EMBEDDING CLOSED HYPERBOLIC 3-MANIFOLDS IN SMALL VOLUME HYPERBOLIC 4-MANIFOLDS

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Abstract. In this paper we study existence and lack thereof of closed embedded orientable co-dimension one totally geodesic submanifolds of minimal volume cusped orientable hyperbolic manifolds.

1. Introduction

Let $W^n$ denote a minimal volume, orientable, cusped hyperbolic $n$-manifold. In this paper we will be concerned with the existence of closed embedded orientable totally geodesic hyperbolic submanifolds $M^{n-1} \hookrightarrow W^n$.

When $n = 2$, $W^2$ is either the thrice-punctured sphere or a once-punctured torus, and in the former case there are no such submanifolds, whilst in the latter there are many. In dimension 3, by work of Cao and Meyerhoff, [CM], the manifolds in question are the complement of the figure-eight knot and its sister. These do not contain any closed embedded orientable totally geodesic surfaces (although they do contain infinitely many immersed closed totally geodesic surfaces [MR, Chapter 9]). Indeed, they do not contain any closed embedded essential surfaces (see [Th], [CJR] and [FH]). These 3-manifolds are arithmetic hyperbolic 3-manifolds, as is the case of the thrice-punctured sphere. The once-punctured torus has a positive dimensional Teichmüller space, but there are a finite number of examples which are arithmetic.

This paper is concerned with the following conjecture. As we discuss in Section 2.2, Conjecture 1.1 is easily reduced to the only non-trivial case, that of dimension 4.

Conjecture 1.1. Let $W^n$ denote a minimal volume, orientable, cusped arithmetic hyperbolic $n$-manifold. If $W^n$ contains a co-dimension one, closed embedded orientable totally geodesic submanifold, then $n = 2$ and $W^2$ is an arithmetic once-punctured torus.

In [RT] the authors provide a census of 1171 so-called integral congruence two hyperbolic 4-manifolds that are all obtained from face-pairings of the ideal 24-cell in $\mathbb{H}^4$. These are all commensurable cusped, arithmetic, hyperbolic 4-manifolds of Euler characteristic 1 (i.e. minimal volume). Amongst these, only 22 are orientable, and these are listed in Section 8 together with some information that we will make use of. Towards a positive resolution of Conjecture 1.1 we prove the following result.

Theorem 1.2. Let $W$ denote one of the 22 manifolds mentioned above. Then $W$ does not contain a closed embedded orientable totally geodesic hyperbolic 3-manifold.
As remarked upon, the 1171 integral congruence two hyperbolic 4-manifolds are all commensurable. In a private communication J. Ratcliffe and S. Tschantz have informed us that there are many more manifolds obtained by side-pairings of the ideal 24-cell in this commensurability class, namely 13, 108 side-pairings of the ideal 24-cell (up to symmetry of the 24-cell) yield a cusped hyperbolic 4-manifold of Euler characteristic 1. Only 675 of these side-pairings provide orientable manifolds (which include the 22 orientable ones of Theorem 1.2). In addition, in [RiSl] it is shown that there is at least one more commensurability class of cusped arithmetic hyperbolic 4-manifolds that contains an orientable cusped hyperbolic 4-manifold with Euler characteristic 1.

At present we cannot say anything about Conjecture 1.1 for these other orientable examples, nor do we have a classification of the finite number of commensurability classes of cusped arithmetic hyperbolic 4-manifolds that contain a manifold with Euler characteristic 1 (although in principle this should be doable).

Our methods also apply to another situation. In [I] the author provides an example of a cusped, orientable, hyperbolic 4-manifold of Euler characteristic 2 that is the complement of five 2-tori in $S^4$ (with the standard smooth structure, [I2]). This link complement arises as the orientable double cover of the non-orientable manifold 1011 in the census of integral congruence two hyperbolic 4-manifolds mentioned above (see also [I]). We prove the following result.

**Theorem 1.3.** Let $W$ be the link complement in $S^4$ described above. Then $W$ contains an embedded orientable totally geodesic cusped hyperbolic 3-manifold isometric to the complement of the link $S_3^0$ (shown in Figure 1) in $S^3$, but no closed embedded orientable totally geodesic hyperbolic 3-manifold.

