Profinite rigidity, Grothendieck pairs and low-dimensional topology

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Abstract The extent to which a finitely generated or finitely presented residually finite group is profinitely rigid amongst the class of finitely generated or finitely presented residually finite groups has seen a resurgence of activity in the last decade. This paper will survey some recent progress, in both the positive and negative directions, as well as surveying properties of groups that are, or are not, profinite properties.

1 Introduction

It is an old and natural idea to try to distinguish finitely presented groups via their finite quotients. For instance, one might prove that a group presentation does not represent the trivial group by exhibiting a map onto a non-trivial finite group. Recently, there has been renewed interest in the question of when the set of finite quotients of a finitely generated residually finite group determines the group up to isomorphism. In more sophisticated language, one wants to develop a complete understanding of the circumstances in which finitely generated residually finite groups have isomorphic *profinite completions*. Motivated by this, we say that a residually finite group is *profinitely rigid*, if whenever $\widehat{\Delta} \cong \widehat{\Gamma}$, then $\Delta \cong \Gamma$.

This article is based on two lectures the author gave at K.I.A.S. in September 2023, and its purpose is to survey some recent work and progress on profinite rigidity motivated by Remeslennikov's question (stated here as Conjecture 1) on the profinite rigidity of a free group, related groups as well as properties of discrete groups that are, or are not, profinite properties. A survey of previous work up until 2018 is available at [51], so this article will focus on more recent developments. However, the article will have some overlap with a recent Seminar Bourbaki article by B. Remy [52].

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We finish the Introduction by briefly outlining the remainder of the paper. In §2 we review some basics on profinite groups and profinite completions of discrete groups, in §3 we recall the notions of profinite rigidity and being a profinite property, as well stating the main conjectures about profinite rigidity of free groups, surface groups and the fundamental groups of finite volume hyperbolic 3-manifolds. In §4 we then discuss recent progress on these conjectures. In the other direction, in §5 we describe how so-called Grothendieck pairs can be constructed, as well as their utility in constructing examples of groups with isomorphic profinite completions with properties not preserved by the profinite completion. Finally in §7, we gather some remarks and pose some questions that arise out of this survey article.

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2 Preliminaries

We will provide some background on profinite groups and profinite completions of discrete groups, and refer the reader [53] for a more detailed account of the topics covered here.

2.1 Profinite groups

A *directed set* is a partially ordered set I such that for every $i, j \in I$ there exists $k \in I$ such that $k \geq i$ and $k \geq j$. An *inverse system* is a family of sets $\{X_i\}_{\{i \in I\}}$, where I is a directed set, and a family of maps $\phi_{ij} : X_i \longrightarrow X_j$ whenever $i \geq j$, such that:

- $\phi_{ii} = id_{X_i}$;
- $\phi_{ij}\phi_{jk} = \phi_{ik}$, whenever $i \geq j \geq k$.

Denoting this system by (X_i, ϕ_{ij}, I), the *inverse limit* of the inverse system (X_i, ϕ_{ij}, I) is the set

$$\varprojlim X_i = \{(x_i) \in \prod_{i \in I} X_i | \phi_{ij}(x_i) = x_j, \text{ whenever } i \geq j\}.$$

If (X_i, ϕ_{ij}, I) is an inverse system of non-empty compact, Hausdorff, totally disconnected topological spaces (resp. topological groups) over the directed set I, then $\varprojlim X_i$ is a non-empty, compact, Hausdorff, totally disconnected topological space (resp. topological group).

2.2 Profinite completion

Let Γ be a finitely generated group (not necessarily residually finite for this discussion), and let $\mathcal{N}$ denote the collection of all finite index normal subgroups of Γ. Note that $\mathcal{N}$ is non-empty as $\Gamma \in \mathcal{N}$, and we can make $\mathcal{N}$ into directed set by declaring that

$$\text{For } M, N \in \mathcal{N}, M \leq N \text{ whenever } M \textit{ contains } N.$$

In this case, there are natural epimorphisms $\phi_{NM} : \Gamma/N \to \Gamma/M$, and the inverse limit of the inverse system $(\Gamma/N, \phi_{NM}, \mathcal{N})$ is denoted $\widehat{\Gamma}$ and defined to be to the *profinite completion* of Γ.

Note that there is a natural map $\iota : \Gamma \to \widehat{\Gamma}$ defined by

$$g \mapsto (gN) \in \varprojlim \Gamma/N,$$

and it is easy to see that ι is injective if and only if Γ is residually finite.

An alternative, perhaps more concrete way of viewing the profinite completion is as follows. If, for each $N \in \mathcal{N}$, we equip each Γ/N with the discrete topology, then $\prod\{\Gamma/N : N \in \mathcal{N}\}$ is a compact space and $\widehat{\Gamma}$ can be identified with $\overline{j(\Gamma)}$ where $j : \Gamma \to \prod\{\Gamma/N : N \in \mathcal{N}\}$ is the map $g \mapsto (gN)$.

3 Profinite rigidity

Henceforth all discrete groups will be assumed to be *finitely generated and residually finite*.

Definition 1 The genus of a finitely generated group Γ is: $\mathcal{G}(\Gamma) = \{\Delta : \widehat{\Delta} \cong \widehat{\Gamma}\}$.

In addition, if $\mathcal{L}$ is a class of groups, then we define $\mathcal{G}_{\mathcal{L}}(\Gamma) = \{\Delta \in \mathcal{G}(\Gamma) : \Delta \in \mathcal{L}\}$.

Definition 2 We define Γ to be profinitely rigid if $\mathcal{G}(\Gamma) = \{\Gamma\}$.

In addition, if $\Gamma \in \mathcal{L}$, a class of groups, we say Γ is profinitely rigid in $\mathcal{L}$ if $\mathcal{G}_{\mathcal{L}}(\Gamma) = \{\Gamma\}$.

We will not give a detailed survey of profinite rigidity, we refer the reader to [51] for this, however, we will discuss some recent progress on profinite rigidity, and its failure below. The context for the discussion remains Remeslennikov's question as to whether the free group F_n of rank $n \geq 2$ is profinitely rigid. This remains open, but we feel at this stage, Remeslennikov's question should be promoted to have a conjectured positive answer.

