Arithmetic of Hyperbolic 3-Manifolds

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Hyperbolic 3-Manifolds and Discrete Groups

Hyperbolic 3-space can be defined as

$$\mathbb{H}^3 = \{(z,t) \in \mathbb{C} \times \mathbb{R} : t > 0\}$$

and equipped with the metric $ds = \frac{ds_{\mathbb{E}}}{t}$.
Geodesics are vertical lines perpendicular to $\mathbb{C}$ or semi-circles perpendicular to $\mathbb{C}$.
Codimension 1 geodesic submanifolds are Euclidean planes in $\mathbb{H}^3$ orthogonal to $\mathbb{C}$ or hemispheres centered on $\mathbb{C}$.

The full group of orientation-preserving isometries of $\mathbb{H}^3$ can be identified with $\mathrm{PSL}(2,\mathbb{C})$.

Abuse of notation

We will often view Kleinian groups as groups of matrices!

A discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$ is called a *Kleinian group*.

Such a group acts *properly discontinuously* on $\mathbb{H}^3$.
If Γ is torsion-free (does not contain elements of finite order) then Γ acts freely.
In this latter case we get a quotient manifold $\mathbb{H}^3/\Gamma$ and a covering map $\mathbb{H}^3 \to \mathbb{H}^3/\Gamma$ which is a local isometry.

We will be interested in the case when $\mathbb{H}^3/\Gamma$ has finite volume.

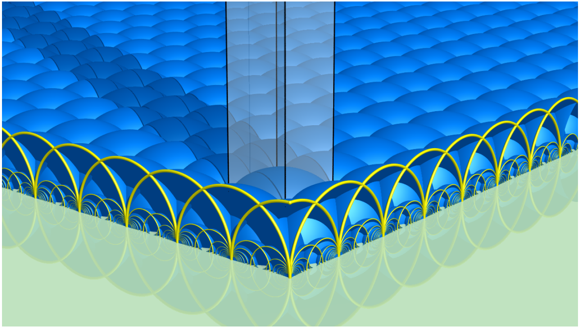
Note: If Γ is a finitely generated subgroup of $\mathrm{PSL}(2, \mathbb{C})$ there is always a finite index subgroup that is torsion-free.

Examples

1. Let d be a square-free positive integer, and O_d the ring of algebraic integers in $\mathbb{Q}(\sqrt{-d})$. Then $\mathrm{PSL}(2, \mathrm{O_d})$ is a Kleinian group.

Indeed $\mathbb{H}^3/\mathrm{PSL}(2, \mathrm{O_d})$ has finite volume but is non-compact.

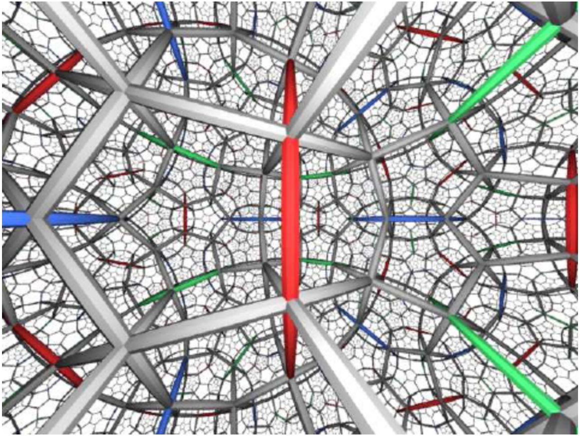
These are known as the Bianchi groups.



$d = 1$

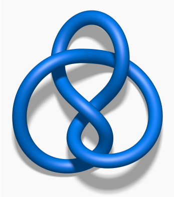
(picture by Jos Leys)

2. Reflection groups.



Tessellation by all right dodecahedra (from the Not Knot video)

3.Knot Complements



The Figure-Eight Knot

Thurston's hyperbolization theorem shows that many knots have complements that are hyperbolic 3-manifolds of finite volume.

e.g. Out of approximately 1.7 million knots with ≤ 16 crossings only 32 are not hyperbolic.

Important example: K the figure-eight knot above, $S^3 \setminus K = \mathbb{H}^3/\Gamma$ where

$$\Gamma = < \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} >, \text{ where } \omega^2 + \omega + 1 = 0$$

So $\Gamma < \mathrm{PSL}(2, \mathrm{O}_3)$ of index 12 (R. Riley, 1974).