A simple, but elegant argument (see [I0, Proposition 4.10]) shows that if $X$ is a hyperbolic link complement of 2-tori in $S^4$, then $\chi(X) = \chi(S^4) = 2$, and so there are only finitely many hyperbolic link complements of 2-tori in $S^4$. This statement holds more generally for link complements of 2-tori and Klein bottles in other fixed 4-manifolds.

By way of comparison, Thurston’s hyperbolization theorem shows many links in $S^3$ have hyperbolic complements, and although it is known that many hyperbolic link complements in $S^3$ do not contain a closed embedded totally geodesic surface (e.g. alternating links [MeR]), examples do exist (see [Lei] and [MeR]).

It is also the case that in dimension 4, there are obstructions to smoothly embedding closed orientable 3-manifolds in $S^4$; for example it is a result of Hantzsche [Han] that if a closed orientable 3-manifold $M$ embeds in $S^4$, then $\text{Tor}(H_1(M,\mathbb{Z})) \cong A \oplus A$ for some finite abelian group $A$. In fact, the Kirby Problem List Question 3.20 [Kir] asks: *Under what conditions does a closed, orientable 3-manifold $M$ smoothly embed in $S^4$?* We refer the reader to [BB] for examples, for more discussion of this question, and additional references.

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2. Cusped arithmetic hyperbolic manifolds

We will mainly work with the hyperboloid model of $\mathbb{H}^n$ defined using the quadratic form $j_n$ defined as $x_0^2 + x_1^2 + \ldots x_{n-1}^2 - x_n^2$; i.e.

$$\mathbb{H}^n = \{ x \in \mathbb{R}^{n+1} : j_n(x) = -1, x_n > 0 \}$$

equipped with the Riemannian metric induced from the Lorentzian inner product associated to $j_n$. The full group of isometries of $\mathbb{H}^n$ is then identified with $O^+(n,1)$, the subgroup of $O(n,1) = \{ A \in GL(n+1, \mathbb{R}) : A^t J_n A = J_n \}$ preserving the upper sheet of the hyperboloid $j_n(x) = -1$, and where $J_n$ is the symmetric matrix associated to the quadratic form $j_n$. The full group of orientation-preserving isometries is given by $SO^+(n,1) = \{ A \in O^+(n,1) : \det(A) = 1 \}$.

2.1. Constructing cusped arithmetic hyperbolic manifolds. Cusped, arithmetic, hyperbolic $n$-manifold are constructed as follows (see [VS] for example). Suppose that $X = \mathbb{H}^n/\Gamma$ is a finite volume cusped hyperbolic $n$-manifold. Then $X$ is arithmetic if $\Gamma$ is commensurable with a group $\Lambda < SO^+(n,1)$ as described below.

Let $f$ be a non-degenerate quadratic form defined over $\mathbb{Q}$ of signature $(n,1)$, which we can assume is diagonal and has integer coefficients. Then $f$ is equivalent over $\mathbb{R}$ to the form $j_n$ defined above; i.e. there exists $T \in GL(n+1, \mathbb{R})$ such that $T^t F T = J_n$, where $F$ and $J_n$ denote the symmetric matrices associated to $f$ and $j_n$ respectively. Then $T^{-1} SO(f, \mathbb{Z}) T \cap SO^+(n,1)$ defines the arithmetic subgroup $\Lambda < SO^+(n,1)$.

Note that the form $f$ is anisotropic (i.e. does not represent 0 non-trivially over $\mathbb{Q}$) if and only if the group $\Gamma$ is cocompact, otherwise the group $\Gamma$ is non-cocompact (see [BHCh]). By Meyer’s Theorem [Se, §IV.3.2, Corollary 2], the case that $f$ is anisotropic can only occur when $n = 2, 3$.

2.2. Reducing Conjecture 1.1 to dimension 4. We include a quick proof of the following, likely well-known, result. Clearly Theorem 2.1 then reduces Conjecture 1.1 to dimension 4.