Conjecture 1: *F_n is profinitely rigid.*

The group F_n arises in many guises in low-dimensional geometry and topology, and hence affords several natural ways in which to generalize. In the light of this, natural generalizations of Remeslennikov's question, which we also promote to conjectures are the following.

Conjecture 2: *Let Σ_g be a closed orientable surface of genus $g \geq 2$, then $\pi_1(\Sigma_g)$ is profinitely rigid.*

Conjecture 3: *Let M be a complete orientable hyperbolic 3-manifold of finite volume, then $\pi_1(M)$ is profinitely rigid.*

In §4 we will describe some recent progress on these conjectures as well as other related questions and conjectures.

We will also discuss profinite properties of finitely generated groups: i.e. a property $\mathcal{P}$ of a group is said to be a *profinite property*, if whenever Δ is finitely generated with $\Delta \in \mathcal{G}(\Gamma)$, if Γ has $\mathcal{P}$ then Δ has $\mathcal{P}$. It is easy to see that being abelian or nilpotent are profinite properties, as is the abelianization of a group. A more subtle profinite property is the first ℓ^2-Betti number [16, Corollary 3.3]. On the other hand, having Property T is not a profinite property [3], and an alternative way to formulate Remeslennikov's question is the question as to whether being free is a profinite property. Some recent work on properties that are not profinite properties is provided in §5

4 Progress on the Conjectures in §3

In this section we describe some recent advances on the conjectures raised in §3.

4.1 The free group

Until recently, there has been little progress on whether the free group F_n is profinitely rigid. However, in [38], it was proved that if there exists a finitely generated residually finite group Γ with $\widehat{\Gamma} \cong \widehat{F_n}$, then Γ is *parafree*: that is to say, Γ is residually nilpotent and has the same lower central series quotients as F_n. Apart from the properties in the definition of a parafree group, it is known that parafree groups do share other properties with free groups: for example any non-trivial finitely generated normal subgroup has finite index, they satisfy a Freiheitssatz, they have non-zero $b_1^{(2)}$ and are C^*-simple (see [15] for a discussion of all of these results).

Beyond this, what is needed at this point is to develop a structure theory for finitely generated parafree groups, which we now discuss and list some questions. We begin with the following question.

? Question

Let Γ be a finitely generated parafree group with the property that every subgroup of finite index is parafree. Is Γ free?

Proposition 1 *Suppose that Question 7 has a positive answer. Then free groups are profinitely rigid.*

Proof As noted above, if there exists a finitely generated residually finite group Γ with $\widehat{\Gamma} \cong \widehat{F_n}$, then Γ is parafree. Given a subgroup P of finite index of Γ, then it follows that $\widehat{P} \cong \widehat{F}$ for some finitely generated free group F. Applying [38] once again implies that P is parafree. Hence all subgroups of finite index in Γ are parafree, and so the assumption of a positive answer to Question 7 implies that Γ is free.

Question 7 is similar in spirit to what has become known as Mel'nikov's conjecture. Recall that a group Γ is called a *Mel'nikov group* if it is a non-free infinite one-relator group, all of whose finite index subgroups are one-relator groups. Then the conjecture is that a Mel'nikov group is a surface group or isomorphic to the Baumslag-Solitar group $\mathrm{BS}(1, \mathrm{n})$ for some $n \neq 0$.

There has been recent progress on the Mel'nikov conjecture: for example, [31] proved it in the case of two-generator one-relator groups, [40] proved it in the case when $H_2(\Gamma, \mathbb{Z}) \neq 0$, and [29] proved it in the case when Γ is residually free. This contrasts strongly with Question 7 in which there has been (to the best of the author's knowledge) no progress.

Going back to work of Baumslag [6], many constructions of parafree groups provide examples of one-relator parafree groups. Moreover, a basic result about parafree groups proved by Baumslag in [5] is that parafree groups are 2-free; i.e. every 2-generator subgroup of a parafree group is free. Using this, together with work of Louder and Wilton [42], Linton proved in [41, Corollary 7.4] that one-relator parafree groups are hyperbolic and virtually compact special (in particular they are linear). Given this, a recent result of Wilton [62] proves that if Γ is a one-relator group, then Γ contains a one-ended, quasi-convex subgroup S of infinite index, and so we have the following.

Corollary 1 *In the notation above, if Γ is a one-relator parafree group for which the quasi-convex subgroup S is a surface group, then $\widehat{\Gamma}$ is not isomorphic to $\widehat{F}$ for any finitely generated free group F.*

Proof This follows from [16, Proposition 4.9] on noting that S is good in the sense of Serre, and that the virtual compact special assumption ensures that quasi-convex subgroups of Γ are separable in Γ. This latter condition then implies that $\widehat{S} \to \widehat{\Gamma}$ is injective, and so completes the proof.

Remark 1 We note that this argument was also employed by Wilton [61] when he proved (amongst other things) that the profinite completion of a non-free limit group could not be isomorphic to a free profinite group.

Remark 2 If the one-ended subgroup S as provided by Wilton could be shown to be a Poincaré duality group over some finite field $\mathbb{F}_p$, then the same argument could be made to prove that the profinite completion of a one-relator parafree group could not be isomorphic to a free profinite group.

Indeed, in the light of the importance of the framework of virtually special groups, an approach based on the discussion above (albeit seemingly out of reach at present without a better understanding of a structure theory for parafree groups) to Conjecture 1 is presented by a solution to the following conjecture:

Conjecture 4: *Let Γ be a finitely generated parafree group. Then Γ is hyperbolic, virtually compact special and contains a quasi-convex surface subgroup.*

We close this section by noting that in [6], Baumslag constructs an *infinitely generated* parafree group with the same set of finite quotients as a free group of rank 2.

4.2 A general template for proving profinite rigidity

Here we describe a general template that can be used to *attempt to prove profinite rigidity*. This was successfully implemented in [17] and [18] with an improvement in [19] in some (as yet) restricted class of examples (see below). We will discuss in more detail in certain situations the steps in the template.

