addition, if $\mathcal{P}$ is a class of groups, then we define $\mathcal{G}_{\mathcal{P}}(\Gamma) = \{\Delta \in \mathcal{G}(\Gamma) : \Delta \in \mathcal{P}\}$. For convenience we restate the definition of profinite rigidity.

Definition 2.4. Let Γ be a finitely generated group. Say that Γ is profinitely rigid if $\mathcal{G}(\Gamma) = \{\Gamma\}$.

For convenience we often say that Γ is *profinitely flexible* if it is not profinitely rigid.

In addition when $\Gamma = \pi_1(M)$ where M is a compact 3-manifold we occasionally abuse notation and refer to M as being profinitely rigid or flexible.

The basic questions we are interested in are the following (and also within classes of groups $\mathcal{P}$).

Question 2.5. *Which finitely generated (resp. finitely presented) groups Γ are profinitely rigid (resp. profinitely flexible)?*

Question 2.6. *How large can $|\mathcal{G}(\Gamma)|$ be for finitely generated (resp. finitely presented) groups?*

Question 2.7. *What group theoretic properties are shared by (resp. are different for) groups in the same genus?*

These questions (and ones where the class of groups is restricted) provide the motivation and focus of this article, with particular attention paid to Question 2.5.

2.4 Inducing the full profinite topology. Let Γ be a finitely generated residually finite group and $H < \Gamma$. The profinite topology on Γ determines some pro-topology on H and therefore some completion of H. To understand what happens in certain cases that will be of interest to us, we recall the following. Since we are assuming that Γ is residually finite, H injects into $\widehat{\Gamma}$ and determines a subgroup $\overline{H} \subset \widehat{\Gamma}$. Hence there is a natural epimorphism $\widehat{H} \to \overline{H}$. This need not be injective. For this to be injective (i.e. the full profinite topology is induced on H) it is easy to see that the following needs to hold:

(*) For every subgroup H_1 of finite index in H, there exists a finite index subgroup $\Gamma_1 < \Gamma$ such that $\Gamma_1 \cap H < H_1$.

A important case where the full profinite topology is induced is when the ambient group Γ is *LERF*, the definition of which we recall here. Suppose that Γ is a group and H a subgroup of Γ, then Γ is called H-separable if for every $g \in G \smallsetminus H$, there is a subgroup K of finite index in Γ such that $H \subset K$ but $g \notin K$; equivalently, the intersection of all finite index subgroups in Γ containing H is precisely H. The group Γ is called LERF (or *subgroup separable*) if it is H-separable for every finitely-generated subgroup H, or equivalently, if every finitely-generated subgroup is a closed subset in the profinite topology.

Lemma 2.8. *Let Γ be a finitely-generated group, and H a finitely-generated subgroup of Γ. Suppose that Γ is H_1-separable for every finite index subgroup H_1 in H. Then the profinite topology on Γ induces the full profinite topology on H; that is, the natural map $\widehat{H} \to \overline{H}$ is an isomorphism.*

Proof. Since Γ is H_1 separable, the intersection of all subgroups of finite index in Γ containing H_1 is H_1 itself. From this it easily follows that there exists $\Gamma_1 < \Gamma$ of finite index, so that $\Gamma_1 \cap H = H_1$. The lemma follows from (*) above. □

Immediately from this we deduce.

Corollary 2.9. *Let Γ be a finitely generated group that is LERF. Then if $H < \Gamma$ is finitely generated then the profinite topology on Γ induces the full profinite topology on H; that is, the natural map $\widehat{H} \to \overline{H}$ is an isomorphism.*

3 Two simple examples

We provide two elementary examples that already indicate a level of complexity in trying to understand profinite rigidity and lack thereof. In addition, some consequences of these results and techniques will be helpful in what follows.

Proposition 3.1. *Let Γ be a finitely generated Abelian group, then $\mathcal{G}(\Gamma) = \{\Gamma\}$.*

Proof. Suppose first that $\Delta \in \mathcal{G}(\Gamma)$ and Δ is non-abelian. We may therefore find a commutator $c = [a, b] \in \Delta$ that is non-trivial. Since Δ is residually finite there is a homomorphism $\phi : \Delta \to Q$, with Q finite and $\phi(c) \neq 1$. However, $\Delta \in \mathcal{G}(\Gamma)$, so Q is abelian and therefore $\phi(c) = 1$, a contradiction.

Thus Δ is Abelian, so we can assume that $\Gamma \cong \mathbb{Z}^r \oplus T_1$ and $\Delta \cong \mathbb{Z}^s \oplus T_2$, where T_i $(i = 1, 2)$ are finite Abelian groups. It is easy to see that $r = s$, for if $r > s$ say, we can choose a large prime p such that p does not divide $|T_1||T_2|$, and construct a finite quotient $(\mathbb{Z}/p\mathbb{Z})^r$ that cannot be a quotient of Δ.

In addition if T_1 is not isomorphic to T_2, then some invariant factor appears in T_1 say, but not in T_2. One can then construct a finite abelian group that is a quotient of T_1 (and hence Γ_1) but not of Γ_2. □

Remark 3.2. The proof of Proposition 3.1 also proves the following. Let Γ be a finitely generated group, and suppose that $\Delta \in \mathcal{G}(\Gamma)$. Then $\Gamma^{ab} \cong \Delta^{ab}$. In particular $b_1(\Gamma) = b_1(\Delta)$.

Somewhat surprisingly, moving only slightly beyond abelian groups (indeed $\mathbb{Z}$) to groups that are virtually $\mathbb{Z}$, the situation is dramatically different. The following result is due to Baumslag [1974].

Theorem 3.3. *There exists non-isomorphic meta-cyclic groups Γ_1 and Γ_2 for which $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$. Both of these groups are virtually $\mathbb{Z}$ and defined as extensions of a fixed finite cyclic group F by $\mathbb{Z}$.*

A more precise form of what Baumslag actually proves in Baumslag [ibid.] is the following:

Let F be a finite cyclic group with an automorphism of order n, where n is different from 1, 2, 3, 4 and 6. Then there are at least two non-isomorphic cyclic extensions of F, say Γ_1 and Γ_2 with $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$.

A beautiful, and useful observation, that is used in the proof that the constructed groups Γ_1 and Γ_2 lie in the same genus is the following that goes back to Hirshon Hirshon [1977]: Suppose that A and B are groups with $A \times \mathbb{Z} \cong B \times \mathbb{Z}$, then $\widehat{A} \cong \widehat{B}$.

Remark 3.4. Moving from meta-cyclic to meta-abelian provides even more striking examples of profinite flexibility. Pickel [1974] constructs finitely presented meta-abelian groups Γ for which $\mathcal{G}(\Gamma)$ is infinite.

4 Profinite rigidity and flexibility in low-dimensions

In connection with Question 2.5 perhaps the most basic case is the following that goes back to Noskov, Remeslennikov, and Romankov [1979, Question 15] and remains open:

Question 4.1. Let F_n be the free group of rank $n \geq 2$. Is F_n profinitely rigid?

The group F_n arises in many guises in low-dimensional topology and affords several natural ways to generalize. In the light of this, natural generalizations of Question 4.1 are the following (which remain open):

Question 4.2. Let Σ_g be a closed orientable surface of genus $g \geq 2$. Is $\pi_1(\Sigma_g)$ profinitely rigid?

As we will discuss in more detail below, profinite rigidity in the setting of 3-manifold groups is different, however, one generalization that we will focus on below is the following question:

Question 4.3. Let M be a complete orientable hyperbolic 3-manifold of finite volume. Is $\pi_1(M)$ profinitely rigid?

In this section we describe some recent progress on Questions 4.1, 4.2 and 4.3, as well as other directions that generalize Question 4.1. However, we begin by recalling some necessary background from the geometry and topology of 3-manifolds.

4.1 Some 3-manifold topology. For the purposes of this subsection, M will always be a compact connected orientable 3-manifold whose boundary is either empty, or consists of a disjoint union of incompressible tori. The Geometrization Conjecture of Thurston was established by Perelman (see Morgan and Tian [2014] for a detailed account) and we state what is needed here in a convenient form. We refer the reader to Bonahon [2002] or Thurston [1997] for background on geometric structures on 3-manifolds.