Theorem 2.1. Let $X = \mathbb{H}^n/\Gamma$ be a cusped, arithmetic, hyperbolic $n$-manifold with $n \geq 5$. Then $X$ does not contain any co-dimension one, immersed, closed, totally geodesic hyperbolic manifold. On the other hand it contains infinitely many co-dimension one, immersed, cusped, totally geodesic hyperbolic manifolds.

Proof. The first part now follows easily from the discussion in Section 2.1, since any co-dimension one immersed totally geodesic hyperbolic manifold is arithmetic arising from a quadratic form of signature $(n-1,1)$. Since $n \geq 5$, such a form is isotropic by Meyers theorem, and so the submanifold in question is non-compact.

For the last claim we argue as follows. Assume that $f$ is a diagonal form defined over $\mathbb{Q}$ of signature $(n,1)$, $n \geq 5$. Then, we can restrict to a sub-quadratic form $f_1$ of signature $(n-1,1)$, which by hypothesis is isotropic. Using Section 2.1 we can use $f_1$ to build a non-cocompact subgroup $H < \Gamma$ with $H$ an arithmetic subgroup of $O^+(n-1,1)$. Now use the density of the commensurator of $\Gamma$ to construct infinitely many such groups $H$. □
3. Some background from [RT]

3.1. Integral congruence two hyperbolic 4-manifolds. For convenience we now set $J = j_4$. The manifolds of Theorem 1.2 are all obtained by face-pairings of the regular ideal 24-cell in $\mathbb{H}^4$ (with all dihedral angles $\pi/2$), and arise as regular $(\mathbb{Z}/2\mathbb{Z})^4$ covers of the orbifold $\mathbb{H}^4/\Gamma(2)$ where $\Gamma(2)$ is the level two congruence subgroup of the group $O^+(J, \mathbb{Z}) = O^+(4, 1) \cap O(J, \mathbb{Z})$. These are the manifolds referred to as integral congruence two hyperbolic 4-manifolds in [RT].

It will be useful to describe the $(\mathbb{Z}/2\mathbb{Z})^4$ action, and this is best described in the ball model as follows. Locate the 24-cell in the ball model of hyperbolic space with vertices $(\pm 1, 0, 0, 0), (0, \pm 1, 0, 0), (0, 0, \pm 1, 0), (0, 0, 0, \pm 1)$ and $(\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2)$.

The four reflections in the co-ordinate planes of $\mathbb{R}^4$ can be taken as generators of this $(\mathbb{Z}/2\mathbb{Z})^4$ group of isometries. Passing to the hyperboloid model, these reflections are elements of $\Gamma(2)$ and are listed as the first four matrices in [RT, page 110]. Following [RT] we denote this $(\mathbb{Z}/2\mathbb{Z})^4$ group of isometries by $K < \Gamma(2)$. Note from [RT, Table 2], only one of the 22 examples under consideration admits a larger group of isometries (of order 48) than that given by $K$. In particular, none of these 22 manifolds are regular covers of the orbifold $\mathbb{H}^4/O^+(J, \mathbb{Z})$ (since $[O^+(J, \mathbb{Z}) : \Gamma(2)] = 120$).

As noted in [RT] (see also [RT0]) all of the face-pairings of any of the integral congruence two hyperbolic 4-manifolds are invariant under the group $K$. This implies that each of the coordinate hyperplane cross sections of the 24-cell extends in each of the integral congruence two hyperbolic 4-manifolds to a totally geodesic hypersurface which is the fixed point set of one of the reflections described above. Following [RT0] we call these hypersurfaces, cross sections. As described in [RT0], these cross sections, can be identified with integral congruence two hyperbolic 3-manifolds which are also described in [RT]. Moreover, it is possible to use [RT] to identify these explicitly in any given example. The following can be deduced from [RT] or [RT0]

Lemma 3.1. Any orientable cross-section is isometric to one of the complement in $S^3$ of the link $6_2^3$ (the Borromean rings), the link $8_0^3$, or the link $8_1^4$ (see Figure 1).

