Constructing thin groups

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1 Introduction

Let G be a semi-simple Lie group, and $\Gamma < G$ a lattice. Following Sarnak (see [25]), a subgroup Δ of Γ is called *thin* if Δ has infinite index in Γ, but is Zariski dense. Since it is straightforward to exhibit free subgroups of lattices that are Zariski dense, we shall, in addition insist that a thin group Δ is finitely generated and does not decompose as a free product.

There has been a good deal of interest recently in thin groups. This has been motivated in part by work on expanders, and in particular the so-called "affine sieve" of Bourgain, Gamburd and Sarnak [5]. Other recent work that study thin groups can be found in [13], [14], and [15]. We also refer the reader to other papers in this volume.

The aim of this article is to summarize work in [20], [21] and [22], as well as discussing some other results regarding (sometimes conjectural) constructions of thin subgroups in lattices.

We begin by outlining our strategy for generating some thin subgroups. We do this first in broad terms, and then specialise to some cases that we describe in more detail. In the remainder of this section, Γ will denote a Fuchsian group (i.e. a discrete subgroup of $\mathrm{PSL}(2,\mathbf{R})$) or a Kleinian group (i.e. a discrete subgroup of $\mathrm{PSL}(2,\mathbf{C})$) of finite co-volume, and G a semi-simple Lie group not locally isomorphic to $\mathrm{PSL}(2,\mathbf{R})$ or $\mathrm{PSL}(2,\mathbf{C})$.

Let

$$\mathrm{Hom}_0(\Gamma,\mathrm{G}) = \{\rho : \Gamma \to \mathrm{G} : \rho \text{ is an irreducible representation}\}$$

and $X(\Gamma,G)$ the quotient of $\mathrm{Hom}_0(\Gamma,\mathrm{G})$ under G-conjugation. Then, depending on G, $X(\Gamma,G)$ is a real or complex algebraic set.

Basic idea: *If $X(\Gamma,G)$ contains a component of positive dimension then look for specialisations that result in* **integral** *representations.*

In broad terms, integrality is necessary for the image to live in a lattice; however this does not suffice and subsequent work is usually necessary to arrange the image to lie in a lattice of a prescribed Lie group G. Zariski density is typically fairly simple to arrange. The main issue is exhibiting a certificate to demonstrate that the image has infinite index; once this is done, the nature of the constructions here will ensure the free indecomposability of the image.

In general the question of whether the image has infinite index in the lattice touches on decidablity issues. For example, the following question is raised in [7]: *Is the finite presentation problem unsolvable in* $\mathrm{SL}(\mathrm{n},\mathbf{Z})$ *for some n?* (where one says that the finite presentation problem is solvable if there is an algorithm that, given a finite set of matrices of $\mathrm{SL}(\mathrm{n},\mathbf{Z})$ generating a finitely presentable

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subgroup Γ of $\mathrm{SL}(n, \mathbf{Z})$ outputs a finite presentation for Γ.) However the geometrical constructions offered below give extra information that one can exploit.

Although this article is for the most part survey, one new result that exploits the strategy outlined above is the following (we refer to §5 for an explanation of the notation).

Theorem 1.1. *Let d be a square-free positive integer and let $L = \mathbf{Q}(\sqrt{d})$ be a real quadratic number field.*

Then the lattice $\mathrm{SU}(\mathrm{J}, \mathcal{O}_{\mathrm{L}}, \tau) < \mathrm{SL}(3, \mathbf{R})$ contains a thin subgroup isomorphic to the fundamental group of some closed orientable surface of genus ≥ 2.

This result, taken together with [20] and the classification of non-uniform lattices in $\mathrm{SL}(3, \mathbf{R})$ (see §5 for more on this) shows that every non-uniform lattice inside $\mathrm{SL}(3, \mathbf{R})$ contains a thin hyperbolic surface group.

All the constructions that we outline here produce thin subgroups in non-uniform lattices, i.e. those lattices Γ for which G/Γ has finite volume, but is not compact. Finding thin subgroups in uniform lattices is apparently a good deal harder. One can certainly attempt to follow the strategy given by the Basic Idea, but the additional conditions for arithmeticity of uniform lattices thus far seem prohibitive.

Indeed, it is worth remarking that only recently has the existence of thin surface subgroups in all lattices in $\mathrm{PSL}(2, \mathbf{C})$ been completed (by the recent work of Kahn and Markovic [17] in the uniform case and previously in [10] in the non-uniform case).

2 $G = \mathrm{SL}(3, \mathbf{R})$ and Γ a cocompact Fuchsian group

We first consider the case where $G = \mathrm{SL}(3, \mathbf{R})$ with Γ a cocompact Fuchsian group so that $\Sigma = \mathbf{H}^2/\Gamma$ is a closed orientable hyperbolic 2-orbifold. Let $|\Sigma|$ denote the underlying space, k_c the number of cone points on Σ and b_c the number of cone points with cone angle π.