Recall that M is *irreducible* if every embedded 2-sphere in M bounds a 3-ball, and if M is *prime* (i.e. does not decompose as a non-trivial connect sum), then M is irreducible or is covered by $S^2 \times S^1$, in which case M admits a geometric structure modeled on $S^2 \times \mathbb{R}$.

Theorem 4.4. *Let M be an irreducible 3-manifold.*

1. *If $\pi_1(M)$ is finite, then M is covered by S^3.*
2. *If $\pi_1(M)$ is infinite, then M is either:*

   (i) hyperbolic and so arises as $\mathbb{H}^3/\Gamma$ where $\Gamma < \mathrm{PSL}(2, \mathbb{C})$ is a discrete torsion-free subgroup of finite co-volume, or;

   (ii) a Seifert fibered space and has a geometry modeled on $\mathbb{E}^3$, $\mathbb{H}^2 \times \mathbb{R}$, NIL or $\widetilde{SL}_2$, or;

   (iii) a $SOLV$ manifold, or;

   (iv) a manifold that admits a collection of essential tori that decomposes M into pieces that are Seifert fibered spaces with incompressible torus boundary, or have interior admitting a finite volume hyperbolic structure. In this case, we will say that M has a non-trivial JSJ decomposition.

An important well-known consequence of geometrization for us is the following corollary.

Corollary 4.5. *Let M be compact 3-manifold, then $\pi_1(M)$ is residually finite.*

A manifold M that admits a geometric structure modeled on $\mathbb{E}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, NIL or $SOLV$ all *virtually fiber*. That is to say, given M admitting such a structure then there is a finite cover $M_f \to M$ with M_f constructed as the mapping torus of a surface homeomorphism $f : \Sigma_g \to \Sigma_g$, where $g = 0$ in the case of $S^2 \times \mathbb{R}$, $g = 1$ in the case of $\mathbb{E}^3$, NIL or $SOLV$ and $g > 1$ when the geometry is $\mathbb{H}^2 \times \mathbb{R}$. If M is a compact Seifert fibered space with incompressible torus boundary, then M also virtually fibers. On the other hand, it is known (see Gabai [1986]) that closed manifolds admitting a geometric structure modeled on $\widetilde{SL}_2$ do not virtually fiber.

Regarding virtual fibering of hyperbolic manifolds, a major breakthrough came with Agol's work in Agol [2008], which, taken together with work of Agol [2013] and Wise [2009] (see also Groves and Manning [2017]) leads to the following.

Theorem 4.6 (Virtual fibering)**.** *Let M be a finite volume hyperbolic* 3*-manifold. Then M has a finite cover that fibers over the circle.*

For manifolds with a non-trivial JSJ decomposition, it was known previously that there were *graph manifolds* (i.e. all pieces in the decomposition are Seifert fibered spaces) that do not virtually fiber Neumann [1997], whilst more recently it was shown in Przytycki and Wise [2018] that *mixed 3-manifolds* (i.e. those in Theorem 4.4 2(iv) that have a decomposition containing a hyperbolic piece) are all virtually fibered.

4.2 Profinite completions of 3-manifold groups after Agol and Wise.

The remarkable work of Agol [2013] and Wise [2009] has had significant implications on our understanding of the profinite completion of the fundamental groups of finite volume hyperbolic 3-manifolds. We refer the reader to the excellent book Aschenbrenner, Friedl, and Wilton [2015] for a detailed discussion of the many consequences of Agol [2013] and Wise [2009] for 3-manifold groups. One such concerns LERF (recall Section 2.4). The following result summarizes work of Scott [1978] for Seifert fibered spaces, Agol [2013] and Wise [2009] in the hyperbolic setting, and Sun [2016] who showed that non-geometric irreducible 3-manifolds had non-LERF fundamental group

Theorem 4.7. *Let M be an irreducible 3-manifold (as in Section 4.1). Then $\pi_1(M)$ is LERF if and only if M is geometric (i.e covered by Theorem 4.4 1, 2(i), (ii), (iii)).*

Lemma 2.8 together with Theorem 4.7 yields the following consequence.

Corollary 4.8. *Let M be a finite volume hyperbolic 3-manifold and $H < \pi_1(M)$ a finitely generated subgroup. Then the full profinite topology on H is induced by the profinite topology of $\pi_1(M)$. In particular the closure of H in $\widehat{\pi_1(M)}$ is isomorphic to $\widehat{H}$.*

We now turn to *goodness* in the sense of Serre [1997]. Let G be a profinite group, M a discrete G-module (i.e. an abelian group M equipped with the discrete topology on which G acts continuously) and let $C^n(G, M)$ be the set of all continous maps $G^n \to M$. One defines the coboundary operator $d : C^n(G, M) \to C^{n+1}(G, M)$ in the usual way thereby defining a complex $C^*(G, M)$ whose cohomology groups $H^q(G; M)$ are called the continuous cohomology groups of G with coefficients in M.

Now let Γ be a finitely generated group. Following Serre [ibid.], we say that a group Γ is *good* if for all $q \geq 0$ and for every finite Γ-module M, the homomorphism of cohomology groups

$$H^q(\widehat{\Gamma}; M) \to H^q(\Gamma; M)$$

induced by the natural map $\Gamma \to \widehat{\Gamma}$ is an isomorphism between the cohomology of Γ and the continuous cohomology of $\widehat{\Gamma}$.

Example 4.9. Finitely generated free groups are good.

In general goodness is hard to establish, however, one can establish goodness for a group Γ that is LERF (indeed a weaker version of separability is all that is needed) and in addition has a "well-controlled splitting of the group" as a graph of groups Grunewald, Jaikin-Zapirain, and Zalesskii [2008]; for example that coming from the virtual special technology Wise [2009]. In addition, a useful criterion for goodness is provided by the next lemma due to Serre [1997, Chapter 1, Section 2.6].

Lemma 4.10. *The group Γ is good if there is a short exact sequence*

$$1 \to N \to \Gamma \to H \to 1,$$

such that H and N are good, N is finitely-generated, and the cohomology group $H^q(N, M)$ is finite for every q and every finite Γ-module M.

Coupled with Theorem 4.6 (the virtual fibering theorem) and commensurability invariance of goodness Grunewald, Jaikin-Zapirain, and Zalesskii [2008], this proves that the fundamental groups of all finite volume hyperbolic 3-manifolds are good. Indeed, more is true using Agol [2013] and Wise [2009] (as noticed by Cavendish [2012], see also Reid [2015]):

Theorem 4.11. *Let M be a compact 3-manifold, then $\pi_1(M)$ is good.*

Several notable consequences of this are recorded below.

Corollary 4.12. *Let M be a closed irreducible orientable 3-manifold, and N a compact 3-manifold with $\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)}$. Then:*

1. *$\widehat{\pi_1(M)}$ is torsion-free.*

2. *N is closed, orientable and can have no summand that has finite fundamental group.*

Proof. Let $\Gamma = \pi_1(M)$ and $\Delta = \pi_1(N)$. Since $\mathrm{cd}(\Gamma) = 3$, $H^3(\Gamma, \mathbb{F}_p) \neq 0$ for every prime p, and $H^q(\Gamma, M) = 0$ for every Γ-module M and every $q > 3$. By goodness, these transfer to the profinite setting in the context of finite modules. It follows from standard results about the cohomology of finite groups, that goodness forces $\widehat{\pi_1(M)}$ to be torsion-free. Hence Δ is also torsion-free, and so N cannot have a summand that has finite fundamental group.

In addition, N must be closed, since $H^3(\Gamma, \mathbb{F}_2) \neq 0$ implies $H^3(\widehat{\Gamma}, \mathbb{F}_2) \neq 0$, and if N is not closed we have, $H^3(\Delta, \mathbb{F}_2) = H^3(\widehat{\Delta}, F_2) = 0$. Orientability follows in a similar fashion using $H^3(\Gamma, \mathbb{F}_p) \neq 0$ for $p \neq 2$. □

Remark 4.13. In Lubotzky [1993], it is shown that there are torsion-free subgroups $\Gamma < \mathrm{SL}(\mathrm{n}, \mathbb{Z})$ ($n \geq 3$) of finite index, for which $\widehat{\Gamma}$ contains torsion of all possible orders. It follows that $\mathrm{SL}(\mathrm{n}, \mathbb{Z})$ is not good for $n \geq 3$).