Proof. Each of the 3-dimensional cross sections must be isometric to one of the 107 possibilities encoded in [RT, Page 115]. However, these 107 are classified into 13 equivalence classes corresponding to isometry classes of the corresponding 3-manifolds [RT, Theorem 5], out of which only 3 are orientable 3 manifolds. The 3 orientable possibilities are described in [RT, pages 108-109] and are the complement in $S^3$ of the link $6_2^3$, the link $8_0^3$ or the link $8_1^4$. \hfill \Box

3.2. More about the links $6_2^3$, $8_0^3$, and $8_1^4$. Let $L$ denote one of the links $6_2^3$, $8_0^3$ or $8_1^4$. Note that the complements of these links share the same 3-dimensional hyperbolic volume which is approximately 7.3277247... .

Lemma 3.2. $S^3 \setminus L$ does not contain a closed embedded totally geodesic surface.

Proof. It is shown in [Loz] that the complement of the Borromean rings is small (i.e. they do not contain any closed embedded essential surface), and in particular does not contain a closed embedded totally geodesic surface.
The link $8_2^4$ is the Montesinos link $K\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$ (in the notation of [Oe]) and so [Oe, Theorem 1] implies that any closed embedded essential surface in the complement of $8_2^4$ arises from tubing the obvious 4-punctured spheres separating pairs of tangles. In particular such a surface carries an accidental parabolic, and so cannot be totally geodesic. Indeed, in this case, it can be shown that in fact $8_2^4$ is small, as these tubed surfaces compress.

Now consider the case of $L = 8_3^9$, and suppose that $S\setminus L$ contains a closed embedded totally geodesic surface $S$. Trivial filling on the component $J$ of Figure 1 provides a split link, and hence $S$ compresses in this filling. In addition, it can be checked that $\pm 1$ filling yields a manifold homeomorphic to the complement of the Whitehead link, which being a 2-bridge link, does not contain any closed embedded essential surface [HT]. Thus $S$ compresses in both $\pm 1$ filling. Now the surface $S$ is totally geodesic (so does not carry an accidental parabolic element), and so an application of [Wu, Theorem 1] (following [CGLS]) provides a contradiction since the slopes $\pm 1$ have distance 2. □

4. Volume from tubular neighbourhoods

We will make use of a result of Basmajian [Bas] which provides disjoint collars about closed embedded orientable totally geodesic hypersurfaces in hyperbolic manifolds. We state this only the case of interest, namely for hyperbolic 4-manifolds.

Following [Bas], let $r(x) = \log \coth(x/2)$, let $V(r)$ denote the volume of a ball of radius $r$ in $\mathbb{H}^3$ and denote by $c_4(A) = \left(\frac{1}{2}\right)(V \circ r)^{-1}(A)$, defined as the 4-dimensional tubular neighbourhood function in [Bas]. It is noted in [Bas] that, $V(r) = \omega_3 \int_0^r \sinh^2(r)dr$, where $\omega_3$ is the area of the unit sphere in $\mathbb{R}^3$ (i.e. $\omega_3 = 4\pi$).

From [Bas, pages 213–214], we can compute the volume of a tubular neighbourhood of a closed embedded orientable totally geodesic hyperbolic 3-manifold of 3-dimensional hyperbolic volume $A$ in a hyperbolic manifold as follows.

**Lemma 4.1.** Let $X$ be an orientable finite volume hyperbolic 4-manifold containing a closed embedded orientable totally geodesic hyperbolic 3-manifold of 3-dimensional hyperbolic volume $A$. Then $X$ contains a tubular neighbourhood of $M$ of volume

$$ \mathcal{V}(A) = 2A \int_0^{c_4(A)} \cosh^3(t)dt $$
As noted in [Bas, Remark 2.1], when the totally geodesic submanifold separates, an improved estimate is obtained using the tubular neighbourhood function

\[ d_4(A) = \frac{1}{2}(V \circ r)^{-1}(A/2) \]

resulting in a formula

\[ V'(A) = 2A \int_0^{d_4(A)} \cosh^3(t) dt. \]

Moreover, [Bas] also proves that disjoint embedded closed orientable totally geodesic hyperbolic 3-manifolds in an orientable finite volume hyperbolic 4-manifold have disjoint collars, thereby contributing additional volume. For our purposes we summarize what we need in the following.