For each such Γ, there is a natural embedding of $\Gamma \hookrightarrow \mathrm{SL}(3, \mathbf{R})$ given by:

$$\Gamma < \mathrm{PSL}(2, \mathbf{R}) \cong \mathrm{SO}_0(2, 1) \hookrightarrow \mathrm{SL}(3, \mathbf{R}).$$

Thus the Teichmüller component of $X(\Gamma, \mathrm{SO}_0(2, 1))$ sits naturally in $X(\Gamma, \mathrm{SL}(3, \mathbf{R}))$, and determines the so-called Hitchin component $X^{Hit}(\Gamma, \mathrm{SL}(3, \mathbf{R}))$. It was proven in [16] that *all* the characters on this component correspond to discrete and faithful representations of Γ. In fact, Theorem A of [8] shows that $X^{Hit}(\Gamma, \mathrm{SL}(3, \mathbf{R}))$ is homeomorphic to a cell of dimension $-8\chi(|\Sigma|) + (6k_c - 2b_c)$, and so unless Γ is a triangle group of type (p, q, r) with one of p, q and r equal 2, then $X(\Gamma, \mathrm{SL}(3, \mathbf{R}))$ will contain a component as in the Basic Idea.

Moreover we also have (cf. the proof of Theorem 2.1 of [20]):

Theorem 2.1. *Suppose $\chi_\rho \in X^{Hit}(\Gamma, \mathrm{SL}(3, \mathbf{R}))$ is not the character of the hyperbolic structure.*

Then the image $\rho(\Gamma)$ is Zariski dense in $\mathrm{SL}(3, \mathbf{R})$.

One needs to address the issue of specialisations that have images that lie in a lattice in $\mathrm{SL}(3, \mathbf{R})$, but if this can be arranged then the image is necessarily thin, since a faithful representation of a Fuchsian group must have infinite index in a rank two lattice.

2.1

The most obvious lattice in $\mathrm{SL}(3, \mathbf{R})$ is $\mathrm{SL}(3, \mathbf{Z})$ and in [20] we analyzed the above discussion in much more detail for the case of the triangle group of type $(3, 3, 4)$. We proved the following result:

Theorem 2.2. *The family of representations of the triangle group*

$$\Delta(3,3,4) = < a, b \mid a^3 = b^3 = (a.b)^4 = 1 >$$

given by

$$a \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$b \mapsto \begin{pmatrix} 1 & 2-t+t^2 & 3+t^2 \\ 0 & -2+2t-t^2 & -1+t-t^2 \\ 0 & 3-3t+t^2 & (-1+t)^2 \end{pmatrix}$$

are discrete and faithful for every $t \in \mathbf{R}$.

It follows that for all integral values of t the image groups are subgroups of $\mathrm{SL}(3,\mathbf{Z})$, and using Theorem 2.1 can be shown to be thin. In fact, this line of representations determines a line of characters which embeds into $X^{Hit}(\Gamma, \mathrm{SL}(3,\mathbf{R}))$, so that for distinct values of t, the images are non-conjugate.

The proof of this result relies on exploiting the method of [11], which allows for computation of the representation variety and identifies this component explicitly. This, and some Diophantine analysis yields the curve of representations defined above.

2.2

By Margulis's arithmeticity theorem [23], all lattices in $\mathrm{SL}(3,\mathbf{R})$ are arithmetic, and in the case of $\mathrm{SL}(3,\mathbf{R})$ the totality of the commensurability classes of non-uniform arithmetic lattices in $\mathrm{SL}(3,\mathbf{R})$ is described by Witte [28] (this is discussed in §5 below). We now discuss the proof of the result stated in §1. As mentioned, this will provide thin surface subgroups in all non-uniform lattices in $\mathrm{SL}(3,\mathbf{R})$ (again we refer the reader to §5 for notation).

Theorem 2.3. *Let d be a square-free positive integer and let $L = \mathbf{Q}(\sqrt{d})$ be a real quadratic number field. Then the lattice $\mathrm{SU}(\mathrm{J}, \mathcal{O}_\mathrm{L}, \tau)$ contains a thin subgroup isomorphic to the fundamental group of some closed orientable surface of genus ≥ 2.*

Proof: The proof is very much in the spirit of [20], applied in this case to the $(3,4,4)$ triangle group.

We follow the Basic Idea, and so first note that from the discussion above, $X^{Hit}(\Gamma, \mathrm{SL}(3,\mathbf{R}))$ is again 2-dimensional, and following the analysis in [11] and [20], one gets the following description of $X^{Hit}(\Gamma, \mathrm{SL}(3,\mathbf{R}))$:

$$a \mapsto \begin{pmatrix} 1 & 1 & \dfrac{a_1}{\eta} \\ 0 & \dfrac{(1-v)}{(-1+u)} & 1 \\ 0 & \dfrac{(-1+u-u^2+v+uv-v^2)}{(-1+u)^2} & \dfrac{(-u+v)}{(-1+u)} \end{pmatrix}$$

$$b \mapsto \begin{pmatrix} \dfrac{(-1+v)}{(-1+u)} & 0 & \dfrac{b_1}{\eta} \\ \dfrac{1-3u+(2+u+u^2)v-2v^2-(-1+u)\sqrt{D})}{(2(-1+u)^2)} & 1 & -1 \\ \dfrac{-b_2}{2(-1+u)^3} & 0 & \dfrac{(1-v)}{(-1+u)} \end{pmatrix}$$