Template: *Let Γ be a lattice in a semi-simple Lie group G and suppose that Λ is a finitely generated residually finite group with $\widehat{\Lambda} \cong \widehat{\Gamma}$.*

1. *(Representation rigidity/Galois rigidity) Arrange that Γ has the* **minimal number** *of Zariski dense representations into G, together with additional constraints (discussed below). Use this to construct a homomorphism $\rho : \Lambda \to \Gamma$ with Zariski dense image (sometimes one can get away with not being quite into Γ but commensurable with Γ).*
2. *Arrange for the following statement to hold: Any proper Zariski-dense subgroup $L < \Gamma$ has a finite quotient that Γ does not.*

Given this set up, it follows that the homomorphism ρ is surjective, and then the Hopfian property for profinite groups [53], implies the induced epimorphism $\hat{\rho} : \widehat{\Lambda} \to \widehat{\Gamma}$ is an isomorphism, from which profinite rigidity follows.

We now discuss some cases where the above template can be followed, in some cases to completion, and others only partially.

Galois rigidity for Fuchsian and Kleinian groups: Let $\phi\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$ be the quotient homomorphism, and suppose that H is a finitely generated subgroup of $\mathrm{PSL}(2,\mathbb{C})$. Set $H_1 = \phi^{-1}(H)$, then we say that H is *Zariski dense* in $\mathrm{PSL}(2,\mathbb{C})$ when what H_1 is a Zariski dense subgroup of $\mathrm{SL}(2,\mathbb{C})$. The *trace-field* of H is defined to be the field

$$K_H = \mathbb{Q}(\mathrm{tr}(\gamma)\,:\,\gamma \in H_1).$$

If K_H is a number field with ring of integers R_{K_H}, we say that H has *integral traces* if $\mathrm{tr}(\gamma) \in \mathrm{R_{K_H}}$ for all $\gamma \in H_1$.

Suppose that H is a finitely generated group and $\rho\colon H \to \mathrm{PSL}(2,\mathbb{C})$ a Zariski dense representation with $K = K_{\rho(H)}$ a number field of degree n_K. If $K = \mathbb{Q}(\theta)$ for some algebraic number θ, then the Galois conjugates of θ, say $\theta = \theta_1, \ldots, \theta_{n_K}$, provide embeddings $\sigma_i\colon K \to \mathbb{C}$ defined by $\theta \mapsto \theta_i$. These in turn can be used to build n_K Zariski dense non-conjugate representations $\rho_{\sigma_i}\colon H \to \mathrm{PSL}(2,\mathbb{C})$ with the property that $\mathrm{tr}(\rho_{\sigma_\mathrm{i}}(\gamma)) = \sigma_\mathrm{i}(\mathrm{tr}\rho(\gamma))$ for all $\gamma \in H$. We refer to these as *Galois conjugate representations*. The existence of these Galois conjugates shows that $|\mathrm{X_{zar}}(H,\mathbb{C})| \geq n_{K_{\rho(H)}}$, where $\mathrm{X_{zar}}(H,\mathbb{C})$ denotes the set of Zariski dense representations $H \to \mathrm{PSL}(2,\mathbb{C})$ up to conjugacy.

Definition 3 Let Γ be a finitely generated group and $\rho\colon \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ a Zariski dense representation whose trace field $K_{\rho(\Gamma)}$ is a number field. If $|\mathrm{X_{zar}}(\Gamma,\mathbb{C})| = n_{K_{\rho(\Gamma)}}$, we say that Γ is *Galois rigid* (with associated field K_Γ).

Zariski dense representations $\beta : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ are irreducible, so it follows that they are determined up to conjugacy by their character. With this in mind, we shall sometimes abuse notation by writing $\beta \in \mathrm{X_{zar}}(\Gamma,\mathbb{C})$ when what we mean is that β is a Zariski dense representation. Likewise, it is sometimes convenient to refer to elements of $\mathrm{X_{zar}}(\Gamma,\mathbb{C})$ as if they were representations.

Examples: (1) Let $p > 5$ be a prime number then the $(2,3,p)$ triangle group $\Delta(p) = \langle a,b | a^2 = b^3 = (ab)^p = 1\rangle$ is Galois rigid (this can be done by direct calculation, or see [1] for example).

Note that all triangle groups have 0-dimensional $\mathrm{PSL}(2,\mathbb{C})$-character varieties, but most are not Galois rigid; e.g. if $\Delta = \langle a,b | a^l = b^m = (ab)^n = 1\rangle$, then using divisors of l, m and n we can arrange for surjections of Δ to other hyperbolic triangle groups (depending on the divisors).

(2) The Bianchi group $\mathrm{PSL}(2,\mathbb{Z}[\omega])$ is Galois rigid [17, Section 6]. However, all other Bianchi groups are not Galois rigid, since Thurston's Dehn surgery theorem produces high (i.e. ≥ 1) dimensional components in the $\mathrm{PSL}(2,\mathbb{C})$-character variety.

(3) If M_W denotes the Weeks manifold (i.e. the smallest volume closed orientable hyperbolic 3-manifold [30]), then $\pi_1(M_W)$ is Galois rigid [17, Proposition 5.3].

(4) Let M_1 (resp. M_2) denote the manifold obtained by 0-surgery on the knot 6_2 (resp. 6_3). Then $\pi_1(M_i)$ is Galois rigid for $i = 1,2$ [23]. Note that these manifolds

are fibered over the circle, hence $\pi_1(M_i)$ admits a homomorphism onto $\mathbb{Z}$, and so has a positive dimensional character variety. However, the Zariski dense representations are simply the ones given as described as above using Galois conjugate embeddings.

(5) Let $K \subset S^3$ denote the figure-eight knot, let Q_n be the orbifold obtained by $(n, 0)$-Dehn filling on K and let M_n be the n-fold cyclic branched cover of K. Note that M_n can also be regarded as an n-fold cyclic (orbifold) cover of Q_n; this is the maximal abelian cover. When $n = 2$, M_2 is the Lens Space $L(5, 3)$, and so Q_2 is an orbifold with finite fundamental group; when $n = 3$, M_3 is the flat manifold known as the Hantzsche-Wendt manifold and so Q_3 is a Euclidean crystallographic orbifold; when $n \geq 4$, M_n and Q_n are hyperbolic (see [27] for example). In addition, it was shown in [36] that $\Gamma_n = \pi_1(M_n) \cong F(2, 2n)$ where $F(2, 2n)$ is one of the Fibonacci groups of [26]. Let Δ_n be the orbifold fundamental group of Q_n. Then $Q_n = \mathbb{H}^3/\Delta_n$ and $\Gamma_n = [\Delta_n, \Delta_n]$ is a subgroup of index n.

The following theorem is proved in [19].