Number Theory

Let $\Gamma < \mathrm{PSL}(2, \mathbb{C})$ be finitely generated. The trace-field of Γ is the field:

$$\mathbb{Q}(\mathrm{tr}\ \Gamma) = \{\mathrm{tr}(\gamma) : \gamma \in \Gamma\}$$

e.g. With Γ the figure-eight knot group,

$$\mathbb{Q}(\mathrm{tr}\ \Gamma) = \mathbb{Q}(\omega) = \mathbb{Q}(\sqrt{-3}).$$

Some trace-fields of other hyperbolic knot groups. (thanks to SnapPy, Sage and Matthias Goerner).

They are all number fields; ie finite extensions of $\mathbb{Q}$!

Rigidity implies that if Γ has finite co-volume, $\mathbb{Q}(\mathrm{tr}\ \Gamma)$ is always a number field.
Question: Which number fields arise? How about just for knot complements?

Trace-fields of Knot Complements

1. If two hyperbolic knot complements are commensurable (i.e. have a common finite sheeted cover) they have the same trace-field.

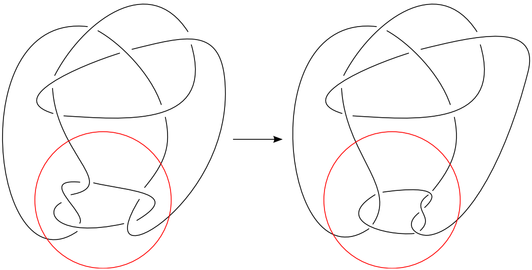
In general commensurable Kleinian groups need not have the same trace-field. However there is an invariant trace-field. For knots, the invariant trace-field is the trace-field.

2. The figure-eight knot complement is the only arithmetic knot complement.
(i.e. trace field quadratic imaginary and all traces are algebraic integers)
So the figure-eight knot complement is incommensurable with any other knot complement.

Question: Is there any other knot whose complement has quadratic imaginary trace-field?

3. There are many knots with the same trace-field.

Mutants



Question *Are there only finitely many knots with the same trace-field?*

4. Consider the knots 6_1 and 7_7 from the table.
Their trace-fields are generated by a root of the polynomial $x^4 + x^2 - x + 1$ (which has two pairs of complex conjugate roots).
However, the trace-fields are generated by different pairs.

So these knot complements are not commensurable.

Conjecture There are at most three hyperbolic knot complements in a commensurability class.

Known to be true "generically" (when the knot has no hidden symmetries—Boileau-Boyer-Cebanu-Walsh).

4. Suppose $S^3 \setminus K = \mathbb{H}^3/\Gamma$ is a hyperbolic knot with k_Γ of odd prime degree and tr Γ consists of algebraic integers.

e.g. 5_2, 6_2, 7_1, 7_2, 7_4, 8_5, 8_{10}, 8_{15}, 8_{16}, 8_{20},.....

$S^3 \setminus K$ does not contain any immersed closed totally geodesic surfaces. Proof uses ideas from later in the lecture.

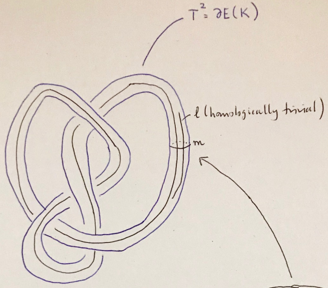
When K is the figure-eight knot, $S^3 \setminus K$ contains infinitely many distinct immersed closed totally geodesic surfaces. This is a consequence of arithmeticity!

Thanks to recent work of Margulis-Mohammadi, Bader-Fisher-Miller-Stover this characterizes the figure-eight knot amongst hyperbolic knots.

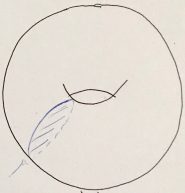
$S^3 \setminus 8_{20}$ contains no immersed totally geodesic surface (closed or otherwise). (more subtle, work of D. Calegari).

Some more constructions of hyperbolic 3-manifolds Dehn Surgery

K



Every essential simple closed curve $r \subset T^2$ can be represented as $m^p \ell^q$ for (p,q) coprime integers (or $(1,0)$, $(0,1)$)



V solid torus.

Form $\boxed{K(r) := r\text{-Dehn Surgery}}$

Glue V to $E(K)$ identifying $T^2 \hookrightarrow \partial V$ s.t. $\mu \hookrightarrow r = m^p \ell^q$.

Thurston's Dehn Surgery Theorem

If K is a hyperbolic knot then all but finitely many Dehn surgeries give hyperbolic 3-manifolds.

Example: The figure-eight knot

All $p/q \notin \{1/0, 0/1, \pm 1, \pm 2, \pm 3, \pm 4\}$ give closed hyperbolic manifolds.

Theorem (Hodgson, Long-R)

Let K be a hyperbolic knot. If r_j is any infinite sequence of distinct hyperbolic Dehn surgeries on K, then the degree of the trace-field of $\pi_1(K(r_j)) \to \infty$.