**Corollary 4.2.** Let \( X \) be an orientable finite volume hyperbolic 4-manifold of euler characteristic \( \chi \) containing \( K \) disjoint copies of a closed embedded orientable totally geodesic hyperbolic 3-manifold of 3-dimensional hyperbolic volume \( A \).

1. \( \text{Vol}(X) = \left( \frac{4\pi^2}{3} \right) \chi \geq K V(A) \).
2. If all of these submanifolds manifolds separate in \( X \), then

\[ \text{Vol}(X) = \left( \frac{4\pi^2}{3} \right) \chi \geq K V'(A). \]

Finally, as noted in [Bas], the assumption that the submanifold be closed can be weakened with some additional hypothesis. For example, the following can be proved using the method of proof of [Bas, Theorem 1.2].

**Lemma 4.3.** Let \( X \) be an orientable finite volume hyperbolic 4-manifold containing \( K \) disjoint closed embedded orientable totally geodesic hyperbolic 3-manifolds of 3-dimensional hyperbolic volume \( A \) that are all disjoint from an embedded orientable totally geodesic cusped hyperbolic 3-manifold of 3-dimensional hyperbolic volume \( B \). Then \( \text{Vol}(X) \geq K V(A) + V(B) \).

5. Co-dimension one closed totally geodesic submanifolds in cusped arithmetic hyperbolic 4-manifolds

In dimension 3, any cusped arithmetic hyperbolic 3-manifold contains infinitely many immersed closed totally geodesic surfaces (see [MR, Chapter 9]). In this section, we show that the situation in dimension 4 is similar, providing a contrast with Theorem 2.1 in dimensions \( \geq 5 \).

5.1. Immersed closed totally geodesic hyperbolic 3-manifolds in integral congruence two hyperbolic 4-manifolds. We first show that the integral congruence two hyperbolic 4-manifolds of [RT] all contain many immersed closed totally geodesic hyperbolic 3-manifolds (indeed any manifold in the commensurability class of these integral congruence two hyperbolic 4-manifolds). To that end, let \( p \equiv -1 \pmod{8} \) be a prime, and \( q_p \) the quadratic form (over \( \mathbb{Q} \)) given by \( x_1^2 + x_2^2 + x_3^2 - px_4^2 \). The congruence condition on \( p \) implies that this form is anisotropic, and so as in Section 2.1, the group \( \text{SO}(q_p, \mathbb{Z}) \) determines a co-compact arithmetic lattice in \( \text{SO}^+(3,1) \).

**Proposition 5.1.** With \( p \) as above, and given any integral congruence two hyperbolic 4-manifold \( N \), there is a finite index subgroup \( \Lambda_N < \text{SO}(q_p, \mathbb{Z}) \) so that \( \mathbb{H}^3/\Lambda_N \hookrightarrow \mathbb{H}^4/N \) is an immersed closed totally geodesic hyperbolic 3-manifold. In particular any such \( N \) contains...
infinitely many commensurability classes of immersed closed totally geodesic hyperbolic 3-manifolds.

Proof. The proof of the first claim will follow using standard arguments on equivalences of quadratic forms over \( \mathbb{Q} \) yielding commensurable arithmetic lattices. In particular, we need to show that the quadratic form \( q_p \perp <p> \) is equivalent to the form \( J \) of Section 3.1 over \( \mathbb{Q} \) (see [ALR, \S 5,6] for example). In this case, the equivalence can be seen directly as follows.

Let \( p = 8k - 1 \), and let
\[
A_p = \begin{pmatrix} 4k & 4k - 1 \\ 4k - 1 & 4k \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad D_p = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}
\]

A simple calculation shows that \( A_pDA_p^t = D_p \), and from this the required equivalence can be deduced.