where

$$D = -7 - 4u^3 + 4v - 4v^2 - 4v^3 + 2u(2+7v) + u^2(-4+v^2),$$

$$\eta = 2(2 + u^4 - 4v + 5v^2 - 3v^3 + v^4 - u^3(3+v) + u^2(5 - v + 2v^2) - u(4 - 2v + v^2 + v^3)),$$

$$a_1 = -(-1+u)(-1+u^3(-2+v) + 2v - 2v^2 + 2v^3 + \sqrt{D} + u(v - 4v^2 - 2\sqrt{D} + u^2(1 + 2v + \sqrt{D})),$$

$$b_1 = (3-8u+11u^2-8u^3+2u^4+(-5+5u-u^2+u^3)v-(-6+7u-2u^2+u^3)v^2+2(-1+u)v^3-(-1+u)^2(-1+v)\sqrt{D}),$$

and

$$b_2 = (-3 + 5u - 6u^2 + 2u^3 + (5+u^2)v - (6 - u + u^2)v^2 + 2v^3 + (-1+u)(-1+v)\sqrt{D}).$$

The hyperbolic representation occurs at $u = 7$, $v = 7$.

The reader may easily verify that upon setting $u = v$, the representation above may be conjugated to the representation:

$$\rho_v(a) = \begin{pmatrix} \frac{1}{4}\left(-v + \sqrt{(v-7)(v+1)} - 1\right) & \frac{1}{4}\left(v - \sqrt{(v-7)(v+1)} + 5\right) & 0 \\ \frac{1}{4}\left(-v + \sqrt{(v-7)(v+1)} - 1\right) & \frac{1}{4}\left(v - \sqrt{(v-7)(v+1)} + 1\right) & 1 \\ -\frac{4}{v+\sqrt{(v-7)(v+1)}-3} & \frac{1}{4}\left(v - \sqrt{(v-7)(v+1)} + 1\right) & 0 \end{pmatrix}$$

$$\rho_v(b) = \begin{pmatrix} 1 & 0 & \frac{1}{2}\left(-v - \sqrt{(v-7)(v+1)} - 1\right) \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

We will require that these matrices have entries in the ring of integers of a real quadratic number field. Hence, assuming that a square-free positive integer d is given, we require that $(v-7)(v+1) = Y^2 d$ for a positive integer Y. Notice that $(v-7)(v+1)$ can be rewritten as $(v-3)^2 - 16$, and so we need to solve $(v-3)^2 - 16 = Y^2 d$; i.e. the the equation $X^2 - Y^2 d = 16$. In particular, given any square-free positive integer d, choosing a solution $(X_0, Y_0) \in \mathbf{N}^2$ of the Pell's equation $X^2 - Y^2 d = 1$, then $v = 4X_0 + 3$ is the desired solution (which is clearly odd).

To conclude the proof we need to establish that these constraints can be arranged so that these matrices also preserve a Hermitean form of the required type. Further computation shows that the image of ρ_v (where we assume that $v = 4X_0 + 3$ is an odd positive integer) preserves the Hermitean form H_v defined by the matrix:

$$J_v = \begin{pmatrix} 2(v+1) & -\frac{1}{2}(v+1)\left(v + \sqrt{v^2-6v-7} + 1\right) & -\frac{1}{2}(v+1)\left(v + \sqrt{v^2-6v-7} + 1\right) \\ -\frac{1}{2}(v+1)\left(v - \sqrt{v^2-6v-7} + 1\right) & 2(v+1) & (v+1)^2 \\ -\frac{1}{2}(v+1)\left(v - \sqrt{v^2-6v-7} + 1\right) & (v+1)^2 & 2(v+1) \end{pmatrix}$$

As it stands the form J_v is not of the type described in §5, however, we can argue as follows to show that J_v is equivalent to a matrix as in §5. To see this, first note that H_v determines a 6-dimensional quadratic form over $\mathbf{Q}$ (i.e. the form q_v defined by $H_v(x,x)$ for $x \in \mathbf{Q}(\sqrt{v^2-6v-7})$). Checking the characteristic polynomial of J_v shows that H_v has signature $(2,1)$ and so q_v is a 6-dimensional indefinite quadratic form over $\mathbf{Q}$, and as such is isotropic. Hence H_v is isotropic as a Hermitean form over $\mathbf{Q}(\sqrt{v^2-6v-7})$. It follows from the classification of Hermitean forms (see for example [28] §15J) that H_v is equivalent to a form as described in §5.

Note that by Theorem 2.1, away from the solution $u = v = 7$, the surface subgroups will be thin. □

Example: For concreteness we describe an example in the case when $d = 30$.