4.3 Profinite flexibility of 3-manifold groups. We now describe some recent progress on identifying 3-manifold groups by their profinite completions restricted to the class of 3-manifold groups. To that end let

$$\mathfrak{M} = \{\pi_1(M) : M \text{ is a compact 3-manifold}\}.$$

We note that unlike in the previous subsection M need not be prime, can be non-orientable, may have boundary other than tori and this boundary may be compressible. By capping off 2-sphere boundary components with 3-balls, we can exclude S^2 boundary components (and $\mathbb{RP}^2$ boundary components). Also note that included in $\mathfrak{M}$ are the fundamental groups of non-compact finite volume hyperbolic 3-manifolds where such a manifold is viewed as the interior of a compact 3-manifold with boundary consisting of tori or Klein bottles.

Example 4.14 (Profinitely flexible Seifert fibered spaces)**.** We record a construction of Hempel [2014] that provides examples of closed Seifert fibered spaces M_1 and M_2 that are not homeomorphic but $\widehat{\pi_1(M_1)} \cong \widehat{\pi_1(M_2)}$. This builds on the idea of Baumslag mentioned in Section 3.

Let $f : S \to S$ be a periodic, orientation-preserving homeomorphism of a closed orientable surface S of genus at least 2, and let k be relatively prime to the order of f. Let M_f (resp. M_{f^k}) denote the mapping torus of f (resp. f^k), and let $\Gamma_f = \pi_1(M_f)$ (resp. $\Gamma_{f^k} = \pi_1(M_{f^k})$).

Hempel shows that $\widehat{\Gamma}_f \cong \widehat{\Gamma}_{f^k}$ by proving that $\Gamma_f \times \mathbb{Z} \cong \Gamma_{f^k} \times \mathbb{Z}$ (c.f. the example of Baumslag in Section 3). The proof is elementary group theory, but Hempel also notes that, interestingly, the isomorphism $\Gamma_f \times \mathbb{Z} \cong \Gamma_{f^k} \times \mathbb{Z}$ follows from Kwasik and Rosicki [2004] where it is shown that (in the notation above) $M_f \times S^1 \cong M_{f^k} \times S^1$.

Of course some additional work is needed to prove that the groups are not isomorphic, but in fact typically this is the case as Hempel describes in Hempel [2014]. Note that these examples admit a geometric structure modeled on $\mathbb{H}^2 \times \mathbb{R}$.

More recently it was shown by Wilkes [2017] that the construction of Hempel is the only occasion in which profinite rigidity fails in the closed case (there are also results in the bounded case). More precisely:

Theorem 4.15 (Wilkes)**.** *Let M be a closed Seifert fibered space with infinite fundamental group. Then $\mathcal{G}_{\mathfrak{M}}(\pi_1(M)) = \{\pi_1(M)\}$ unless M is as in Example 4.14 and the failure is precisely given by the construction in Example 4.14. In this case, $\mathcal{G}_{\mathfrak{M}}(\pi_1(M))$ is finite.*

The proof of this relies on some beautiful work of Wilton and Zalesskii [2017a] that remarkably detects geometric structure from finite quotients. We discuss this in more detail below in Section 4.4. but first give some other examples of profinite flexibility in the setting of closed 3-manifolds.

Example 4.16 (Profinitely flexible torus bundles)**.** Profinite flexibility for the fundamental groups of torus bundles admitting a $SOLV$ geometry was studied in detail in

Funar [2013]. These torus bundles arise as the mapping torus of a self-homeomorphism $f : T^2 \to T^2$ which can be identified with an element of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ with $|a + d| > 2$. In Funar [ibid.] it is shown that for any $m \geq 2$ there exist m torus bundles admitting $SOLV$ geometry whose fundamental groups have isomorphic profinite completions but are pairwise non-isomorphic.

A particular pair of examples of such torus bundles are give by the mapping tori of the following homeomorphisms:

$$f_1 = \begin{pmatrix} 188 & 275 \\ 121 & 177 \end{pmatrix} \text{ and } f_1 = \begin{pmatrix} 188 & 11 \\ 3025 & 177 \end{pmatrix}.$$

The methods of proof are very different from that used in Example 4.14. In particular it does not use the ideas in Baumslag's examples in Section 3, using instead, number theoretic techniques arising in understanding "local conjugacy" of matrices in $\mathrm{SL}(2, \mathbb{Z})$. Briefly, the fundamental groups of torus bundles M_f and M_g have isomorphic profinite completions if and only if the cyclic subgroups $< f >, < g > \subset \mathrm{SL}(2, \mathbb{Z})$ are locally conjugate, namely their images modulo m are conjugate in $\mathrm{GL}(2, \mathbb{Z}/\mathrm{m}\mathbb{Z})$, for any positive integer m (see Funar [ibid.]).

Interestingly, as described in Funar [ibid.] the issue of profinite flexibility in this case is related to problems arising from understanding quantum TQFT invariants of the torus bundles.

Example 4.17 (Profinitely flexible 3-manifolds with non-trivial JSJ decomposition)**.** The fundamental groups of the manifolds occurring in Theorem 4.4 2(iv) were investigated in Wilkes [2016]. We will not go into this in any detail here, other than to say that it is shown in Wilkes [ibid.] that there are non-homeomorphic closed graph manifolds whose fundamental groups have isomorphic profinite completions, and that graph manifolds can be distinguished from mixed 3-manifolds by the profinite completion of their fundamental groups. In addition it is shown that if M is a graph manifold that is profinitely flexible, then $|\mathcal{G}_{\mathfrak{M}}(\pi_1(M))| < \infty$.

4.4 Profinite completions of 3-manifold groups and geometric structures.

We now turn to the work of Wilton and Zalesskii [2017a,b] that describes a beautiful connection between the existence of a particular geometric structure on a 3-manifold and the profinite completion of its fundamental group. We begin with a mild strengthening of Wilton and Zalesskii [2017a, Theorem 8.4]

Theorem 4.18 (Wilton-Zalesskii)**.** *Let M be a closed orientable 3-manifold with infinite fundamental group admitting one of Thurston's eight geometries and let $N \in \mathfrak{M}$ with $\pi_1(N) \in \mathcal{G}_{\mathfrak{M}}(\pi_1(M))$. Then N is closed and admits the same geometric structure.*

Proof. This is proved in Wilton and Zalesskii [ibid., Theorem 8.4] with N assumed to be closed, orientable and irreducible. However, the version stated in Theorem 4.18 quickly reduces to this. Briefly, by Theorem 4.11 $\pi_1(M)$ is good, so immediately we have N is closed and orientable by Corollary 4.12.

Furthermore, $\widehat{\pi_1(M)}$ is torsion-free by Corollary 4.12 and so if N is not prime, the summands must all have torsion-free fundamental group. However, in this case we can use the fact that the first L^2-betti number $b_1^{(2)}$ is a profinite invariant by Bridson, Conder, and Reid [2016], and this, together with the work of Lott and Lück [1995] shows that aspherical geometric 3-manifolds have $b_1^{(2)} = 0$, whilst manifolds that are not prime and have torsion-free fundamental group have $b_1^{(2)} \neq 0$. Note that their theorem is stated only for orientable manifolds but this is not a serious problem because, by Lück approximation Lück [1994], if X is a non-orientable compact 3-manifold with infinite fundamental group and $Y \to X$ is its orientable double cover, then $b_1^{(2)}(Y) = 2\,b_1^{(2)}(X)$. We can now use Wilton and Zalesskii [2017a] to complete the proof. □

Given Theorem 4.18, Theorem 4.15, is reduced to the consideration of Seifert fiber spaces. However, the proof still entails some significant work using Bridson, Conder, and Reid [2016] as well as the delicate issue of recovering the euler number of the Seifert fibration from the profinite completion.

In the context of hyperbolic manifolds, a corollary of Theorem 4.18 that is worth recording is.