The second part follows from the fact that there are infinitely many primes \( \equiv -1 \pmod{8} \), and as noted above, all these quadratic forms being anisotropic over \( \mathbb{Q} \) provide closed hyperbolic 3-manifolds. \( \square \)

5.2. **Immersed closed totally geodesic hyperbolic 3-manifolds in arithmetic hyperbolic 4-manifolds.** In this section we prove that the conclusion of Proposition 5.1 holds much more broadly for cusped arithmetic hyperbolic 4-manifolds.

**Theorem 5.2.** Let \( W \) be an arithmetic hyperbolic 4-manifold. Then \( M \) contains infinitely many commensurability classes of immersed closed totally geodesic hyperbolic 3-manifolds.

We begin with some preliminaries on non-degenerate diagonal quaternary quadratic forms
\[
f_p = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2
\]
over the local field \( \mathbb{Q}_p \) for \( p \) a prime or \( p = -1 \), with the understanding that \( \mathbb{Q}_{-1} = \mathbb{R} \).

Let \( c_p(f) \) denote the Hasse-Minkowski invariant which is defined as
\[
c_p(f) = \prod_{i<j}(a_i, a_j)_p
\]
where \((\cdot, \cdot)_p\) denotes the Hilbert symbol. This invariant depends only on the equivalence class of \( f \) and not on the choice of orthogonal basis.

We collect some useful statements about Hilbert symbols and quadratic forms over local fields taken from [Se]. Throughout \((\frac{u}{p})_p = (u, p)_p\) denotes the Legendre symbol, which as in [Se, II.3.3], is extended to be defined for \( u \in \mathbb{Z}_p^* \).

**Lemma 5.3.** (a) If \( p \neq 2 \), the image of the integer \( x = p^n u \) is a square in \( \mathbb{Q}_p^* \) if and only if \( n \) is even and \((\frac{u}{p})_p = 1 \) [Se, \S II.3.3, Theorem 3, page 17].
(b) The image of the integer \( x = 2^n u \) is a square in \( \mathbb{Q}_2^* \) if and only if \( n \) is even and \( u \equiv 1 \pmod{8} \) [Se, II.3.3, Theorem 4, page 18].
(c) The Hilbert symbol satisfies the formulas [Se, III.1.1, Proposition 2, page 19]
\[
\begin{align*}
(i) \quad (a, b)_p &= (b, a)_p \\
(ii) \quad (a, b^2)_p &= 1 \\
(iii) \quad (a, -a)_p &= 1. \\
(iv) \quad (-1, -1)_p &= \begin{cases} -1 & \text{if } p = -1, 2 \\ 1 & \text{if } p \text{ is odd.} \end{cases}
\end{align*}
\]
(d) If \(a = p^\alpha u\) and \(b = p^\beta v\) then 
\[
(a, b)_p = \begin{cases} 
(-1)^{\alpha + \beta}(u \cdot v)^\alpha (p) \cdot (p) \beta & \text{if } p \neq 2 \\
(-1)^{\epsilon(u)\epsilon(v) + \omega(u) + \omega(v)} & \text{if } p = 2 
\end{cases}
\]
where \(\epsilon(u)\) denotes the class modulo 2 of \(\frac{u^2 - 1}{2}\) and \(\omega(u)\) denotes the class modulo 2 of \(\frac{u^2 - 1}{8}\) \cite{Se, III.1.2, Theorem 1, page 20}.

(e) By Dirichlet’s Theorem, if \(a\) and \(m\) are relatively prime positive integers, there exists infinitely many primes \(q\) such that \(p \equiv a \mod m\) see \cite{Se, III.2.2, Lemma 3 page 25}.

(f) A quadratic form \(f\) over \(\mathbb{Q}_p\) is anisotropic if and only if its determinant \(d(f)\) is in \((\mathbb{Q}_p^\star)^2\) and \(c_p(f) = -(-1, -1)_p\) \cite{Se, IV.2.2, Theorem 6, page 36}.

(g) By the Hasse principle, a quadratic form \(f\) over \(\mathbb{Q}\) is anisotropic if there is some prime \(p\) for which the local form \(f_p\) over \(\mathbb{Q}_p\) is anisotropic \cite{Se, IV.3, Theorem 8, page 41}.