Theorem 1 *In the notation established above,*

1. Δ_p *is Galois rigid for all primes* $p \geq 5$*;*
2. Δ_4, Δ_6 *and* Δ_9 *are Galois rigid;*
3. Γ_4 *is Galois rigid.*

To pass from Galois rigidity to profinite rigidity requires a good deal more control of the groups in question. In particular, we need the Fuchsian and Kleinian groups in question to be *arithmetic* (which we will not define here, but refer the reader to [45] for a detailed discussion), but in particular requires integral traces. Furthermore, additional control on the defining number field and quaternion algebra is also required. For complete details we refer the reader to [17, Theorem 4.8 & Corollary 4.11].

The endgame for Fuchsian and Kleinian groups: We now discuss cases where Step (2) of the template can be executed, and thereby, in some cases establish profinite rigidity. The following result can be proved using [18] and [19]. We provide a sketch of the proof.

Theorem 2 *1. Let* Γ *be a Kleinian group of finite co-volume. If* $H < \Gamma$ *is a finitely generated, proper subgroup, then* $\widehat{\Gamma}$ *and* $\widehat{H}$ *are not isomorphic.*
2. Let $\Delta(p, q, r) = \langle a, b | a^p = b^q = (ab)^r = 1\rangle$ *be a Fuchsian triangle group, and* H *a non-elementary Fuchsian group that is not a triangle group. Then there exists a finite field* $\mathbb{F}$ *and a representation* $H \to \mathrm{PSL}(2, \mathbb{F})$ *whose image is not a quotient group of* $\Delta(p, q, r)$*.*

Some ideas in the proof: For (1), if H is infinite index then $b_1^{(2)}(\Gamma) = 0$ whilst $b_1^{(2)}(H) \neq 0$, which contradicts the profinite invariance of $b_1^{(2)}$ (as noted in §3). If H has finite index, then we use a result of Liu [43] which establishes that only finitely many finite volume hyperbolic 3-manifolds can have fundamental groups with isomorphic profinite completions; the key point is that the Correspondence

Theorem between finite index subgroups of the discrete group and the profinite completion [53] can then be used to build an infinite sequence of subgroups $H_n < \Gamma$ with $[\Gamma : H_n] = d^n$ such that $\widehat{H_n} \cong \widehat{\Gamma}$, which if Γ is torsion-free contradicts Liu's result. The torsion case is easy to deduce from this.

For (2), the key point is that if H is not a triangle group then the $\mathrm{PSL}(2, \mathbb{C})$-character variety is positive dimensional, and this can be leveraged into building representations in characteristic p.

Remark 3 It is also worth pointing out that (in the notation on Theorem 2(2)), that if $H \neq \Delta(p, q, r)$ is a triangle group then it is known that H can be explicitly distinguished from $\Delta(p, q, r)$ by finite quotients (see [16], [22] and [25]).

We now summarize some of the known examples of Fuchsian and Kleinian groups that can be proved to be profinitely rigid using the template, and the ideas described above.

Theorem 3 *1.* $\mathrm{PSL}(2, \mathbb{Z}[\omega])$ *and* $\pi_1(M_W)$ *are profinitely rigid [17].*
2. In the notation of Example (5) above, the manifold M_4 *and the* 2*-fold quotient known as* Vol 3 *have fundamental groups that are profinitely rigid [19].*
3. The fundamental groups of the manifolds obtained by 0*-surgery on* 6_2 *and* 6_3 *are profinitely rigid [23].*
4. The following arithmetic triangle groups are profinitely rigid (see [18] and the erratum):

$$\Delta(3,3,4), \Delta(3,3,5), \Delta(3,3,6), \Delta(2,5,5), \Delta(4,4,4)$$
$$\Delta(2,3,8), \Delta(2,3,10), \Delta(2,3,12), \Delta(2,4,5), \Delta(2,4,8).$$

We finish this subsection with some challenges that would be interesting to answer.

1. *Show that* $\Delta(2, 3, 7)$ *is profinitely rigid.*
   Note that $\Delta(2, 3, 7)$ is Galois rigid, and running the ideas in the template one finds that if Λ is a finitely generated residually finite group with $\widehat{\Lambda} \cong \hat{\Delta}(2, 3, 7)$, then; either $\Lambda \cong \Delta(2, 3, 7)$ or there is a representation $\rho : \Lambda \to \mathrm{PSL}(2, \mathrm{R})$ as in (1) of the template, where R is the ring of integers in $\mathbb{Q}(\cos(\pi/7))$.
2. *Construct infinitely many profinitely rigid Kleinian groups of finite co-volume.*

4.3 Other 3-manifold groups

We refer to [51] for earlier work on profinite rigidity, as well as its failure, for the fundamental groups of compact non-hyperbolic 3-manifolds (as well as in the relative setting of the fundamental groups of compact 3-manifolds). Of most relevance to what follows in this article is the following result proved in [20], and for which we mention some of the key ideas in the proof.

Theorem 4 *Let $\Delta(p,q,r)$ be a triangle group listed in Theorem 3, and let M be a Seifert fibred space with base $S^2(p,q,r) = \mathbb{H}^2/\Delta(p,q,r)$. Then $\pi_1(M)$ is profinitely rigid.*

Sketch of the ideas in the proof: Underlying the proof are some of the ideas described above, namely the use of *Galois rigidity* in the context of $\Gamma = \pi_1(M)$ with M as in the statement of Theorem 10; namely if Λ is a finitely generated residually finite group with $\widehat{\Lambda} \cong \widehat{\pi_1(M)}$, then Galois rigidity can be applied to deduce that Λ admits an epimorphism onto Δ with central kernel isomorphic to $\mathbb{Z}$. Since $\pi_1(M)$ is torsion free, and good in the sense of Serre, this implies that Λ is also torsion-free.

The next two results then complete the proof, the first of which reduces the problem to the relative setting of deciding profinite rigidity amongst the fundamental groups of closed orientable Seifert fibred spaces. The second is a result of Wilkes [60] which classifies exactly when the fundamental groups of a Seifert fibred space is (or is not) profinitely rigid within the class of fundamental groups of compact 3-manifolds, and in our setting an easy calculation shows that the euler number $e(M) \neq 0$.