Corollary 4.19. *Let M be a closed orientable hyperbolic 3-manifold and $N \in \mathfrak{M}$ with $\pi_1(N) \in \mathcal{G}_{\mathfrak{M}}(\pi_1(M))$, then N is closed orientable and hyperbolic.*

More recently Wilton and Zalesskii [2017b] have established a cusped version of this result, namely.

Theorem 4.20. *Let M be a finite volume non-compact orientable hyperbolic 3-manifold and $N \in \mathfrak{M}$ with $\pi_1(N) \in \mathcal{G}_{\mathfrak{M}}(\pi_1(M))$, then N is a finite volume non-compact orientable hyperbolic 3-manifold.*

The proofs of Theorems 4.18 and 4.20 also use the work of Agol and Wise, as well as crucially using "nice" actions of profinite groups on profinite trees which are transferred from the discrete setting using LERF and other parts of the virtual special technology of Wise [2009] (see Wilton and Zalesskii [2017a] and Wilton and Zalesskii [2017b] for details).

Actually what is really at the heart of Corollary 4.19 is a profinite analogue of the Hyperbolization Theorem, which asserts that M is hyperbolic if and only if $\pi_1(M)$ does not contain a copy of $\mathbb{Z} \oplus \mathbb{Z}$. The main part of the proof of Corollary 4.19 is to show that if M is a closed hyperbolic 3-manifold, then $\widehat{\pi_1(M)}$ does not contain a subgroup isomorphic to $\widehat{\mathbb{Z}} \oplus \widehat{\mathbb{Z}}$.

Remark 4.21. One might wonder about the extent to which the full profinite completion of the fundamental group of a hyperbolic 3-manifold is actually needed to distinguish the fundamental group. With that in mind, it is easy to give infinitely many examples of links $L \subset S^3$ (so-called homology boundary links) with hyperbolic complement for which $\pi_1(S^3 \setminus L)$ all have the same pro-p completion (namely the free pro-p group of rank 2) for all primes p, see Bridson and Reid [2015a, Section 8.4] for an explicit example.

4.5 Profinite rigidity amongst 3-manifold groups. We now turn to the issue of profinite rigidity. Given the discussion in Section 4.3 about the failure of profinite rigidity (even amongst 3-manifold groups) for Seifert fibered spaces, torus bundles admitting $SOLV$ geometry, and manifolds admitting a non-trivial JSJ decomposition, the case that needs to be understood is that of finite volume hyperbolic 3-manifolds. We focus on this case in the remainder of this section. We first deal with the case of $\mathcal{G}_{\mathfrak{M}}(\pi_1(M))$, where M is a finite volume hyperbolic 3-manifold. In the light of Theorem 4.6, a natural class of hyperbolic 3-manifolds to attempt to establish rigidity for are hyperbolic 3-manifolds that fiber over the circle, since, as we now explain, this can be used to help organize an approach to profinite rigidity of the fundamental groups of hyperbolic 3-manifolds.

Proposition 4.22. *Suppose that for any orientable finite volume hyperbolic 3-manifold M that fibers over the circle we have $\mathcal{G}_{\mathfrak{M}}(\pi_1(M)) = \{\pi_1(M)\}$. Then if N is a finite volume hyperbolic 3-manifold and Y a compact 3-manifold with $\pi_1(Y) \in \mathcal{G}_{\mathfrak{M}}(\pi_1(N))$, then Y is commensurable to N.*

Proof. Note that from Corollary 4.19 and Theorem 4.20, Y is a finite volume hyperbolic 3-manifold. By Theorem 4.6, we can pass to finite covers N_f and Y_f of N and Y respectively, that are both fibered, and with $\widehat{\pi_1(N_f)} \cong \widehat{\pi_1(Y_f)}$. By the "rigidity hypothesis" of Proposition 4.22, it follows that $\pi_1(N_f) \cong \pi_1(Y_f)$, and so N and Y share a common finite sheeted cover $N_f \cong Y_f$. □

Thus, it is natural to focus on the case of surface bundles. The following rigidity result is proved in Bridson, Reid, and Wilton [2017] (see also Bridson and Reid [2015b] and Boileau and Friedl [2015] for the the case of the figure-eight knot complement). This is the first family of hyperbolic 3-manifolds that fiber over the circle for which the rigidity required in Proposition 4.22 has been carried to completion. An approach to handle other fibered hyperbolic 3-manifolds is described in Bridson, Reid, and Wilton [2017].

Theorem 4.23. *Let M be a once-punctured torus bundle over the circle (hyperbolic or otherwise). Then $\mathcal{G}_{\mathfrak{M}}(\pi_1(M)) = \{\pi_1(M)\}$.*

Some ideas in the proof: We only discuss the hyperbolic case, and refer the reader to Bridson, Reid, and Wilton [ibid.] for the remaining (simpler) cases. In this case $b_1(M) = 1$. From Theorem 4.20 we can assume that if N is a compact 3-manifold with $\pi_1(N) \in \mathcal{G}_{\mathfrak{M}}(\pi_1(M))$, then N is a cusped hyperbolic 3-manifold with $b_1(N) = 1$ (recall Remark 3.2). The proof can be broken down into two main steps as follows:

Step 1: Prove that N is fibered with fiber a once-punctured torus.

Step 2: Since M is a once-punctured torus bundle, given Step 1, a simple analysis gives finitely many possibilities for N. Distinguish these finitely many.

We will make no further comment on Step 2 and refer the reader to Bridson, Reid, and

Wilton [ibid.]. The proof of Step 1 follows Bridson and Reid [2015b] and we briefly comment on this (a different proof of this is given in Boileau and Friedl [2015]). The main difficulty is in establishing that N is fibered. Once this is done, the fact that the fiber is a once-punctured torus follows routinely.

Note that in Bridson and Reid [2015b] the cases that N is hyperbolic or not hyperbolic were treated separately (since Theorem 4.20 was unavailable at the time of writing). As noted above, using Theorem 4.20 we can now reduce to the case that N is hyperbolic. Regardless of this development, we still need to follow the argument of Bridson and Reid [ibid.] to complete the proof. The key point is that if N is not fibered, then using B. Freedman and M. H. Freedman [1998] we can build a surface subgroup $H < K = \ker\{\pi_1(\mathrm{N}) \to \mathbb{Z}\}$ (this homomorphism is unique since $b_1(N) = 1$). By Corollaries 4.8 and 2.9 we deduce that $\widehat{H} \cong \overline{H} < \overline{K} < \widehat{\pi_1(N)}$. Now by uniqueness of the homomorphism $\pi_1(M) \to \mathbb{Z}$, which has kernel a free group F of rank 2, we get $\widehat{H} < \overline{K} \cong \widehat{F}$. However, using cohomological dimension in the context of profinite groups (see Serre [1997]) we get a contradiction: the cohomological dimension of $\widehat{H}$ is 2 and it is 1 for $\widehat{F}$. □

Since Bridson, Reid, and Wilton [2017] was written, the fact that fibering is a profinite invariant has been established by Jaikin-Zapirain [n.d.] without the restriction on $b_1(M)$. The proof of this uses very different methods to those outlined above.

Theorem 4.24 (Jaikin-Zapirain)**.** *Let M be a compact irreducible 3-manifold and let $\Gamma = \pi_1(M)$.*

1. *If $\widehat{\Gamma}$ is isomorphic to the profinite completion of free-by-cyclic group, then M has non-empty boundary consisting of a disjoint union of incompressible tori and Klein bottles, and fibers over the circle with fiber a compact surface with non-empty boundary.*

2. *If $\widehat{\Gamma}$ is isomorphic to the profinite completion of the fundamental group of a closed* 3*-manifold that fibers over the circle, then M is a surface bundle over the circle with fiber a closed surface.*

One can distill from the cohomological dimension argument used at the end of the proof of Theorem 4.23 the following useful proposition.

Proposition 4.25. *Let Γ be a finitely generated residually finite group that contains a subgroup $H \cong \pi_1(\Sigma_g)$ for some $g \geq 1$ and for which $\overline{H} \cong \widehat{H}$ in $\widehat{\Gamma}$. Then $\Gamma \notin \mathcal{G}(F_n)$ for any $n \geq 2$.*

Remark 4.26. It is worth remarking that $\widehat{F}_n$ contains a subgroup isomorphic to some $\pi_1(\Sigma_g)$ which is dense in $\widehat{F}_n$ (see Breuillard, Gelander, Souto, and Storm [2006]).