Quaternion algebras

The Hamiltonian quaternions can be described as a 4-dimensional real vector space $\mathbf{H}$ with basis $\{1, i, j, k = ij\}$ and multiplication defined by $i^2 = j^2 = -1$ and $ij = -ji$.
We can encode this with a symbol:

$$\left(\frac{-1\,,\ -1}{\mathbb{R}}\right)$$

Now let k be any field (characteristic $\neq 2$) and $a, b \in k$ non-zero elements. Then a Quaternion Algebra B/k can be described by a Hilbert symbol

$$\left(\frac{a\,,\ b}{k}\right)$$

Meaning: There is a 4-dimensional k-vector space B with basis $\{1, i, j, ij\}$ with multiplication defined by $i^2 = a, j^2 = b$ and $ij = -ji$.

Quaternion algebras associated to Kleinian groups

Let Γ be a Kleinian group of finite co-volume (which we view as a subgroup of $SL(2,\mathbb{C})$) and with trace-field k_Γ.
Set

$$A_0(\Gamma) = \left\{ \sum_{i=1}^{n} \alpha_i \gamma_i \ : \ \alpha_i \in k_\Gamma, \gamma_i \in \Gamma \right\}.$$

Then $A_0(\Gamma)$ is a quaternion algebra defined over k_Γ.
As a basis one can take any $x, y \in \Gamma$ non-commuting elements and then $\{1, x, y, xy\}$ is a basis.

Examples

1. K to be the figure eight knot. Seen

$$\Gamma =< \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} >\subset \mathrm{SL}(2, \mathrm{O}_3) \subset \mathrm{M}(2, \mathbb{Q}(\sqrt{-3})).$$

$A_0\Gamma = M(2, \mathbb{Q}(\sqrt{-3}))$.

2. K any hyperbolic knot, and Γ the knot group.
Can conjugate Γ so that:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma \text{ and } \Gamma \text{ lies in } \mathrm{SL}(2, \mathrm{k}_\Gamma).$$

It follows that $A_0\Gamma = M(2, k_\Gamma)$.

3. $M = K(5/1)$ where K is the figure eight knot.
$A_0\Gamma = \left(\frac{-1\,,\,-1}{k}\right)$ where $k = \mathbb{Q}(t)$ and t a complex root of $t^4 - t - 1 = 0$.

Return to:

Suppose $S^3 \setminus K = \mathbb{H}^3/\Gamma$ is a hyperbolic knot with k_Γ of odd prime degree and tr Γ consists of algebraic integers.

$S^3 \setminus K$ does not contain any immersed closed totally geodesic surfaces.

If it did then Γ contains a Fuchsian surface group F whose traces are real. Hence $k_F \cap k_\Gamma \subset \mathbb{R} \cap k_\Gamma = \mathbb{Q}$.(prime degree)
Moreover, tr Γ consists of algebraic integers.

Hence trF $\subset \mathbb{Z}$.

This implies F is an arithmetic Fuchsian surface group and $A_0(F)$ is a division algebra defined over $\mathbb{Q}$.

But: $A_0(F) \otimes_{\mathbb{Q}} k_\Gamma = M(2, k_\Gamma)$.

This is impossible! odd degree extension cant split a quaternion division algebra.

Ramification

Theorem (Classification of quaternion algebras over local fields)

- *$M(2,\mathbb{C})$ is the unique isomorphism class of quaternion algebras over $\mathbb{C}$.*
- **H** *and $M(2,\mathbb{R})$ represent the isomorphism classes of quaternion algebras over $\mathbb{R}$.*
- *If L is a finite extension of $\mathbb{Q}_p$ then there two isomorphism classes of quaternion algebras over L: $M(2,L)$ and a unique division algebra of quaternions D_L.*

Let k be a number field and B/k a quaternion algebra given by the Hilbert Symbol
$$\left(\frac{a\,,\ b}{k}\right).$$
Let $\sigma : k \to \mathbb{C}$ be a field embedding. Define:
$$B^{\sigma} = \left(\frac{\sigma(a)\,,\ \sigma(b)}{\sigma(k)}\right).$$
Say B is ramified at σ if $B^{\sigma} \otimes_{\sigma(k)} \mathbb{R} \cong \mathbf{H}$.

Examples

1. $B = \left(\frac{-1\,,\ -1}{\mathbb{Q}}\right)$ is ramified at the identity embedding of $\mathbb{Q}$ but $B = \left(\frac{-2\,,\ 3}{\mathbb{Q}}\right)$ is not.