Lemma 1 *[33]*

Suppose that G fits into a short exact sequence

$$1 \to \mathbb{Z} \to G \to F \to 1,$$

where F is a cocompact Fuchsian group. Then G is the fundamental group of a Seifert fibred space if and only if G is torsion-free.

Theorem 5 *Let M be a closed aspherical Seifert fibred space, and let N be a compact 3-manifold with $\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)}$. Then, either $N \cong M$ or else N is a Seifert fibred space and both M and N are among the surface bundles over the circle with periodic monodromy that arise in the construction of [35]. In particular, when $\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)}$ and M and N are not homeomorphic, both M and N have $\mathbb{H}^2 \times \mathbb{R}$ geometry and $e(M) = e(N) = 0$.*

Remark 4 It is worth remarking that the relationship between the centre of a group Γ and the centre of $\widehat{\Gamma}$ can be somewhat complicated in general. For example, if $\Gamma = \mathrm{SL}(3,\mathbb{Z})$, then $Z(\Gamma) = \{1\}$ is trivial. On the other hand, there are infinitely many p for which $\mathbb{Z}_p$ (the p-adic integers) contains a primitive cube root of unity, in which case the centre of $\mathrm{SL}(3,\mathbb{Z}_\mathrm{p})$ is a cyclic group C_3 of order 3. Therefore $Z(\widehat{\Gamma}) = Z(\prod_p \mathrm{SL}(3,\mathbb{Z}_\mathrm{p}))$ contains a countable direct product of copies of C_3. However, in our setting things are much better, and we can gain control of centres.

4.4 Other lattices

As we have described above, the template can be implemented and pushed to a successful conclusion in the setting of certain Fuchsian and Kleinian groups. As far as the author is aware this is the extent of our knowledge of successfully executing the template to completion. There are many situations in which one can make progress on (1) of the template; e.g. using superrigidity, and the congruence subgroup property (and we list a few of these below). However, the major obstacle to progressing is simply: we do not understand the subgroup structure of lattices in other semi-simple Lie groups well enough! This is already visible in the case of $\Delta(2,3,7)$, where the “or" conclusion in the discussion of $\Delta(2,3,7)$ at the end of §4.2 is the issue.

As an example of the progress mentioned above, in [55] it is shown that if Λ is a finitely generated residually finite group with the same profinite completion as Γ being one of $\mathrm{SL}(\mathrm{n},\mathbb{Z})$ with $n \geq 3$ or $\mathrm{SL}(\mathrm{n},\mathbb{Z}[1/\mathrm{p}])$ with $n = 2, 3$ and p a prime, then there exists *an injective homomorphism* $\Lambda \hookrightarrow \Gamma$ which induces the isomorphism of profinite completions.

Remark 5 Interestingly, $\mathrm{SL}(4,\mathbb{Z}[1/\mathrm{p}])$ is *not* profinitely rigid [24]. Indeed, [24] produces a group Λ which is not isomorphic to a subgroup of $\mathrm{SL}(4,\mathbb{Z}[1/\mathrm{p}])$ with the same profinite completion. This example will be discussed further in §5.2 Application 3.

5 Grothendieck pairs

In this section we describe a construction that provides many interesting example of groups with the same profinite completion, and also a mechanism for certifying certain properties are not profinite properties.

5.1 Grothendieck's question

We begin by posing the following question: *Let* Γ *be a finitely presented residually finite group and let* $u : P \hookrightarrow \Gamma$ *be the inclusion of a subgroup* P*. Suppose that* u *induces an isomorphism* $\hat{u} : \widehat{P} \hookrightarrow \widehat{\Gamma}$*. Is* u *an isomorphism?*

When u fails to be an isomorphism we call $(\Gamma, P)_u$ a *Grothendieck pair* (we often suppress the subscript). We say that Γ is *Grothendieck rigid* if no proper finitely generated subgroup $u : P \to \Gamma$ gives a Grothendieck pair.

The motivation for this terminology is Grothendieck's paper [32], in which he asked about the existence of Grothendieck pairs when P is also finitely presented group. The first such pairs were constructed by Bridson and Grunewald in [13]. The analogous problem for finitely generated groups had been settled earlier by Platonov and Tavgen [48] (and other examples were given by Bass and Lubotzky in [4]). All of the constructions rely on versions of the following result (cf. [48], [13, Theorem 5.2] and [9]).

We first remind the reader of the definition of the *fibre product* (we give a general version): For $i = 1, \ldots, d$, let $p_i : \Gamma_i \to Q_i$ be a homomorphism of groups. The *fibre* *product* of this family of maps is

$$P_d = \{(\gamma_1, \ldots, \gamma_d) \mid p_i(\gamma_i) = p_j(\gamma_j), i, j = 1, \ldots, d\}.$$

When all of the groups Γ_i are isomorphic as well as the groups Q_i, $d = 2$ and $p_1 = p_2$ we get:

Proposition 2 *Let $1 \to N \to \Gamma \to Q \to 1$ be a short exact sequence of groups with Γ finitely generated and let P be the associated fibre product. Suppose that $Q \neq 1$ is finitely presented with $\widehat{Q} = 1$, and $H_2(Q, \mathbb{Z}) = 0$. Then*

1. *$(\Gamma \times \Gamma, P)$ is a Grothendieck pair;*
2. *if N is finitely generated then (Γ, N) is a Grothendieck pair.*

Examples of Grothendieck pairs were constructed so as to provide the first examples of finitely-presented, residually finite groups Γ that contain an infinite sequence of non-isomorphic finitely presented subgroups P_n so that the inclusion maps $u_n : P_n \hookrightarrow \Gamma$ induce isomorphisms of profinite completions. In particular, this provides examples of finitely presented groups Γ for which $\mathcal{G}(\Gamma)$ contains infinitely many finitely presented groups [10].

The Platonov-Tavgen example [48]: In the notation of Proposition 2, the group Γ is F_4 and the group Q is Higman's group which has no finite quotients by construction, and is presented as:

$$\langle a, b, c, d | bab^{-1} = a^2, cbc^{-1} = b^2, dcd^{-1} = c^2, ada^{-1} = d^2 > .$$

That $H_2(Q, \mathbb{Z}) = 0$ follows since Q has a balanced presentation.