4.6 A profinitely rigid Kleinian group.

At present it still remains open as to whether there is *any* finite volume hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ with $\mathcal{G}(\Gamma) = \{\Gamma\}$. However in recent work Bridson, McReynolds, Reid, and Spitler [n.d.] if we allow Γ to be a

Kleinian group (i.e. a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$) containing torsion then this can be done. As far as we can tell, this seems to be first example (indeed we give two) of a group "similar to a free group" that can be proved to be profinitely rigid, and can be viewed as providing the first real evidence towards answering Question 4.3 (and 4.1) in the affirmative. Namely we prove the following theorem in Bridson, McReynolds, Reid, and Spitler [ibid.] (where $\omega^2 + \omega + 1 = 0$).

Theorem 4.27. *The Kleinian groups* $\mathrm{PGL}(2, \mathbb{Z}[\omega])$ *and* $\mathrm{PSL}(2, \mathbb{Z}[\omega])$ *are profinitely rigid.*

The case of $\mathrm{PGL}(2, \mathbb{Z}[\omega])$ follows from that of $\mathrm{PSL}(2, \mathbb{Z}[\omega])$, and we so we limit ourselves to briefly indicating the strategy of the proof of Theorem 4.27 for $\mathrm{PSL}(2, \mathbb{Z}[\omega])$.

There are three key steps in the proof which we summarize below.

Theorem 4.28 (Representation Rigidity)**.** *Let* $\iota : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ *denote the identity homomorphism, and* $c = \bar{\iota}$ *the complex conjugate representation. Then if* $\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ *is a representation with infinite image,* $\rho = \iota$ *or* c.

Using Theorem 4.28 we are able to get some control on $\mathrm{PSL}(2, \mathbb{C})$ representations of a finitely generated residually finite group with profinite completion isomorphic to $\widehat{\Gamma}$, and to that end we prove:

Theorem 4.29. *Let* Δ *be a finitely generated residually finite group with* $\widehat{\Delta} \cong \widehat{\Gamma}$. *Then* Δ *admits an epimorphism to a group* $L < \Gamma$ *which is Zariski dense in* $\mathrm{PSL}(2, \mathbb{C})$.

Finally, we make use of Theorem 4.29, in tandem with an understanding of the topology and deformations of orbifolds $\mathbb{H}^3/G$ for subgroups $G < \Gamma$. Briefly, in the notation of Theorem 4.29, the case of L having infinite index can be ruled out using Teichmüller theory to construct explicit finite quotients of L and hence Δ that cannot be finite quotients of Γ. To rule out the finite index case we make use of information about low-index subgroups of Γ, together with the construction of L, and 3-manifold topology to show that L contains the fundamental group of a once-punctured torus bundle over the circle of index 12. We can then invoke Bridson, Reid, and Wilton [2017] to yield the desired conclusion that $\Delta \cong \Gamma$. □

5 Virtually free groups, Fuchsian groups and Limit groups

We now turn from the world of 3-manifold groups to other classes of groups closely related to free groups; virtually free groups (i.e. contains a free subgroup of finite index), Fuchsian groups which are discrete subgroups of $\mathrm{PSL}(2, \mathbb{R})$ and limit groups which we define below. All three classes of these groups contain the class of free groups amongst them. As already noted even groups that are virtually $\mathbb{Z}$ can fail to be profinitely rigid. In Grunewald and Zalesskii [2011] this is extended to give examples of virtually non-abelian free groups in the same genus, as well as providing cases where they show that certain virtually free groups are the only groups in the genus when restricted to virtually free groups.

5.1 Some restricted genus results. Regarding Fuchsian groups, the following is proved in Bridson, Conder, and Reid [2016].

Theorem 5.1. *Let $\mathfrak{L}$ denote the collection of lattices in connected Lie groups and let Γ be a finitely generated Fuchsian group. Then $\mathcal{G}_{\mathfrak{L}}(\Gamma) = \{\Gamma\}$.*

Using the profinite invariance of $b_1^{(2)}$ Bridson, Conder, and Reid [ibid.], it turns out that the hard case of Theorem 5.1 is ruling out non-isomorphic Fuchsian having isomorphic profinite completions. The main part of the proof of this step is to rule out "fake torsion" in the profinite completion, and uses the technology of profinite group actions on profinite trees (see Bridson, Conder, and Reid [ibid.] for details).

By a limit group we mean a finitely-generated group Γ that is fully residually free; i.e. a finitely generated group in which every finite subset can be mapped injectively into a free group by a group homomorphism. In connection with Question 4.1, Wilton [2017] recently proved the following:

Theorem 5.2. *Let Γ be a limit group that is not a free group, and let F be a free group. Then $\widehat{\Gamma}$ is not isomorphic to $\widehat{F}$.*

The key point in the proof of Theorem 5.2 (and indeed the main point of Wilton [ibid.]) is to construct a surface subgroup in a non-free limit group. One can then follow an argument in Bridson, Conder, and Reid [2016] that uses Wilton [2008] (which proves LERF for limit groups) and Proposition 4.25 to complete the proof.

5.2 Profinite genus of free groups and one-ended hyperbolic groups. We close this section with a discussion of possible groups in the genus for free groups. As noted above, Theorem 5.2 uses the existence of surface subgroups to show that non-free limit groups do not lie in the same genus as a free group. The next result from Bridson, Conder, and Reid [2016] takes up this theme, and connects to two well-known open problems about word hyperbolic groups, namely:

(A) Does every 1-ended word-hyperbolic group contain a surface subgroup?

(B) Is every word-hyperbolic group residually finite?

The first question, due to Gromov, was motivated by the case of hyperbolic 3-manifolds, and in this special case the question was settled by Kahn and Markovic [2012]. Indeed, given Kahn and Markovic [ibid.], a natural strengthening of (A) above is to ask:

(A′) Does every 1-ended word-hyperbolic group contain a quasi-convex surface subgroup?

Theorem 5.3. *Suppose that every 1-ended word-hyperbolic group is residually finite and contains a quasi-convex surface subgroup. Then there exists no 1-ended word-hyperbolic group Γ and free group F such that $\widehat{\Gamma} \cong \widehat{F}$.*

Proof. Assume the contrary, and let Γ be a counter-example, with $\widehat{\Gamma} \cong \widehat{F}$ for some free group F. Let H be a quasi-convex surface subgroup of Γ. Note that the finite-index subgroups of H are also quasi-convex in Γ. Under the assumption that all 1-ended hyperbolic groups are residually finite, it is proved in Agol, Groves, and Manning [2009] that H and all its subgroups of finite index must be separable in Γ. Hence by Lemma 2.8, the natural map $\widehat{H} \to \overline{H} < \widehat{\Gamma} \cong \widehat{F}$ is an isomorphism, and can use Proposition 4.25 to complete the proof. □

Corollary 5.4. *Suppose that there exists a 1-ended word hyperbolic group* Γ *with* $\widehat{\Gamma} \cong \widehat{F}$ *for some free group* F*. Then either there exists a word-hyperbolic group that is not residually finite, or there exists a word-hyperbolic group that does not contain a quasi-convex surface subgroup.*

6 Profinite rigidity and flexibility in other settings

Although our attention has been on groups arising from low-dimensional geometry and topology we think it worthwhile to include a (far from complete) survey of profinite rigidity and flexibility for other classes of finitely generated or finitely presented groups.

6.1 Nilpotent and polycyclic groups. As is already evident from Baumslag's examples of meta-cyclic groups in Section 3, the case of nilpotent groups already shows some degree of subtlety. The case of nilpotent groups more generally is well understood due to work of Pickel [1971]. We will not discuss this in any detail, other than to say that, in Pickel [ibid.] it is shown that for a finitely generated nilpotent group Γ, $\mathcal{G}(\Gamma)$ consists of a finite number of isomorphism classes of nilpotent groups, and moreover, examples where the genus can be made arbitrarily large are known (see for example Segal [1983] Chapter 11). Examples of profinitely rigid nilpotent groups of class 2 are constructed in Grunewald and Scharlau [1979].