In this case an appropriate unit is given by $11+2\sqrt{30}$ which from above implies that we take $v = 47$. With this we have

$$\rho_{47}(a) = \begin{pmatrix} -12+2\sqrt{30} & 13-2\sqrt{30} & 0 \\ -12+2\sqrt{30} & 12-2\sqrt{30} & 1 \\ -11+2\sqrt{30} & 12-2\sqrt{30} & 0 \end{pmatrix},$$

$$\rho_{47}(b) = \begin{pmatrix} 1 & 0 & \left(-24-4\sqrt{30}\right) \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

and the Hermitean form is determined by the matrix:

$$J_{47} = \begin{pmatrix} 1 & -\left(12+2\sqrt{30}\right) & -\left(12+2\sqrt{30}\right) \\ -\left(12-2\sqrt{30}\right) & 1 & 24 \\ -\left(12-2\sqrt{30}\right) & 24 & 1 \end{pmatrix}.$$

3 $G = \mathrm{SL}(4, \mathbf{R})$ and Γ a cocompact Kleinian group

The starting point for this section is to consider certain closed hyperbolic 3-manifolds $M = \mathbf{H}^3/\Gamma$ described in [11] as *flexible*, by which one means that the hyperbolic structure may be deformed when it is regarded as a strictly convex real projective structure. (See [11] for more details)

The certificate for thinness in this case comes from the application of deep results of Koszul [18] and Benoist [2] and [3], which taken together imply that all of these deformations are the holonomy of convex real projective structures and in particular, they are all discrete, faithful representations of the fundamental group in question. (This part of the construction plays the role of the work of Choi-Goldman [8] described in [20].) It is also argued in [22] that the resulting image groups are Zariski dense in $\mathrm{SL}(4, \mathbf{R})$.

This part of the argument is quite general and applies to any flexible hyperbolic 3-manifold (even if these are perhaps quite rare, see [11]), but as in §3 some specialisation is now necessary to ensure that the deformed group lies inside a lattice. To this end, we fix attention upon one particular closed hyperbolic 3-manifold, traditionally known as vol3. Again, the lattices in question are described in §5.

Theorem 3.1. *For infinitely many real quadratic number fields L, there exist lattices* $\mathrm{SU}(\mathrm{J}, \mathcal{O}_{\mathrm{L}}, \tau)$ *that contain a thin subgroup isomorphic to a subgroup of finite index in* $\pi_1(\mathrm{vol3})$*.*

We will not discuss anything about the proof here, nor describe the matrices other than to say that this proof is similar to that described for Theorem 2.3 (although integrality requires some more delicate computation in this case). We refer the reader to [22] for more details.

However, for convenience we record one example of the representation.

Example: The fundamental group of vol3 has a presentation

$$\langle\ a, b \ \mid\ aabbABAbb\ ,\ aBaBabaaab\ \rangle$$

where $A = a^{-1}$ and $B = b^{-1}$. As in [22] (following [11]) we work with an orbifold $Q = \mathrm{vol3}/\langle \mathrm{u} \rangle$ which is four-fold covered by vol3. We denote by Γ_Q the orbifold fundamental group of Q. Notice that a representation of Γ_Q is discrete and faithful only if it is discrete and faithful when restricted to $\pi_1(\mathrm{vol3})$, so that it suffices to work with Γ_Q.

From [11] one sees that Γ_Q is generated by two elements of finite order u and c. The group $\pi_1(\text{vol3})$ is recovered as $a = u^2.c$ and $b = (a.u.a)^{-1}.u$. Then one such representation ρ of Γ_Q given by Theorem 3.1 is:

$$\rho(u) = \begin{pmatrix} 0 & -1 & -4+3\sqrt{3} & -1+2\sqrt{3} \\ 1 & 0 & -2+\sqrt{3} & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho(c) = \begin{pmatrix} 0 & 0 & -1+\sqrt{3} & 0 \\ 0 & 0 & 0 & -1+\sqrt{3} \\ (1+\sqrt{3})/2 & 0 & 0 & 0 \\ 0 & (1+\sqrt{3})/2 & 0 & 0 \end{pmatrix}$$

This does not have integral entries, but there is a surjection from $\pi_1(\text{vol3})$ to the dihedral group with ten elements where one sends a to a reflection and b to a rotation. A direct calculation shows that the kernel of this map consists of elements whose entries lie in $\mathbf{Z}[\sqrt{3}]$. One can also check that a non-degenerate τ-Hermitean form for the image of ρ is

$$J = \begin{pmatrix} 2 & 0 & 2-2\sqrt{3} & -2\sqrt{3} \\ 0 & 2 & 6-4\sqrt{3} & 2-2\sqrt{3} \\ 2+2\sqrt{3} & 6+4\sqrt{3} & -4 & 0 \\ 2\sqrt{3} & 2+2\sqrt{3} & 0 & -4 \end{pmatrix},$$

which in turn can be checked as being equivalent to the diagonal form $\text{diag}(1,1,1,-5)$.

Remarks.