5.2 Some applications

Application 1: *Rank is not a profinite property:* Recall that by the *rank* of a finitely generated group Γ we mean the cardinality of a minimal generating set for Γ. We will make use of Proposition 2(2) in tandem with the Rips construction [54]. Recall that [54] proves the following:

Theorem 6 *Given any finitely presented group Q and any real number $\lambda > 0$, there is a $C'(\lambda)$ group G and a short exact sequence*

$$1 \to N \to G \xrightarrow{p} Q \to 1$$

with N finitely generated. Indeed N can be taken to be 2-generator.

With this in hand, we can proceed as follows (which was explained to me by Martin Bridson):

Take Q to be Higman's group as in the Platonov-Tavgen example above, and let $Q(d)$ denote the free product of d copies of Q, which by Grushko's theorem has rank $4d$. In addition $\widehat{Q(d)} = 1$, and $H_2(Q(d), \mathbb{Z}) = 0$. For each d we now apply Proposition 2(2) to produce Grothendieck pairs $(\Gamma(d), N(d))$ using $p : \Gamma(d) \to Q(d)$ as in Theorem 6. Since $Q(d)$ has rank $4d$, it follows that $\Gamma(d)$ has rank $\geq 4d$, yet, as remarked upon in Theorem 6, $N(d)$ is always 2-generator. This proves that not only is rank not a profinite property, but the difference can be arbitrarily large.

Application 2: *Right Angled Coxeter groups:* In [28] the following result is proved:

Theorem 7 *Every Right Angled Coxeter group is profinitely rigid amongst all Coxeter groups. On the other hand there are Right Angled Coxeter groups which have infinite genus amongst all finitely generated residually finite groups.*

We will not discuss the proof of the first statement, but will discuss the second as the groups in question arise as Grothendieck pairs. To that end, let Γ denote the "universal" Right Angled Coxeter group on four generators; i.e. the free product of four cyclic groups of order 2, where we view each order two generator as a "reflection". We will apply Proposition 2(1) using an epimorphism $\Gamma \to V_n$ where $n \geq 2$ is a Higman-Thompson group. That such an epimorphism exists is the crucial point, and this is proved in [28, Theorem 4.1]. We refer the reader to [28] for explanations of why V_n has the properties required by Proposition 2(1), and in particular why $H_2(V_n, \mathbb{Z}) = 0$. This produces Grothendieck pairs $(\Gamma \times \Gamma, P_n)$, and we note that $\Gamma \times \Gamma$ is also a Right Angled Coxeter group.

Application 3 *Fixed point properties:* As was noted in Remark 4, the group $\mathrm{SL}(4, \mathbb{Z}[1/\mathrm{p}])$ is *not* profinitely rigid [24], although the "other" group, which we denote by Γ in what follows, is not part of a Grothendieck pair. In fact, the main point of [24] is to prove that Property FA is not a profinite property (recall that [3] showed that Property T is not a profinite property, but the groups in question both have Property FA). Indeed, [24] proves the existence of finitely presented groups Γ_1 (which is the group $\mathrm{SL}(4, \mathbb{Z}[1/\mathrm{p}])$) and Γ_2 for which $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$ with Γ_1 having Property FA where as Γ_2 does not. In [11], a different proof of this is given (although in the setting of [11], one of the groups is finitely presented, and the other is only finitely generated), and moreover provides a generalization replacing actions on trees (a 1-dimensional CAT(0) space) to actions on d-dimensional CAT(0) spaces, with $d \geq 1$ arbitrary. To state the result of [11] we recall terminology.

Say that a group G has property $\mathrm{Fix_d}$ if G fixes a point whenever it acts by semisimple isometries on a complete CAT(0) space of dimension at most d.

Theorem 8 *[11]*

For every integer $d \geq 1$, there exist triples of residually finite groups

$$M \hookrightarrow P \hookrightarrow \Gamma \textit{ that}:$$

1. The inclusion maps induce isomorphisms $\widehat{M} \cong \widehat{P} \cong \widehat{\Gamma}$;

2. *M is finitely generated, P is finitely presented, and Γ is of type F_∞;*
3. *M and Γ have property* $\mathrm{Fix_d}$*, but*
4. *P is a nontrivial free product with amalgamatation, and therefore acts on a simplicial tree without a global fixed point.*

The proof again crucially relies on the template for constructing Grothendieck pairs described above.

Application 4: *Free-by-free contains failure of many profinite properties:* The final application is a construction due to Bridson [12], and illustrates how Grothendieck pairs can be used to show that the profinite completions of finitely generated, residually finite free-by-free groups contain, as retracts, the profinite completions of all subgroups of finitely presented groups. In particular this allows for some additional " exotic" behavior in groups with the same profinite completion (we point out some explicitly below). The following is proved in [12].

Theorem 9 *Given an arbitrary, finitely generated, recursively presented group Γ that is residually finite, one can construct a finitely generated, residually finite free-by-free group $M_\Gamma = F_\infty \rtimes F_4$ and an embedding $M_\Gamma \to (F_4 * \Gamma) \times F_4$ that induces an isomorphism of profinite completions.*

Note that $D(\Gamma) := (F_4 * \Gamma) \times F_4$ is residually finite and the obvious retraction $D(\Gamma) \to \Gamma$ induces a retraction $\widehat{M_\Gamma} \cong \widehat{D(\Gamma)} \to \widehat{\Gamma}$. Some consequences of Theorem 9 are:

1. *Examples of groups of cohomological dimension* 2 *whose profinite completions have cohomological dimension $d+1$, where d is arbitrary.*
2. *Examples to show that the profinite completions of free-by-free groups can contain torsion of prescribed order (torsion-free groups with arbitrary torsion in the profinite completion were previously exhibited by Lubotzky [44]).*
3. *Examples that show being locally indicable or being left-orderable or having all 2-generator subgroups being finitely presented are not profinite properties.*

6 Finitely presented versus finitely generated

The main result that we will discuss in this section is another occurrence of Grothendieck pairs, but in this case, the appearance of Grothendieck pairs is connected to a *complete* description of groups which occur in the genus of certain finitely presented groups, and is the following (cf. the discussion above in connection with construction of Platonov and Tavgen).