Similar results are also known for polycyclic groups and we refer the reader to Grunewald and Segal [1978] and Segal [1983]. Note that in the case of nilpotent groups it is straightforward to prove that any finitely generated residually finite group in the same genus as a nilpotent group is nilpotent. The same holds for polycyclic groups (see Sabbagh and Wilson [1991]), but this is a good deal harder.

These results should be compared with the examples of the meta-abelian groups (which are solvable) of Pickel given in Remark 3.4 where the genus is infinite.

6.2 Lattices in semi-simple Lie groups. Let Γ be a lattice in a semi-simple Lie group, for example, in what follows we shall take $\Gamma = \mathrm{SL}(\mathrm{n}, \mathrm{R_k})$ where R_k denotes the ring of integers in a number field k. A natural, obvious class of finite quotients of Γ, are those of the form $\mathrm{SL}(\mathrm{n}, \mathrm{R_k}/\mathrm{I})$ where $I \subset R_k$ is an ideal. Let $\Gamma(I)$ denote the kernel of the reduction homomorphism $\Gamma \to \mathrm{SL}(\mathrm{n}, \mathrm{R_k}/\mathrm{I})$. By Strong Approximation for $\mathrm{SL_n}$ (see Platonov and Rapinchuk [1994] Chapter 7.4 for example) these reduction homomorphisms are surjective for all I. A *congruence subgroup* of Γ is any subgroup

$\Delta < \Gamma$ such that $\Gamma(I) < \Delta$ for some I. A group Γ is said to have the *Congruence Subgroup Property* (abbreviated to CSP) if every subgroup of finite index is a congruence subgroup.

Thus, if Γ has CSP, then $\mathcal{C}(\Gamma)$ is known precisely, and in effect, to determine $\mathcal{C}(\Gamma)$ is reduced to number theory. Expanding on this, since R_k is a Dedekind domain, any ideal I factorizes into powers of prime ideals. If $I = \prod \mathcal{P}_i^{a_i}$, then it is known that $\mathrm{SL}(n, R_k/I) = \prod \mathrm{SL}(n, R_k/\mathcal{P}_i^{a_i})$. Thus the finite groups that arise as quotients of $\mathrm{SL}(n, R_k)$ are determined by those of the form $\mathrm{SL}(n, R_k/\mathcal{P}_i^{a_i})$. Hence we are reduced to understanding how a rational prime p behaves in the extension $k/\mathbb{Q}$. This idea, coupled with the work of Serre [1970] which has shed considerable light on when Γ has CSP allows construction of non-isomorphic lattices in the same genus. We refer the reader to Aka [2012b], Aka [2012a] and Reid [2015] for further details.

6.3 Grothendieck's Problem. A particular case of when discrete groups groups have isomorphic profinite completions is the following (which goes back to Grothendieck [1970]).

Let Γ be a residually finite group and let $u : P \hookrightarrow \Gamma$ be the inclusion of a subgroup P. Then (Γ, P) is called a *Grothendieck Pair* if the induced homomorphism $\widehat{u} : \widehat{P} \to \widehat{\Gamma}$ is an isomorphism but u is not.

We say that Γ is *Grothendieck Rigid* if no proper finitely generated subgroup $u : P \to \Gamma$ gives a Grothendieck Pair.

Grothendieck [ibid.] asked about the existence of Grothendieck Pairs of finitely presented groups and the first such pairs were constructed by Bridson and Grunewald [2004]. The analogous problem for finitely generated groups had been settled earlier by Platonov and Tavgen [1990] (see also Bass and Lubotzky [2000]). Using different methods, Pyber [2004] gave a construction of continuously many finitely generated groups Γ_α with subgroups H_α for which $(\Gamma_\alpha, H_\alpha)$ are Grothendieck Pairs.

The constructions of Platonov and Tavgen [1990] and Bridson and Grunewald [2004] rely on versions of the following result (see also Bridson [2010]). We remind the reader that the *fibre product* $P < \Gamma \times \Gamma$ associated to an epimorphism of groups $p : \Gamma \to Q$ is the subgroup $P = \{(x, y) \mid p(x) = p(y)\}$.

Proposition 6.1. *Let $1 \to N \to \Gamma \to Q \to 1$ be a short exact sequence of groups with Γ finitely generated and let P be the associated fibre product. Suppose that $Q \neq 1$ is finitely presented, has no proper subgroups of finite index, and $H_2(Q, \mathbb{Z}) = 0$. Then*

1. *$(\Gamma \times \Gamma, P)$ is a Grothendieck Pair;*
2. *if N is finitely generated then (Γ, N) is a Grothendieck Pair.*

More recently in Bridson [2016], examples of Grothendieck Pairs were constructed so as to provide the first examples of finitely-presented residually finite groups Γ that contain an infinite sequence of non-isomorphic finitely presented subgroups P_n so that (Γ, P_n) are Grothendieck Pairs. In particular, this provides examples of finitely presented groups Γ for which $\mathcal{G}(\Gamma)$ is infinite. These examples are non-solvable in contrast to those of Pickel in Remark 3.4

Note that if a H is a subgroup of a group Γ and Γ is H-separable then it is easy to see that (Γ, H) cannot be a Grothendieck Pair (since H is not dense in the profinite topology). This was noticed in Platonov and Tavgen [1990] to observe that free groups and Fuchsian groups were Grothendieck Rigid. For 3-manifolds Grothendieck Rigidity was shown in Long and Reid [2011] for the fundamental groups of closed geometric 3-manifolds and finite volume hyperbolic 3-manifolds without appealing to LERF in the hyperbolic case. In Cavendish [2012] and Reid [2015] this was extended to the fundamental groups of all closed irreducible 3-manifolds (as a consequence of Theorem 4.11). This program has been completed by Boileau and Friedl [2017] who proved:

Theorem 6.2. *The fundamental group of any compact, connected, irreducible, orientable* 3*-manifold with empty or toroidal boundary is Grothendieck Rigid.*

7 Final remarks and further questions

As should be clear from this article, the questions posed in Section 4 remain stubbornly open, and even questions about the nature of $\mathcal{G}_{\mathfrak{M}}(\pi_1(M))$ for M a finite volume hyperbolic 3-manifold seem hard to resolve. Never the less, these open problems can be used as platforms for other, perhaps more approachable problems. We discuss a few, other problems for other classes of groups can be found in Reid [2015].

Question 7.1. Let Γ denote the fundamental group of the figure-eight knot complement. It is well-known that Γ has index 12 in the group $\mathrm{PSL}(2, \mathbb{Z}[\omega])$ of Theorem 4.27. Is Γ is profinitely rigid?

As noted in Section 4.5, it was shown in Bridson and Reid [2015b] and Boileau and Friedl [2015] that $\mathcal{G}_{\mathfrak{M}}(\Gamma) = \{\Gamma\}$.

Question 7.2. Let M_W denote the Weeks manifold. This is the smallest volume hyperbolic 3-manifold Gabai, Meyerhoff, and Milley [2009]. Is $\mathcal{G}_{\mathfrak{M}}(\pi_1(M_W) = \{\pi_1(M_W)\}$?

Indeed, one might wonder whether the techniques of Bridson, McReynolds, Reid, and Spitler [n.d.] (as described in Theorem 4.27) can be brought to bear in this example since $\pi_1(M_W)$ exhibits a certain amount of representation rigidity.

Question 7.3. In Section 4.5 it was pointed out that recently Jaikin-Zapirain [n.d.] showed that being fibered is a profinite invariant. Given this, a natural question is:

Is the Thurston norm ball a profinite invariant? That is to say, if M is a closed hyperbolic 3-manifold and N a closed hyperbolic 3-manifold with $\pi_1(N) \in \mathcal{G}_{\mathfrak{M}}(\pi_1(M))$ are the Thurston norm balls isomorphic?

Some progress on this is given in Boileau and Friedl [2015] under an additional condition on the isomorphsim between profinite completions. However, it seems unlikely that this condition will hold in general.

Question 7.4. Is the volume a profinite invariant? That is to say, if M is a finite volume hyperbolic 3-manifold and N a finite volume hyperbolic 3-manifold with $\pi_1(N) \in \mathcal{G}_{\mathfrak{M}}(\pi_1(M))$ does $\mathrm{vol}(M) = \mathrm{vol}(N)$?