(i) The manifold vol3 is rather well understood and it is shown in [24] that vol3 has a finite sheeted cover that fibers over S^1 with fiber a closed surface F. The image of this fibre group will have the same Zariski closure as $\pi_1(\text{vol3})$; i.e. this exhibits a thin surface subgroup H in an $\mathrm{SL}(4,\mathbf{R})$-lattice, which has the additional property that its normaliser N in $\mathrm{SL}(4,\mathbf{R})$ contains an element δ for which $\delta^n \notin \mathrm{H}$ for all $n \in \mathbf{Z} \setminus \{0\}$. Motivated by the situation in $\mathrm{PSL}(2,\mathbf{C})$ we refer to such a surface group as *geometrically infinite*. Summarizing, we have as a corollary of Theorem 3.1:

Corollary 3.2. *For infinitely many real quadratic number fields L, there exist non-uniform lattices* $\mathrm{SU}(\mathrm{J}, \mathcal{O}_{\mathrm{L}}, \tau)$ *in* $\mathrm{SL}(4,\mathbf{R})$ *which contain a geometrically infinite, thin surface subgroup.*

As in the case of $n = 3$, for a surface group $\pi_1(\Sigma_g)$ $(g \geq 2)$, there is a component of $X^{Hit}(\pi_1(\Sigma_g), \mathrm{SL}(4,\mathbf{R}))$ called the Hitchin component. Now Labourie [19] showed that the Mapping Class group of Σ_g acts properly discontinuously on this component. Hence for such surface group representations, there can never be an element δ as above (i.e. they are never geometrically infinite). Hence we are led to ask:

Question 3.3. *Does there exist a (non-uniform) lattice* Γ *in* $\mathrm{SL}(3,\mathbf{R})$ *that contains a thin geometrically infinite surface subgroup?*

(ii) Compared to $\mathrm{SL}(3,\mathbf{Z})$, the structure of the non-uniform lattices of Theorem 3.1 are far less well understood. For example, unlike $\mathrm{SL}(3,\mathbf{Z})$, these lattices are not yet known to be boundedly generated by unipotent elements, nor are they known to be not left-orderable (see [29] for more on this). For general countable groups, being left-orderable is equivalent to having an orientation-preserving, faithful action on $\mathbf{R}$. In the context of lattices in $\mathrm{SL}(\mathrm{n},\mathbf{R})$, $n \geq 3$, this is also equivalent to having no faithful action on S^1. More precisely, if Γ is a lattice in $\mathrm{SL}(\mathrm{n},\mathbf{R})$, $n \geq 3$, then

Proposition 2.8 of [29] shows that the following two conjectures are equivalent:

Γ *is not left-orderable.*

Γ *has no faithful action on* S^1.

We note that the proof of equivalence uses that these statements hold for all lattices, and not just an individual one.

There has also been a great deal of interest recently in left-orderability (and lack thereof) of 3-manifold groups (see for example [6], [9] and [12] for example). In light of the aforementioned conjectures, we note the following. We begin with a lemma.

Lemma 3.4. $\pi_1(\mathrm{vol3})$ *is not left-orderable.*

Proof: Since subgroups of left-orderable groups are left-orderable, the lemma will be deduced from the following observation.

It is shown in [12], Theorem 1 that if $M^{(n)}$ denotes the n-fold cyclic branched cover of the figure-eight knot, then $\pi_1(M^{(n)})$ is not left orderable. In the case when $n = 4$, it is shown in [24] that vol3 is double covered by $M^{(4)}$. Hence $\pi_1(\mathrm{vol}\ 3)$ is not left-orderable as required. $\square$

Now from the discussion above $\pi_1(\mathrm{vol3})$ contains a subgroup of finite index contained in a lattice $\mathrm{SU}(\mathrm{J}, \mathcal{O}_\mathrm{L}, \tau)$. It is now a standard argument (which we sketch below) that $\pi_1(\mathrm{vol3})$ is a subgroup of a lattice Γ commensurable with $\mathrm{SU}(\mathrm{J}, \mathcal{O}_\mathrm{L}, \tau)$. Hence Γ is not left-orderable.

To make this observation more interesting, requires ensuring that Γ is torsion-free, which for the moment seems hard to arrange.

We now sketch the proof that $\pi_1(\mathrm{vol3})$ is contained in a lattice. Throughout this discussion, let Δ denote the image of vol3 constructed implicitly in Theorem 3.1, $\Lambda = \mathrm{SU}(\mathrm{J}, \mathcal{O}_\mathrm{L}, \tau)$ and Δ_0 a normal subgroup of Δ of finite index in $\Delta \cap \Lambda$. Consider

$$\mathcal{D} = \mathcal{O}\Delta_0 = \{\Sigma\, a_i\gamma_i \ \mid\ a_i \in \mathcal{O}_L, \gamma_i \in \Delta_0\ \}$$

where the sums are finite. It is shown in [1] (see Proposition 2.2 and Corollary 2.3), that $\mathcal{D}$ is an order of a central simple algebra B defined over L and contained in $M(4, L)$. Indeed, since Δ_0 is Zariski dense, $B = M(4, L)$. Then $\mathcal{D}^1$ is an arithmetic group commensurable with $\mathrm{SL}(4, \mathcal{O}_\mathrm{L})$. Furthermore, since Δ normalizes Δ_0, Δ is contained in the the normalizer N of $\mathcal{D}$ in $\mathrm{SL}(4, \mathbf{R})$. Then N is the required arithmetic lattice commensurable with $\mathrm{SL}(4, \mathcal{O}_\mathrm{L})$.

4 Further constructions.

In the previous sections, we were able to exhibit thin subgroups, using the certificate of faithfulness. In this section, we describe some other constructions where the representation might not be faithful, but the image group could be thin coming from structural considerations. Nonetheless, the material here is a good deal more speculative.

