Profinite Rigidity

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Γ a finitely generated group:

$$\mathcal{C}(\Gamma) = \{[A] : \Gamma \text{ surjects onto } A, |A| < \infty\}.$$

Basic Question: *To what extent does $\mathcal{C}(\Gamma)$ determine Γ?*

Without some assumptions it is easy to construct examples where $\mathcal{C}(\Gamma)$ does not determine Γ; e.g.

S a finitely generated infinite simple group and Γ any finitely generated group then $\mathcal{C}(\Gamma) = \mathcal{C}(\Gamma * S)$.

Standing assumptions from here on:

Γ is a discrete group, is finitely generated and residually finite.

i.e. for all nontrivial $\gamma \in \Gamma$, there is a finite group A and a homomorphism $\phi : \Gamma \to A$ so that $\phi(\gamma) \neq 1$.

Examples:

1. $\Gamma < \mathrm{GL}(\mathrm{n}, \mathbb{C})$ (a f.g. subgroup) (Malcev, Selberg)

2. $\Gamma = \pi_1(M)$, M a compact 3-manifold (Perelman, Thurston, Hempel).

Definition: *The genus of* Γ *is the set:*

$$\mathcal{G}(\Gamma) = \{\Lambda : \mathcal{C}(\Lambda) = \mathcal{C}(\Gamma)\}$$

Genus—meant to suggest ”*locally the same but globally .....?*”.

If $\mathcal{P}$ is a class of groups then

$$\mathcal{G}_{\mathcal{P}}(\Gamma) = \{\Lambda \in \mathcal{P} : \mathcal{C}(\Lambda) = \mathcal{C}(\Gamma)\}$$

Examples :

1. Γ a finitely generated abelian group, $\mathcal{G}(\Gamma) = \{\Gamma\}$

2. (G. Baumslag) There exist Γ (virtually $\mathbb{Z}$) with $|\mathcal{G}(\Gamma)| > 1$.

What Baumslag actually proves is the following:

Let F be a finite cyclic group with an automorphism of order n, where n is different from 1, 2, 3, 4 *and* 6.

Then there are at least two non-isomorphic cyclic extensions of F, Γ_1 and Γ_2 with $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$.

A beautiful, and useful observation, that is used in the proof that the constructed groups Γ_1 and Γ_2 lie in the same genus is the following (going back to Hirshon):

Suppose that A and B are groups with $A \times \mathbb{Z} \cong B \times \mathbb{Z}$, then

$$\mathcal{C}(A) = \mathcal{C}(B).$$

3. As is already evident from Baumslag's examples, the case of nilpotent groups already shows some degree of subtlety.

However, the nilpotent case is well understood due to work of Pickel (student of Baumslag in the 1970's).

For a finitely generated nilpotent group Γ, $\mathcal{G}(\Gamma)$ consists of a finite number of isomorphism classes of nilpotent groups.

There are examples where the genus can be made arbitrarily large.

4. Using the Congruence Subgroup Property and some number theory, examples of lattices Γ in certain semisimple Lie groups can be constructed with $|\mathcal{G}(\Gamma)| > 1$ (and always finite).

5. There are examples of word hyperbolic (hence finitely presented) groups Γ with $|\mathcal{G}(\Gamma)|$ infinite (Bridson).

6. Restricting the class of groups can be helpful.

By a Fuchsian group we mean a finitely generated discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$.

Let $\mathcal{L}$ denote the class of lattices in connected Lie groups.

Theorem 1 (Bridson-Conder-R)

Let F be a Fuchsian group, then $\mathcal{G}_{\mathcal{L}}(F) = \{F\}$.

Hard case: Distinguishing between Fuchsian groups.

Let Γ be a finitely generated group (not necessarily residually finite for this discussion), and let $\mathcal{N}$ denote the collection of all finite index normal subgroups of Γ.
Note that $\mathcal{N}$ is non-empty as $\Gamma \in \mathcal{N}$, and we can make $\mathcal{N}$ into directed set by declaring that

$$\text{For } M, N \in \mathcal{N}, M \leq N \text{ whenever } M \textit{ contains } N.$$

In this case, there are natural epimorphisms $\phi_{NM} : \Gamma/N \to \Gamma/M$.
The inverse limit of the inverse system $(\Gamma/N, \phi_{NM}, \mathcal{N})$ is denoted $\widehat{\Gamma}$ and defined to be to the profinite completion of Γ.

There is a natural map $\iota : \Gamma \to \widehat{\Gamma}$ defined by

$$g \mapsto (gN) \in \varprojlim \Gamma/N$$

ι is 1-1 if and only if Γ is residually finite.