It follows from well-known properties of the set of volumes of hyperbolic 3-manifolds Thurston [1979] that if Question 7.4 has a positive answer then $|\mathcal{G}_{\mathfrak{m}}(\pi_1(M))|$ is finite.

There does appear to be some conjectural evidence to support a positive answer. Briefly, it is conjectured (roughly) that if $\{\Gamma_n\}$ is a cofinal sequence of subgroups of finite index in $\pi_1(M)$ (as above), then:

$$\frac{\log|\mathrm{Tor}(\mathrm{H}_1(\square_{\mathrm{n}}, \mathbb{Z}))|}{[\pi_1(M) : \Gamma_n]} \to \frac{1}{6\pi}\mathrm{vol}(M) \text{ as } n \to \infty.$$

Note that $\mathrm{Tor}(\mathrm{H}_1(\square_{\mathrm{n}}, \mathbb{Z}))$ is visible in the profinite completions $\widehat{\Gamma}_n$ and so if the above conjecture is true, this would allow one to deduce $\pi_1(N) \in \mathcal{G}_{\mathfrak{m}}(\pi_1(M))$ implies $\mathrm{vol}(M) = \mathrm{vol}(N)$.

References

Ian Agol (2008). "Criteria for virtual fibering". *J. Topol.* 1.2, pp. 269–284. MR: 2399130 (cit. on p. 1197).

– (2013). "The virtual Haken conjecture". *Doc. Math.* 18. With an appendix by Agol, Daniel Groves, and Jason Manning, pp. 1045–1087. MR: 3104553 (cit. on pp. 1191, 1197–1199).

Ian Agol, Daniel Groves, and Jason Fox Manning (2009). "Residual finiteness, QCERF and fillings of hyperbolic groups". *Geom. Topol.* 13.2, pp. 1043–1073. MR: 2470970 (cit. on p. 1207).

Menny Aka (2012a). "Arithmetic groups with isomorphic finite quotients". *J. Algebra* 352, pp. 322–340. MR: 2862189 (cit. on p. 1208).

– (2012b). "Profinite completions and Kazhdan's property (T)". *Groups Geom. Dyn.* 6.2, pp. 221–229. MR: 2914858 (cit. on p. 1208).

Matthias Aschenbrenner, Stefan Friedl, and Henry Wilton (2015). *3-manifold groups*. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, pp. xiv+215. MR: 3444187 (cit. on p. 1198).

Hyman Bass and Alexander Lubotzky (2000). "Nonarithmetic superrigid groups: counterexamples to Platonov's conjecture". *Ann. of Math. (2)* 151.3, pp. 1151–1173. MR: 1779566 (cit. on p. 1208).

Gilbert Baumslag (1974). "Residually finite groups with the same finite images". *Compositio Math.* 29, pp. 249–252. MR: 0357615 (cit. on p. 1196).

Michel Boileau and Stefan Friedl (May 2015). "The profinite completion of 3-manifold groups, fiberedness and the Thurston norm". arXiv: 1505.07799 (cit. on pp. 1203, 1209).

– (Oct. 2017). "Grothendieck rigidity of 3-manifold groups". arXiv: 1710.02746 (cit. on p. 1209).

Francis Bonahon (2002). "Geometric structures on 3-manifolds". In: *Handbook of geometric topology*. North-Holland, Amsterdam, pp. 93–164. MR: 1886669 (cit. on p. 1197).

Emmanuel Breuillard, Tsachik Gelander, Juan Souto, and Peter Storm (2006). "Dense embeddings of surface groups". *Geom. Topol.* 10, pp. 1373–1389. MR: 2255501 (cit. on p. 1204).

Martin R. Bridson (2010). "The Schur multiplier, profinite completions and decidability". *Bull. Lond. Math. Soc.* 42.3, pp. 412–416. MR: 2651935 (cit. on p. 1208).

– (2016). "The strong profinite genus of a finitely presented group can be infinite". *J. Eur. Math. Soc. (JEMS)* 18.9, pp. 1909–1918. MR: 3531665 (cit. on p. 1208).

Martin R. Bridson, M. D. E. Conder, and Alan W. Reid (2016). "Determining Fuchsian groups by their finite quotients". *Israel J. Math.* 214.1, pp. 1–41. MR: 3540604 (cit. on pp. 1201, 1202, 1206).

Martin R. Bridson and Fritz J. Grunewald (2004). "Grothendieck's problems concerning profinite completions and representations of groups". *Ann. of Math. (2)* 160.1, pp. 359–373. MR: 2119723 (cit. on p. 1208).

Martin R. Bridson, D. B. McReynolds, Alan W. Reid, and R. Spitler (n.d.). "Profinite rigidity of $\mathbf{PGL}(2, \mathbb{Z}[\omega])$ and $\mathbf{PSL}(2, \mathbb{Z}[\omega])$". In preparation (cit. on pp. 1204, 1205, 1209).

Martin R. Bridson and Alan W. Reid (2015a). "Nilpotent completions of groups, Grothendieck pairs, and four problems of Baumslag". *Int. Math. Res. Not. IMRN* 8, pp. 2111–2140. MR: 3344664 (cit. on p. 1202).

– (May 2015b). "Profinite rigidity, fibering, and the figure-eight knot". arXiv: 1505.07886 (cit. on pp. 1203, 1204, 1209).

Martin R. Bridson, Alan W. Reid, and Henry Wilton (2017). "Profinite rigidity and surface bundles over the circle". *Bull. Lond. Math. Soc.* 49.5, pp. 831–841. MR: 3742450 (cit. on pp. 1203–1205).

Will Cavendish (2012). *Finite-Sheeted Covering Spaces and Solenoids over 3-manifolds*. Thesis (Ph.D.)–Princeton University. ProQuest LLC, Ann Arbor, MI, p. 96. MR: 3078440 (cit. on pp. 1199, 1209).

John D. Dixon, Edward W. Formanek, John C. Poland, and Luis Ribes (1982). "Profinite completions and isomorphic finite quotients". *J. Pure Appl. Algebra* 23.3, pp. 227–231. MR: 644274 (cit. on p. 1193).

Benedict Freedman and Michael H. Freedman (1998). "Kneser-Haken finiteness for bounded 3-manifolds locally free groups, and cyclic covers". *Topology* 37.1, pp. 133–147. MR: 1480882 (cit. on p. 1204).

Louis Funar (2013). "Torus bundles not distinguished by TQFT invariants". *Geom. Topol.* 17.4. With an appendix by Funar and Andrei Rapinchuk, pp. 2289–2344. MR: 3109869 (cit. on pp. 1200, 1201).

David Gabai (1986). "On 3-manifolds finitely covered by surface bundles". In: *Low-dimensional topology and Kleinian groups (Coventry/Durham, 1984)*. Vol. 112. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, pp. 145–155. MR: 903863 (cit. on p. 1197).

David Gabai, Robert Meyerhoff, and Peter Milley (2009). "Minimum volume cusped hyperbolic three-manifolds". *J. Amer. Math. Soc.* 22.4, pp. 1157–1215. MR: 2525782 (cit. on p. 1209).

Alexander Grothendieck (1970). "Représentations linéaires et compactification profinie des groupes discrets". *Manuscripta Math.* 2, pp. 375–396. MR: 0262386 (cit. on p. 1208).

Daniel Groves and Jason Fox Manning (Aug. 2017). "Quasiconvexity and Dehn filling". arXiv: 1708.07968 (cit. on p. 1197).

Fritz J. Grunewald, A. Jaikin-Zapirain, and Pavel A. Zalesskii (2008). "Cohomological goodness and the profinite completion of Bianchi groups". *Duke Math. J.* 144.1, pp. 53–72. MR: 2429321 (cit. on p. 1199).

Fritz J. Grunewald and Rudolf Scharlau (1979). "A note on finitely generated torsion-free nilpotent groups of class 2". *J. Algebra* 58.1, pp. 162–175. MR: 535851 (cit. on p. 1207).

Fritz J. Grunewald and Daniel Segal (1978). "On polycyclic groups with isomorphic finite quotients". *Math. Proc. Cambridge Philos. Soc.* 84.2, pp. 235–246. MR: 0491962 (cit. on p. 1207).