The basis of this construction is the 4-strand braid group, B_4. It is a standard property of B_4 (see [4]) that there is a surjective homomorphism $B_4 \longrightarrow B_3$ obtained by setting $\sigma_1 = \sigma_3$ and that the kernel K is a finitely generated free group of rank two generated by $\{\sigma_1\sigma_3^{-1}, \sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}\}$.

The connection with matrix representations comes by consideration of the classical reduced Burau representation [4]. This representation does not have determinants equal to one, but one can easily arrange this by scaling and one obtains $\beta : B_4 \longrightarrow \mathrm{SL}(3, \mathbf{Z}[\mathrm{x}, 1/\mathrm{x}])$ as

$$\beta(\sigma_1) = \begin{pmatrix} x^2 & 1/x & 0 \\ 0 & 1/x & 0 \\ 0 & 0 & 1/x \end{pmatrix} \quad \beta(\sigma_2) = \begin{pmatrix} 1/x & 0 & 0 \\ -x^2 & x^2 & 1/x \\ 0 & 0 & 1/x \end{pmatrix} \quad \beta(\sigma_3) = \begin{pmatrix} 1/x & 0 & 0 \\ 0 & 1/x & 0 \\ 0 & -x^2 & x^2 \end{pmatrix}$$

It is an old result of Squier [26] that the Burau representation can be regarded as Hermitean for a certain form. In the case of the representation β, it can be checked that the matrices are Hermitean for the form

$$J = \begin{pmatrix} -2+1/x^3+x^3 & 1-1/x^3 & 0 \\ 1-x^3 & -2+1/x^3+x^3 & 1-1/x^3 \\ 0 & 1-x^3 & -2+1/x^3+x^3 \end{pmatrix}$$

by which we mean that they satisfy $g^*Jg = J$ where g^* means apply the ring automorphism $x \to 1/x$ and then transpose.

Now let k be a real quadratic number field with ring of integers $\mathcal{O}_k$, and $u \in \mathcal{O}_k$ a non-trivial unit for which $\sigma(u) = 1/u$. Setting $x = u$ gives a representation

$$\beta_u : B_4/Z_4 \longrightarrow \mathrm{SL}(3, \mathcal{O}_\mathrm{k})$$

and we will denote the image $\beta_u(B_4/Z_4)$ by $B(u)$ and the form J specialized at u by $J(u)$.

We prove the following (referring to §5 for arithmetic lattices considered).

Theorem 4.1. *(i) $B(u)$ is a subgroup of a non-uniform lattice in* $\mathrm{SL}(3, \mathbf{R})$.

(ii) $B(u)$ is not virtually free.

(iii) $B(u)$ is Zariski dense in $\mathrm{SL}(3, \mathbf{R})$.

Proof: It follows from our choice of u and Squier's result that $B(u)$ is unitary for the non-degenerate form $J(u)$. However as it stands, this is not a form defined over $\mathbf{Z}$: the diagonal entries are rational integers since they are σ-invariant integers of $\mathcal{O}_k$, but the off diagonal entries need not be.

This can be rectified by doing the Gram-Schmidt process which is a $\mathrm{GL}(3, \mathrm{k})$ change of basis. Denoting the integer $1 - 1/u^3 = \tau$, one checks easily that the matrix given by mapping the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\mathbf{e}_1, \sigma(\tau)\mathbf{e}_2, \sigma(\tau^2)\mathbf{e}_3\}$ gives a similarity of $J(u)$ with a non-degenerate symmetric $\mathbf{Z}$-matrix, $\Psi(u)$. Identifying $B(u)$ with its conjugate preserving the form $\Psi(u)$, this discussion shows that $B(u)$ contains a subgroup of finite index contained in the lattice $\mathrm{SU}(\Psi(\mathrm{u}), \mathcal{O}_\mathrm{k}, \tau)$. Arguing as in Remark (ii) of §3, we see that $B(u)$ is also contained in a lattice. That the lattices are non-uniform can be checked by an argument similar to that used in the proof of Theorem 2.3; i.e. checking that the associated 6-dimensional form over $\mathbf{Q}$ is isotropic. This proves part (i).

To prove (ii), we note that in a free group, commuting elements lie in a cyclic subgroup. However, the matrices $\beta_u(\sigma_1)$ and $\beta_u(\sigma_3)$ commute but $< \beta_u(\sigma_1), \beta_u(\sigma_3) > \cong \mathbf{Z} \times \mathbf{Z}$. This persists in a finite index subgroup.

Finally, using the proof of (ii), we deduce that the Zariski closure of $B(u)$ is a non-compact Lie subgroup of $\mathrm{SL}(3, \mathbf{R})$ of $\mathbf{R}$-rank two. Hence it must be $\mathrm{SL}(3, \mathbf{R})$. $\square$

We ask:

Question 4.2. *Are the groups $B(u)$ thin?*

Potential certificates come from the following observation:

Lemma 4.3. *In the notation established above*

$$\beta(B_4)/\beta(K) \cong B_3$$

Proof: As remarked above, $B_4/K \cong B_3$. Hence, the required statement follows from the first isomorphism theorem and the fact that $\ker(\beta) < K$, see [4]. $\square$

In particular, this bypasses the long-standing open problem as to whether the Burau representation of B_4 is faithful, since, independently of whether β is faithful, if there is any algebraic specialisation for which the map $\beta(B_4) \to B(u)$ is faithful, the image group $B(u)$ has an infinite quotient and therefore cannot be a rank 2 lattice. It seems rather likely that such specialisations exist, but identifying one seems formidably difficult.