Theorem 10 *There exist finitely presented, residually finite groups Γ with the following properties:*

1. $\Gamma \times \Gamma$ *is profinitely rigid among all finitely presented, residually finite groups.*

2. *There exist infinitely many non-isomorphic finitely generated groups Λ such that $\widehat{\Lambda} \cong \widehat{\Gamma} \times \widehat{\Gamma}$.*
3. *If Λ is as in (2), then there is an embedding $\Lambda \hookrightarrow \Gamma \times \Gamma$ that induces the isomorphism $\widehat{\Lambda} \cong \widehat{\Gamma \times \Gamma}$; in other words, $(\Gamma \times \Gamma, \Lambda)$ is a Grothendieck pair.*

The groups Γ in question can be taken to be the fundamental group of any Seifert fibered space M with base orbifold $S^2(3,3,4)$ or $S^2(3,3,6)$ or $S^2(2,5,5)$ described in Theorem 4.

Sketch of ideas in the proof: We highlight the key steps in the proof. We begin with a comment: Although Galois rigidity does not hold in the strict sense when applied to representations of $\Gamma \times \Gamma$ to $\mathrm{PSL}(2,\mathbb{C})$, enough control is gained to execute an endgame similar to that described in Step (2) of the template.
Throughout this discussion, Λ is a finitely generated, residually finite group with $\widehat{\Lambda} \cong \widehat{\Gamma \times \Gamma}$.

Step 1: Show that there is an embedding $\Lambda \hookrightarrow \Gamma \times \Gamma$ inducing the isomorphism at the level of profinite completion.

A crucial part of the proof, and where Galois rigidity is used, is in proving the following.

Proposition 3 *Let Γ be the fundamental group of a Seifert fibred space whose base orbifold $S^2(p,q,r)$ is one of those in Theorem 10 and let $\Delta = \Delta(p,q,r)$ be the orbifold fundamental group of this base. Let Λ be a finitely generated group with $\widehat{\Lambda} = \widehat{\Gamma}_1 \times \widehat{\Gamma}_2$, where $\Gamma_1 \cong \Gamma_2 \cong \Gamma$. If Λ_i is the projection of Λ to $\widehat{\Gamma}_i$, then there exist epimorphisms $g_i : \Lambda_i \to \Delta$ and hence a homomorphism*

$$g : \Lambda \hookrightarrow \Lambda_1 \times \Lambda_2 \stackrel{(g_1,g_2)}{\to} \Delta \times \Delta$$

with image a full subdirect product.

With this in hand, and with some additional argument, we can show that the epimorphsims $g_i : \Lambda_i \to \Delta$ fit into a short exact sequence: $1 \to Z \to \Lambda_i \to \Delta \to 1$ with Z central and infinite cyclic. As in the sketch of the proof of Theorem 4, Λ_i can then be shown to be the fundamental group of a Seifert fibred space, and indeed $\Lambda_1 \cong \Lambda_2 \cong \Gamma$.

Step 2: We prove that there are no *finitely presented* Λ as in the statement of Theorem 10. To do this we prove the following result (that builds on arguments of [14]). We provide a sketch of some of the ideas in the proof.

Theorem 11 *Let M be a Seifert fibred space with hyperbolic base orbifold, let $\Gamma = \pi_1 M$, let D be the direct product of finitely many copies of Γ, and let $\Lambda < D$ be a subgroup such that the inclusion induces an isomorphism of profinite completions. If Λ is finitely presented, then $\Lambda = D$.*

Some ideas in the proof: We restrict to the case that the base orbifold is closed.

We first project Λ to the product of the base groups $B = \Delta \times \cdots \times \Delta$, and note that the image group is still finitely presented since the kernel is finitely generated (being in the centre). Using a variation of an argument of [14], we can deduce that this mapping is onto (since the image of Λ will be dense in $\widehat{B}$.

We now have an epimorphism $\widehat{\Lambda} \to \widehat{B}$. The key point is to show that this implies that Λ contains the centre $\mathcal{Z} = \mathbb{Z}^r < D$ (where r is the number of factors). To see this, first note that if $\Lambda \cap \mathcal{Z}$ has finite index in $\mathcal{Z}$, then $\Lambda \cap \mathcal{Z} = \mathcal{Z}$, because otherwise $[D : \Lambda] < \infty$, and this contradicts $\widehat{\Lambda} = \widehat{D}$. With this observation, the only concern is that Λ meets one of the direct factors of $\mathcal{Z}$ trivially. However, this cannot happen.

The reason is this: since M is a Seifert fibred space, Γ is good in the sense of Serre, hence D is also good by [34, Proposition 3.4], and so in particular the fact that D has cohomological dimension $3r$ is witnessed by the continuous cohomology of $\widehat{D}$ with finite field coefficients. If Λ meets one of the factors of $\mathcal{Z}$ trivially, then Λ will be an extension of B by $\mathbb{Z}^s$, with $s < r$, and in particular is again good in the sense of Serre, by another application of [34, Proposition 3.4]). But, in this case Λ has cohomological dimension less than $3r$, and therefore so does $\widehat{\Lambda}$, a contradiction.

Step 3: Finally we construct Grothendieck pairs $(\Gamma \times \Gamma, \Lambda)$ with Λ finitely generated. This is achieved using the following result (but we will not discuss in detail how it is implemented).

Theorem 12 *Let Δ be a non-elementary hyperbolic group and let Γ be a group with $H_2(\Gamma, \mathbb{Z}) = 0$ that maps onto Δ. Let G be a finitely generated group that maps onto a subgroup of finite index in $[\Gamma, \Gamma]$. Then,*

1. *there exists an infinite sequence of distinct finitely generated subgroups $P_n < G \times G$ such that each inclusion $u_n : P_n \hookrightarrow G \times G$ induces an isomorphism of profinite completions.*
2. *If G is a central extension of a hyperbolic group and centralizers of elements in that hyperbolic group are virtually cyclic, then P_n is not abstractly isomorphic to P_m when $n \neq m$.*

The subgroups P_i will not be finitely presented in general, even if G is finitely presented (cf. [10]).

In our setting we would like to apply Theorem 12 to the fundamental groups of the Seifert fibre spaces over the base orbifolds $S^2(p, q, r)$ listed in Theorem 10. More specifically, in the notation of Theorem 12, we would like to take $\Delta = \Delta(p, q, r)$ and $G = \Gamma = \pi_1 M$. But we cannot do this because $H_2(M, \mathbb{Z})$, although finite, is not trivial. Instead, we construct an auxiliary group B with finite abelianisation and with $H_2(B, \mathbb{Z}) = 0$ so that B maps onto a non-elementary hyperbolic group and $\pi_1 M$ maps onto a subgroup of finite index in $[B, B]$. The key point here is the following result (which gives an indication as to why there is a restriction on the Seifert fibred spaces considered in Theorem 10).