Fritz J. Grunewald and Pavel A. Zalesskii (2011). "Genus for groups". *J. Algebra* 326, pp. 130–168. MR: 2746056 (cit. on p. 1205).

John Hempel (Sept. 2014). "Some 3-manifold groups with the same finite quotients". arXiv: 1409.3509 (cit. on p. 1200).

R. Hirshon (1977). "Some cancellation theorems with applications to nilpotent groups". *J. Austral. Math. Soc. Ser. A* 23.2, pp. 147–165. MR: 0447410 (cit. on p. 1196).

A. Jaikin-Zapirain (n.d.). "Recogniton of fibering in compact 3-manifolds". In preparation (cit. on pp. 1204, 1209).

Jeremy Kahn and Vladimir Markovic (2012). "Immersing almost geodesic surfaces in a closed hyperbolic three manifold". *Ann. of Math. (2)* 175.3, pp. 1127–1190. MR: 2912704 (cit. on p. 1206).

Sławomir Kwasik and Witold Rosicki (2004). "On stability of 3-manifolds". *Fund. Math.* 182.2, pp. 169–180. MR: 2100626 (cit. on p. 1200).

Darren D. Long and Alan W. Reid (2011). "Grothendieck's problem for 3-manifold groups". *Groups Geom. Dyn.* 5.2, pp. 479–499. MR: 2782181 (cit. on p. 1209).

John Lott and Wolfgang Lück (1995). "L^2-topological invariants of 3-manifolds". *Invent. Math.* 120.1, pp. 15–60. MR: 1323981 (cit. on p. 1201).

Alexander Lubotzky (1993). "Torsion in profinite completions of torsion-free groups". *Quart. J. Math. Oxford Ser. (2)* 44.175, pp. 327–332. MR: 1240475 (cit. on p. 1199).

W. Lück (1994). "Approximating L^2-invariants by their finite-dimensional analogues". *Geom. Funct. Anal.* 4.4, pp. 455–481. MR: 1280122 (cit. on p. 1202).

John Morgan and Gang Tian (2014). *The geometrization conjecture*. Vol. 5. Clay Mathematics Monographs. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, pp. x+291. MR: 3186136 (cit. on p. 1197).

Walter D. Neumann (1997). "Commensurability and virtual fibration for graph manifolds". *Topology* 36.2, pp. 355–378. MR: 1415593 (cit. on p. 1198).

Nikolay Nikolov and Dan Segal (2007). "On finitely generated profinite groups. I. Strong completeness and uniform bounds". *Ann. of Math. (2)* 165.1, pp. 171–238. MR: 2276769 (cit. on p. 1193).

G. A. Noskov, V. N. Remeslennikov, and V. A. Romankov (1979). "Infinite groups". In: *Algebra. Topology. Geometry, Vol. 17 (Russian)*. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, pp. 65–157, 308. MR: 584569 (cit. on p. 1196).

Grisha Perelman (Nov. 2002). "The entropy formula for the Ricci flow and its geometric applications". arXiv: math/0211159.

– (Mar. 2003). "Ricci flow with surgery on three-manifolds". arXiv: math/0303109.

P. F. Pickel (1971). "Finitely generated nilpotent groups with isomorphic finite quotients". *Trans. Amer. Math. Soc.* 160, pp. 327–341. MR: 0291287 (cit. on p. 1207).

– (1974). "Metabelian groups with the same finite quotients". *Bull. Austral. Math. Soc.* 11, pp. 115–120. MR: 0364455 (cit. on p. 1196).

Vladimir P. Platonov and Andrei Rapinchuk (1994). *Algebraic groups and number theory*. Vol. 139. Pure and Applied Mathematics. Translated from the 1991 Russian original by Rachel Rowen. Academic Press, Inc., Boston, MA, pp. xii+614. MR: 1278263 (cit. on p. 1207).

Vladimir P. Platonov and O. I. Tavgen (1990). "Grothendieck's problem on profinite completions and representations of groups". *K-Theory* 4.1, pp. 89–101. MR: 1076526 (cit. on pp. 1208, 1209).

Piotr Przytycki and Daniel T. Wise (2018). "Mixed 3-manifolds are virtually special". *J. Amer. Math. Soc.* 31.2, pp. 319–347. MR: 3758147 (cit. on p. 1198).

László Pyber (2004). "Groups of intermediate subgroup growth and a problem of Grothendieck". *Duke Math. J.* 121.1, pp. 169–188. MR: 2031168 (cit. on p. 1208).

Alan W. Reid (2015). "Profinite properties of discrete groups". In: *Groups St Andrews 2013*. Vol. 422. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, pp. 73–104. MR: 3445488 (cit. on pp. 1199, 1208, 1209).

Luis Ribes and Pavel A. Zalesskii (2000). *Profinite groups*. Vol. 40. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, Berlin, pp. xiv+435. MR: 1775104 (cit. on pp. 1192, 1193).

Gabriel Sabbagh and John S. Wilson (1991). "Polycyclic groups, finite images, and elementary equivalence". *Arch. Math. (Basel)* 57.3, pp. 221–227. MR: 1119894 (cit. on p. 1207).

Peter Scott (1978). "Subgroups of surface groups are almost geometric". *J. London Math. Soc. (2)* 17.3. See correction in Scott [1985], pp. 555–565. MR: 0494062 (cit. on pp. 1198, 1213).

– (1985). "Correction to: Subgroups of surface groups are almost geometric". *J. London Math. Soc. (2)* 32.2. Correction to Scott [1978], pp. 217–220. MR: 811778 (cit. on p. 1213).

Daniel Segal (1983). *Polycyclic groups*. Vol. 82. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, pp. xiv+289. MR: 713786 (cit. on p. 1207).

Jean-Pierre Serre (1970). "Le problème des groupes de congruence pour SL2". *Ann. of Math. (2)* 92, pp. 489–527. MR: 0272790 (cit. on p. 1208).

Jean-Pierre Serre (1997). *Galois cohomology*. Translated from the French by Patrick Ion and revised by the author. Springer-Verlag, Berlin, pp. x+210. MR: 1466966 (cit. on pp. 1198, 1199, 1204).

Hongbin Sun (Aug. 2016). "NonLERFness of arithmetic hyperbolic manifold groups". arXiv: 1608.04816 (cit. on p. 1198).

William P. Thurston (1979). "The Geometry and Topology of 3-Manifolds". Princeton University mimeographed notes (cit. on p. 1210).

– (1997). *Three-dimensional geometry and topology. Vol. 1*. Vol. 35. Princeton Mathematical Series. Edited by Silvio Levy. Princeton University Press, Princeton, NJ, pp. x+311. MR: 1435975 (cit. on p. 1197).

Gareth Wilkes (May 2016). "Profinite rigidity of graph manifolds and JSJ decompositions of 3-manifolds". arXiv: 1605.08244 (cit. on p. 1201).

– (2017). "Profinite rigidity for Seifert fibre spaces". *Geom. Dedicata* 188, pp. 141–163. MR: 3639628 (cit. on p. 1200).

Henry Wilton (2008). "Hall's theorem for limit groups". *Geom. Funct. Anal.* 18.1, pp. 271–303. MR: 2399104 (cit. on p. 1206).

– (Jan. 2017). "Essential surfaces in graph pairs". arXiv: 1701.02505 (cit. on p. 1206).

Henry Wilton and Pavel A. Zalesskii (2017a). "Distinguishing geometries using finite quotients". *Geom. Topol.* 21.1, pp. 345–384. MR: 3608716 (cit. on pp. 1200–1202).

– (2017b). "Pro-p subgroups of profinite completions of 3-manifold groups". *J. Lond. Math. Soc. (2)* 96.2, pp. 293–308. arXiv: 1703.03701. MR: 3708951 (cit. on pp. 1201, 1202).

Daniel T. Wise (2009). "Research announcement: the structure of groups with a quasiconvex hierarchy". *Electron. Res. Announc. Math. Sci.* 16, pp. 44–55. MR: 2558631 (cit. on pp. 1191, 1197–1199, 1202).

Received 2017-11-29.

Alan W. Reid
Department of Mathematics
Rice University
Houston, TX 77005, USA
alan.reid@rice.edu