Definition: Say Γ is profinitely rigid if whenever $\widehat{\Lambda} \cong \widehat{\Gamma}$ then $\Lambda \cong \Gamma$.

if and only if $\mathcal{G}(\Gamma) = \{\Gamma\}$.

Main Focus: Profinite rigidity and low-dimensional topology

Perhaps the most basic example is the following that goes back to Remeslennikov and remains open:

Question 1

Let F_n be the free group of rank $n \geq 2$. Is F_n profinitely rigid?

The group F_n arises in many guises in low-dimensional topology and affords several natural ways to generalize.

The following are natural generalizations of Question 1 (which remain open):

Question 2

Let Σ_g be a closed orientable surface of genus $g \geq 2$. Is $\pi_1(\Sigma_g)$ profinitely rigid?

As we will discuss, profinite rigidity in the setting of 3-manifold groups is different, however, one generalization that we will focus on is:

Question 3

Let M be a complete orientable hyperbolic 3-manifold of finite volume. Is $\pi_1(M)$ profinitely rigid?

Caution: There are 3-manifold groups that are not profinitely rigid.

Examples:(Funar)

Torus bundles with SOLV geometry arise as the mapping torus of a self-homeomorphism $f : T^2 \to T^2$ which can be identified with an element of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ with $|a + d| > 2$.
Funar shows that for any $m \geq 2$ there exist m torus bundles admitting *SOLV* geometry whose fundamental groups have isomorphic profinite completions although they are pairwise non-isomorphic.

Examples:(Hempel)

Let $f : S \to S$ be a periodic, orientation-preserving homeomorphism of a closed orientable surface S of genus at least 2, and let k be relatively prime to the order of f.

Let M_f (resp. M_{f^k}) denote the mapping torus of f (resp. f^k), and let $\Gamma_f = \pi_1(M_f)$ (resp. $\Gamma_{f^k} = \pi_1(M_{f^k})$).

Hempel shows that $\widehat{\Gamma}_f \cong \widehat{\Gamma}_{f^k}$ by proving that $\Gamma_f \times \mathbb{Z} \cong \Gamma_{f^k} \times \mathbb{Z}$ (c.f. the example of Baumslags).

Steps towards profinite rigidity

Geometrization from profinite completion: Seeing geometry from finite quotients

Theorem 2 (Wilton-Zalesskii)

Let $\mathcal{M}$ denote the class of fundamental groups of compact 3-manifolds.
Let M be a closed orientable 3-manifold with infinite fundamental group admitting one of Thurston's eight geometries and let $\pi_1(N) \in \mathcal{M}$ with $\pi_1(N) \in \mathcal{G}_{\mathcal{M}}(\pi_1(M))$. Then N is closed and admits the same geometric structure.

Important in this work and many other recent developments on profinite rigidity in 3-manifold groups is the work of Agol and Wise.

As an example of this:

Theorem 3 (Bridson-R-Wilton)

Let M be a 1-punctured torus bundle over the circle. Then $\mathcal{G}_{\mathcal{M}}(\pi_1(M)) = \{\pi_1(M)\}$.

Some comments on the proof:

1. If $\pi_1(N) \in \mathcal{M}$ with $\pi_1(N) \in \mathcal{G}(\pi_1(M))$, then N is fibered. (uses Agol and Wise).

2. We have

$$1 \to F \to \pi_1(M) \to \mathbb{Z} \to 1 \text{ and } 1 \to G \to \pi_1(N) \to \mathbb{Z} \to 1,$$

where F is a free group of rank 2 and G is some free group (from the fibering in 1).

(main case is when M is hyperbolic and so in this case $b_1(M) = 1$).

Passing to profinite completions:

$$1 \to \widehat{F} \to \widehat{\pi_1(M)} \to \widehat{\mathbb{Z}} \to 1 \text{ and } 1 \to \widehat{G} \to \widehat{\pi_1(N)} \to \widehat{\mathbb{Z}} \to 1.$$

Left exact comes from the fact that the full profinite topology is induced on F and G.

We know $b_1(M) = b_1(N) = 1$ and $\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)}$.

Hence there is a unique homomorphism to $\widehat{\mathbb{Z}}$ and so $\widehat{F} \cong \widehat{G}$.

Hence $F \cong G$.

Now reduces to analyzing fundamental groups of 1-punctured torus bundles.

This uses properties of $\mathrm{SL}(2, \mathbb{Z})$ viewed as the Mapping Class group of the 1-punctured torus.

Profinite rigidity and hyperbolic geometry

Progress on Question 3 and a step in the right direction on Question 1.