2. $B = \left(\frac{-1\,,\ 1+\sqrt{2}}{\mathbb{Q}(\sqrt{2})}\right)$ is unramified at the identity embedding of $\mathbb{Q}(\sqrt{2})$ and ramified at the non-trivial embedding since $\sigma(1+\sqrt{2}) = 1-\sqrt{2} < 0$.

A notion of ramified can also be made for the non-archimedean places of a number field k "locally one gets the division algebra".

The Classification Theorem for Quaternion Algebras over Number Fields

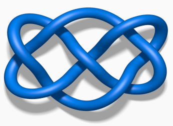
There are a finite even number of ramified places (ie real embeddings and primes). This set determines the isomorphism class.

Dehn surgery and Quaternion Algebras

Let K be a hyperbolic knot. Seen that the quaternion algebra associated to the knot group Γ is always $M(2, k_\Gamma)$.
Apart from finitely many Dehn surgeries, we get closed hyperbolic 3-manifolds which in turn have their associated quaternion algebras over their trace fields.

How do we understand how the algebras are varying?

Two examples: The figure eight knot group and the group of the knot 7_4 (shown below)



The trace-field in this case is $\mathbb{Q}(t)$ where t is a complex root of $x^3 + 2x - 1 = 0$.

What explains this apparently different behavior of ramification at finite places?

Notation:

$\Gamma = \pi_1(S^3 \setminus K)$.

If $K \subset S^3$ is a hyperbolic knot and $K(r)$ is hyperbolic, let $\Gamma_r = \pi_1(K(r))$,

k_r the trace field of Γ_r and B_r the quaternion algebra over k_r associated to Γ_r.

$\Delta_K(t)$ denotes the Alexander polynomial of K.

Theorem (Chinburg-R-Stover)

Let K be a hyperbolic knot and suppose that $\Delta_K(t)$ satisfies:

$(\star)$ *for any root z of $\Delta_K(t)$ and w a square root of z, we have an equality of fields: $\mathbb{Q}(w) = \mathbb{Q}(w + w^{-1})$.*

Then there exists a finite set S of rational primes p so that if some prime $\mathcal{P}$ of k_r ramifies B_r then $\mathcal{P}|p$ for some $p \in S$.

When $(\star)$ fails to hold the set S consists of infinitely many primes (of positive density).

Examples

1. When $\Delta_K(t) = 1$ then $S = \emptyset$ and so B_r as above is unramified at all finite places.

2. The figure-eight knot

$\Delta_K(t) = t^2 - 3t + 1$, and so has roots

$$z = \frac{3 \pm \sqrt{5}}{2},$$

and $z = (\pm w)^2$, where

$$w = \frac{1 \pm \sqrt{5}}{2}.$$

Then $w + 1/w = \pm\sqrt{5}$.

So $\mathbb{Q}(w) = \mathbb{Q}(w + w^{-1})$ in this case and $(\star)$ holds.

In this case $S = \{2\}$.

3. The knot $K = 7_4$

$\Delta_K(t) = 4t^2 - 7t + 4.$

$\Delta_K(t)$ has roots

$$z = \frac{7 \pm \sqrt{-15}}{8}.$$

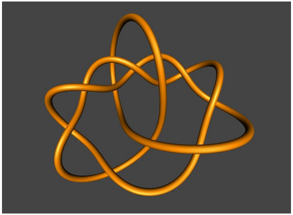
Set $w = \sqrt{z}$.

Then w is a complex number but $w + 1/w = \pm\sqrt{15}/2$ is real.

So in this case condition $(\star)$ does not hold.

The primes in the set S can be identified via congruence conditions obtained from the splitting of primes in the extension $\mathbb{Q}(w = \sqrt{z})$.

4. The $(-2,3,7)$-Pretzel knot



$$\Delta_K(t) = t^{10} - t^9 + t^7 - t^6 + t^5 - t^4 + t^3 - t + 1 = L(-t)$$

where $L(t)$ is Lehmer's polynomial.
So $\Delta_K(t) = 0$ has two (negative) real roots λ and $1/\lambda$ and all other roots lie on the unit circle.
In this case $\mathbb{Q}(w)$ is totally imaginary but $\mathbb{Q}(w + 1/w)$ has a real embedding.

So $(\star)$ does not hold

The $(-2,3,7)$-Pretzel knot is an example of a so-called L-space knot from Heegaard-Floer theory. It seems conjecturally that $(\star)$ never holds for such knots.

Proof involves:

• The theory of Azumaya algebras over curves (organizes the quaternion algebras obtained by Dehn surgery).

• Work of G. De Rham:

Relate roots of the Alexander polynomial to eigenvalues of Non-abelian reducible representations of knot groups.

Introduction aux polynômes d'un noeud, Enseignement Math. **13** (1967).

The End—La Fin—Das Ende