4.1 The figure eight knot group

In some real sense the simplest finite volume hyperbolic 3-manifold group is the fundamental group of the figure eight knot complement. This is a hyperbolic 1-punctured torus bundle so that one can present the fundamental group as

$$\pi_1(M) =< x, y, z | zxz^{-1} = xy, zyz^{-1} = yxy >$$

It's an intriguing question whether this group admits a discrete, faithful representation in $SL(3, \mathbf{R})$ and this is a special case of a question due to Labourie.

Question 4.4. *Does there exist a finite volume hyperbolic 3-manifold M for which $\pi_1(M)$ admits a discrete, faithful representation in* $\mathrm{SL}(3, \mathbf{R})$*?*

Unlike the cases considered in §3 and 4, there is apparently no obvious "natural geometric" representation of a hyperbolic 3-manifold group into $\mathrm{SL}(3, \mathbf{R})$ to attempt to deform. Nevertheless, one can attempt to look for representations as we now describe.

Given the presentation above, the following two propositions can be checked directly by matrix multiplication

Proposition 4.5. *Define a map* $\rho_k : \Gamma \longrightarrow \mathrm{SL}(3, \mathbf{Z}[\mathrm{k}])$ *by*

$$\rho_k(x) = X_k = \begin{pmatrix} 1 & -2 & 3 \\ 0 & k & -1-2k \\ 0 & 1 & -2 \end{pmatrix}$$

$$\rho_k(y) = Y_k = \begin{pmatrix} -2-k & -1 & 1 \\ -2-k & -2 & 3 \\ -1 & -1 & 2 \end{pmatrix}$$

$$\rho_k(z) = Z_k = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -k \\ 0 & 1 & -1-k \end{pmatrix}$$

Then ρ_k is a homomorphism.

Proposition 4.6. *Define a map* $\beta_T : \Gamma \longrightarrow \mathrm{SL}(3, \mathbf{Z}[\mathrm{T}])$ *by*

$$\beta_T(x) = X_T = \begin{pmatrix} -1+T^3 & -T & T^2 \\ 0 & -1 & 2T \\ -T & 0 & 1 \end{pmatrix}$$

$$\beta_T(y) = Y_T = \begin{pmatrix} -1 & 0 & 0 \\ -T^2 & 1 & -T \\ T & 0 & -1 \end{pmatrix}$$

$$\beta_T(z) = Z_T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & T^2 \\ 0 & 1 & 0 \end{pmatrix}$$

Then β_T is a homomorphism.

Note that, in either case an integral specialisation gives:

Corollary 4.7. *For integral k or T, $\rho_k(\Gamma), \beta_T(\Gamma) \leq \mathrm{SL}(3, \mathbf{Z})$.*

Now it is shown in [21] that the representations ρ_k and β_T are each irreducible except possibly for four exceptional values of their parameter, and in particular, ρ_k is irreducible for all $k \in \mathbf{Z}$, and β_T is irreducible for all non-zero $T \in \mathbf{Z}$. Moreover, it is also shown in [21] that for $k \in \mathbf{Z}$ (resp. nonzero $T \in \mathbf{Z}$), the image groups $\rho_k(\Gamma)$ (resp. $\beta_T(\Gamma)$) are Zariski dense subgroups of $\mathrm{SL}(3, \mathbf{Z})$ (and similarly for the images of the fiber group F). The issue is in deciding whether the image is thin. One of the main results of [21] is the following.

Theorem 4.8. *Fix a non-zero integer value of T.*
Then the group $\beta_T(F)$ (and therefore $\beta_T(\Gamma)$) has finite index in $\mathrm{SL}(3, \mathbf{Z})$.

Furthermore, $\bigcap_{T>0} \beta_T(F) = 1$.

From the perspective of thin subgroups, this is undeniably disappointing, however, it is important to note that simply being able to decide that the images of the β_T-representations are finite index in $\mathrm{SL}(3, \mathbf{Z})$ is rather remarkable. For that reason we comment briefly on the proof. Throughout the following discussion, the parameter T is as described in Theorem 4.8.

The natural first question is whether the representations β_T are faithful. Remarkably, one finds the following relation holds for all parameter values T (where $X = \beta_T(x)$ and $Y = \beta_T(y)$).

$$X^{-1}YX^{-1}YX^{-1}X^{-1}YYYXYYXY^{-1}X = XY^{-1}XYYXYYYX^{-1}X^{-1}YX^{-1}YX^{-1}$$

Now it is not difficult to check that the groups $\beta_T(\Gamma)$ contain unipotent elements—for example $\beta_T(y^2)$ is a unipotent. That $\beta_T(\Gamma)$ has finite index in $\mathrm{SL}(3, \mathbf{Z})$ relies on the following result of Venkataramana (see Theorem 3.7 of [27]):