Theorem 4 (Bridson-McReynolds-R-Spitler)

1. *There are profinitely rigid (arithmetic) Kleinian groups.* *These include* $\mathrm{PGL}(2, \mathbb{Z}[\omega])$, $\mathrm{PSL}(2, \mathbb{Z}[\omega])$ *(where* $\omega^2 + \omega + 1 = 0$*),* $\pi_1(M_W)$ *where* M_W *is the Weeks manifold.*
2. *There are profinitely rigid (arithmetic) triangle groups.* *These include* $\Delta(3,3,4)$, $\Delta(2,3,8)$ *and 14 more.*

Remark: We cannot yet handle $\Delta(2,3,7)$.

Some ideas in the proof of Theorem 4

For simplicity let $\Gamma = \mathrm{PSL}(2, \mathbb{Z}[\omega])$.

There are three key steps in the proof.

Theorem 5 (Representation Rigidity)

Let $\iota : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ denote the identity homomorphism, and $c = \bar{\iota}$ the complex conjugate representation. Then if $\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ is a representation with infinite image, $\rho = \iota$ or c.

This is the only Bianchi group $\mathrm{PSL}(2, \mathrm{O_d})$ with this kind of ”representation rigidity”.

Using Theorem 5 we are able to get some control on $\mathrm{PSL}(2, \mathbb{C})$ representations of a finitely generated residually finite group with profinite completion isomorphic to $\widehat{\Gamma}$.

Theorem 6

Let Δ be a finitely generated residually finite group with $\widehat{\Delta} \cong \widehat{\Gamma}$. Then Δ admits an epimorphism to a group $L < \Gamma$ which is Zariski dense in $(\mathrm{P})\mathrm{SL}(2, \mathbb{C})$.

Theorem 6 is proved by patching together local representations and holds in a fairly general setting.

The key point now is in the context of Kleinian groups, we can make use of Theorem 6, in tandem with an understanding of the topology and deformations of orbifolds $\mathbb{H}^3/G$ for subgroups $G < \Gamma$.

Briefly, in the notation of Theorem 6, L has infinite index or finite index.

1. L has infinite index:

   Use Teichmüller theory to construct explicit finite quotients of L and hence Δ that cannot be finite quotients of Γ.

2. L has finite index:

   We make use of an understanding of low-index subgroups of Γ, together with the construction of L, and 3-manifold topology to show:

   L contains the fundamental group of a once-punctured torus bundle over the circle of index 12.

We can then make use of Theorem 3 (profinite rigidity of 1-punctured torus bundles amongst 3-manifold groups) to show $L = \Gamma$.

Final Remarks

1. Theorem 6 holds in more generality given some degree of "representation rigidity".

e.g. for $\mathrm{SL}(3,\mathbb{Z})$.

Run the above argument gives an epimorphism from Δ (fake $\mathrm{SL}(3,\mathbb{Z})$) into $\mathrm{SL}(3,\mathbb{Z})$.

What do f.g. infinite index subgroups of $\mathrm{SL}(3,\mathbb{Z})$ look like?

2. Why cant we handle the $(2,3,7)$ triangle group?

The argument gives either:

Δ admits an epimorphism onto a subgroup of $(2,3,7)$, or

Δ admits an epimorphism onto a subgroup of $\mathrm{PSL}(2,\mathrm{R_k})$ where R_k is the ring of integers in $k=\mathbb{Q}(\cos\pi/7)$.

As before (like $\mathrm{SL}(3,\mathbb{Z})$ we know nothing about the structure of f.g. infinite index subgroups of $\mathrm{PSL}(2,\mathrm{R_k})$.

3. Let $M = \mathbb{H}^3/\Gamma$ be a finite volume hyperbolic 3-manifold. Suppose $N = \mathbb{H}^3/\Lambda$ with $\widehat{\Gamma} \cong \widehat{\Lambda}$.

Can we show $\mathrm{Vol(M)} = \mathrm{Vol(N)}$?

There does appear to some conjectural evidence to support a positive answer.

It is conjectured that if $\{N_m\}$ is a cofinal sequence of subgroups of finite index in Γ, then:

$$\frac{\log |\mathrm{Tor}(\mathrm{H}_1(\mathrm{N_m}, \mathbb{Z}))|}{[\Gamma : N_m]} \to \frac{1}{6\pi}\mathrm{Vol(M)} \text{ as } \mathrm{n} \to \infty.$$

$\mathrm{Tor}(\mathrm{H}_1(\mathrm{N_m}, \mathbb{Z}))$ is visible in the profinite completions $\widehat{N}_m$