Proposition 4 *Let Π be the fundamental group of a Seifert fibred space M with base orbifold $S^2(p, q, r)$. If $|H_1(M, \mathbb{Z})|$ is finite and odd, then there is a group $\Lambda > \Pi$ with the following properties:*

1. $\Pi = [\Lambda, \Lambda]$*;*
2. $H_1(\Lambda, \mathbb{Z})$ *is cyclic of order* 2*;*
3. $H_2(\Lambda, \mathbb{Z}) = 0$*;*
4. Λ *maps onto* $\Delta^-(p, q, r) = \Delta(p, q, r) \rtimes C_2$.

Remark 6 In general, for a fixed group Γ, there can be uncountably many finitely generated, residually finite groups H with $\widehat{H} \cong \widehat{\Gamma}$; see [46] and [49] for example. In other settings, for example nilpotent groups [47], there are only finitely many such H, up to isomorphism. The groups $\Gamma \times \Gamma$ in Theorem 10 provide the first examples of groups where the number of such H is countably infinite; this property is assured by parts (2) and (3) of Theorem 10. We have the following.

Corollary 2 *There exist finitely presented, residually finite groups whose profinite genus is countably infinite.*

7 Final remarks and questions

In this final section we gather some remarks and pose some questions that arise out of this survey article. We will not replicate questions that are stated in [50] and [51], nor will we update the status of the questions in these earlier papers.

? Question

Suppose Γ is word hyperbolic (and hence finitely presented) and Λ a finitely presented, residually finite group with $\widehat{\Lambda} \cong \widehat{\Gamma}$. Is Λ word hyperbolic?

As mentioned earlier, [10] provides an example of a finitely presented group which contains infinitely many finitely presented subgroups in its genus. However, these are not word hyperbolic. Constructions of word hyperbolic groups with the same profinte completion are given in [56] (indeed these can be taken to be cocompact lattices in $\mathrm{PU}(\mathrm{n}, 1)$ for all $n \geq 2$). Using the Rips construction, one can construct a hyperbolic group Γ and a finitely generated subgroup $N < \Gamma$ such that (Γ, N) is a Grothendieck pair. However, N will never be hyperbolic.

? Question

Does there exist a Right Angled Coxeter group $\Gamma < \mathrm{Isom}(\mathbb{H}^{\mathrm{n}})$ admitting an epimorphism $\Gamma \to Q$ where Q satisfies the conditions of Proposition 2 and with finitely generated kernel; e.g. Q can be taken to Higman's group or a Higman-Thompson group V_n (both mentioned above)?

If such exists then, by Proposition 2(2), this would give rise to a Grothendieck pair, and would be the first example of a Grothendieck pair where the ambient group is a lattice in $\mathrm{Isom}(\mathbb{H}^n)$.

? Question

Let T denote the ideal simplex in $\mathbb{H}^4$ defined by the Coxeter notation $[4, 3^{2,1}]$, and let $\Gamma < \mathrm{Isom}^+(\mathbb{H}^4)$ denote the index 2 subgroup consisting of orientation preserving isometries of the group generated by reflections in the faces of T.

Is Γ Galois rigid in $\mathrm{SO}(4, 1)$?

Is Γ profinitely rigid?

Note that by [59], the group Γ can be identified with $\mathrm{SO}^+(4, 1; \mathbb{Z})$, and in some ways is similar to the profinitely rigid group $\mathrm{PSL}(2, \mathbb{Z}[\omega])$. Given that we know nothing about profinite rigidity (or lack thereof) of groups of isometries of hyperbolic spaces of dimensions ≥ 4, this would seem like a good test case.

We suspect the answer to both parts of this question is *Yes*, and it seems likely that the first part can be carried out. As for the second part, we will likely find ourselves in a situation similar to that described for $\mathrm{SL}(n, \mathbb{Z})$ described in §4.4 of not having a good enough grasp of finitely generated subgroups to execute the endgame of the template described in §4.2.

? Question

Can one prove a higher dimensional version of Liu's result [43]? i.e. if M is a finite volume hyperbolic n-manifold with $n \geq 4$, then there are only finitely many hyperbolic n-manifolds of finite volume whose fundamental groups have profinite completion isomorphic to $\widehat{\pi_1(M)}$.

If Γ is a free group or a surface group then Γ is well-known to be LERF, and so there are no Grothendieck pairs (Γ, H) (see [48]). In dimension 3, Agol [2] proved that the fundamental groups of finite volume hyperbolic 3-manifolds are LERF. Actually [2] proves GFERF (i.e. separable on geometrically finite subgroups), the LERF conclusion follows from Canary's covering theorem [21]. In particular, there are no Grothendieck pairs (Γ, H) with Γ the fundamental group of a finite volume hyperbolic 3-manifold.

Now in [32] (see also [39]) it is shown that if Γ is a free group or a surface group, then, in addition, there are no Grothendieck pairs (Λ, Γ). Very roughly, this is achieved by exploiting a hyperbolic structure on a punctured or closed surface that is uniformized by an arithmetic Fuchsian group. A similar strategy works to show that if Γ is an arithmetic lattice in a semisimple Lie group then there are no Grothendieck pairs (Λ, Γ).

However, in dimension 3, we do not know whether there are possible Grothendieck pairs (Λ, Γ) with Γ the fundamental group of a finite volume hyperbolic 3-manifold.

In higher dimensions, [7] proved that arithmetic lattices in SO(n, 1) are GFERF, and so there are no Grothendieck pairs (Γ, H) with H geometrically finite. However, arithmetic lattices in SO(n, 1) are not LERF (see [57], [58] and [8]), and indeed the subgroup structure is much more complicated. Hence there could yet be a Grothendieck pair (Γ, H) with H geometrically infinite.

Given the discussion above we raise the following questions:

? Question

Do there exist Grothendieck pairs (Γ, H) with $\Gamma < \mathrm{SO}(n, 1)$ a lattice and $n \geq 4$? Suppose that $n \geq 3$, and Γ is a non-arithmetic lattice. Can Γ be part of a Grothendieck pair (Λ, Γ)?

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