Theorem 4.9. *Suppose that $n \geq 3$ and $x \in SL(n, \mathbf{Z})$ is a unipotent matrix such that $x - 1$ has matrix rank 1. Suppose that $y \in \mathrm{SL}(\mathrm{n}, \mathbf{Z})$ is another unipotent such that x and y generate a free abelian group N of rank 2. Then any Zariski dense subgroup of $\mathrm{SL}(\mathrm{n}, \mathbf{Z})$ containing N virtually, is of finite index in $\mathrm{SL}(\mathrm{n}, \mathbf{Z})$.*

Given this result, in [21] we proceed to exhibit unipotent matrices b_1 and b_2 in $\beta_T(F)$ such that $b_1 - 1$ has rank 1 and $< b_1, b_2 > \cong \mathbf{Z} \oplus \mathbf{Z}$. In terms of X and Y the matrices are:

$$b_1 = X^{-1}.Y.Y.Y.X.Y.Y.X.Y^{-1}.X$$

$$b_2 = X.Y^{-1}.X.Y.Y.X.Y.Y.Y.X^{-1}$$

It is shown in [21] that there is a conjugation of b_1 and b_2 so that they have the form:

$$P^{-1}b_1P = \begin{pmatrix} 1 & 0 & -T^2(-1+2T)(-5+3T^3) \\ 0 & 1 & -T(-1+2T)(-2+3T^3) \\ 0 & 0 & 1 \end{pmatrix}$$

$$P^{-1}b_2P = \begin{pmatrix} 1 & 0 & -3T^2(-1+2T) \\ 0 & 1 & -T(-1+2T)(-2+3T^3) \\ 0 & 0 & 1 \end{pmatrix}$$

It follows from Venkataramana's result that $\beta_T(\Gamma)$ has finite index.

By contrast, the representations ρ_k remain largely mysterious. Apart from a few small values of k, namely $\{0, 2, 3, 4, 5\}$ (where we can follow the idea described for the β_T representations described above), at present the following remains open.

Question 4.10. *For k an integer and $k \neq 0, 2, 3, 4, 5$ is $\rho_k(\Gamma) < \mathrm{SL}(3, \mathbf{Z})$ a thin subgroup?*

Note that experimentation suggests that $\rho_1(\Gamma)$ is virtually free (and so will be an infinite index subgroup of $\mathrm{SL}(3, \mathbf{Z})$).

5 Non-uniform lattices in $\mathrm{SL}(\mathrm{n}, \mathbf{R})$

For convenience we recall a construction of non-uniform arithmetic lattices in $\mathrm{SL}(\mathrm{n}, \mathbf{R})$ for $n \geq 3$. We refer the reader to [28] Propositions 6.42 and 6.55 for more details.

Let L be a real quadratic number field with ring of integers $\mathcal{O}_L$ and non-trivial Galois automorphism τ. For a matrix $A \in \mathrm{SL}(\mathrm{n}, \mathrm{L})$, denote by A^* the matrix obtained by taking the transpose of the matrix obtained from A by applying τ (the non-trivial Galois automorphism) to all its entries.

Theorem 5.1. *Let L, $\mathcal{O}_L$ and τ be as above, and let $b_1, \ldots, b_n$ be nonzero elements of $\mathbf{Z}$. Setting $J = \mathrm{diag}(\mathrm{b}_1, \ldots, \mathrm{b}_n)$, then the group*

$$\mathrm{SU}(\mathrm{J}, \mathcal{O}_{\mathrm{L}}, \tau) = \{\mathrm{A} \in \mathrm{SL}(\mathrm{n}, \mathcal{O}_{\mathrm{L}}) \quad | \quad \mathrm{A}^*\mathrm{JA} = \mathrm{J}\}$$

is a non-uniform lattice in $\mathrm{SL}(\mathrm{n}, \mathbf{R})$.

Note that Proposition 6.42 of [28] deals with the case of a form that is not diagonal. Following the lines of Witte's argument, one can prove the following.

Let J be any matrix of the form $J = \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ a & 0 & 0 \end{pmatrix}$ where $a, b \in \mathbf{Z}$, $ab \neq 0$.

As above let,

$$\mathrm{SU}(\mathrm{J}, \mathcal{O}_{\mathrm{k}}, \tau) = \{\mathrm{A} \in \mathrm{SL}(3, \mathcal{O}_{\mathrm{k}}) \quad | \quad \mathrm{A}^*\mathrm{JA} = \mathrm{J}\}.$$

Theorem 5.2. *$\mathrm{SU}(\mathrm{J}, \mathcal{O}_{\mathrm{k}}, \tau)$ is a (non-uniform) lattice in $\mathrm{SL}(3, \mathbf{R})$.*

We also note that Proposition 6.46 of [28] shows that up to conjugacy the lattices constructed in Theorem 5.1 together with $\mathrm{SL}(3, \mathbf{Z})$ represent the totality of commensurability classes of non-uniform lattices in $\mathrm{SL}(3, \mathbf{R})$. Indeed, as is discussed in §6 of [28], it suffices to take $a = b = 1$ in the matrix J above to describe the totality of commensurability classes of non-uniform lattices in $\mathrm{SL}(3, \mathbf{R})$ (up to conjugacy).

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