Proof of Theorem 5.2. Let \(W = \mathbb{H}^4/\Lambda\) so that \(\Lambda\) is an arithmetic group commensurable to a group \(\text{SO}(f, \mathbb{Z})\) for some non-degenerate quadratic form \(f\) defined over \(\mathbb{Q}\) of signature \((4, 1)\). According to \cite{Mon, Theorems 6, 8}, the commensurability class of \(\text{SO}(f, \mathbb{Z})\) is uniquely determined by the projective equivalence class of \(f\), which in turn, is itself determined by an invariant \(S\) which is a product of \(s\) many distinct odd primes, and another invariant \(e_{-1}(F)\) (which we will not define here). In our case because every \(f\) has signature \((4, 1)\), this invariant is always 2 and so can be ignored. Hence the projective equivalence class of \(f\) is completely determined by \(S\).

Using \cite{Mon, Theorem 10} (actually Claim 2 of the proof of Theorem 10), we may take \(f\) to be the diagonal form (with basis \(\{e_0, e_1, e_2, e_3, e_4\}\)):

\[
(2) \quad f = \begin{cases} 
(-1, 1, 1, aS, a) & \text{if } S \equiv 1 \mod 4 \\
(1, 1, 1, aS, -a) & \text{if } S \equiv -1 \mod 4 
\end{cases}
\]

where \(a\) is an odd prime such that \(a \nmid S\), \(a \equiv (-1)^S \mod 4\) if \(S \equiv 1 \mod 4\) or \(a \equiv (-1)^{S+1} \mod 4\) if \(S \equiv -1 \mod 4\), and \((\frac{-a}{p}) = -1\) for all \(p \mid S\). The proof will be completed as a consequence of Lemmas 5.4 and 5.5 stated and proved below. \(\square\)

Lemma 5.4. Suppose \(f = (-1, 1, 1, aS, a)\) with \(S \equiv 1 \mod 4\) and \(a \nmid S\), \(a \equiv (-1)^S \mod 4\) with \((\frac{-a}{p}) = -1\) for all \(p \mid S\) as in Equation (2). Then \(f\) contains infinitely many projectively inequivalent anisotropic quadratic subforms over \(\mathbb{Q}\) of signature \((3, 1)\).

Proof. Suppose first that \(s\) is even, so from the description of \(a\) given above, \(a \equiv 1 \mod 4\). By Lemma 5.3(e) there exists infinitely many odd primes \(q\) such that \(q \equiv -S \mod 8\). Let \(u = (\frac{q+1}{2})e_0 + (\frac{q-1}{2})e_1\) so \(f(u) = -q\). Then the diagonal form \(f' = (-q, 1, aS, a)\) is a subform of \(f\) with orthogonal basis \(\{u, e_2, e_3, e_4\}\).

Over the local field \(\mathbb{Q}_2\), the determinant \(d(f') = -qa^2S\) is in \((\mathbb{Q}_2^\star)^2\) since \(q\) was chosen so that \(-qS \equiv 1 \mod 8\) (see Lemma 5.3(b)). Using Lemma 5.3(c), \(c_2(f')\) simplifies to \((-q, S)^2(a, -S)^2\). However, since \(S \equiv a \equiv 1 \mod 4\), it follows from Lemma 5.3(d) that \(c_2(f') = 1 = -(-1, -1)^2\). Therefore \(f'\) is anisotropic over \(\mathbb{Q}_2\) and thus also over \(\mathbb{Q}\). Since different choices of \(q\) yield projectively inequivalent forms, we get infinitely many \(\mathbb{Q}\)-inequivalent anisotropic quadratic subforms of signature \((3, 1)\).

Now suppose that \(s\) is odd, so \(a \equiv -1 \mod 4\). Since \(s\) is odd, and \(S \equiv 1 \mod 4\) we can find a prime \(p | S\) with \(p \equiv 1 \mod 4\). Let \(u = (\frac{q+1}{2})e_0 + (\frac{q-1}{2})e_1\) so \(f(u) = -S\). Let
Since \( \{ f \} \) is a subform of \( f \) with orthogonal basis \( \{ u, v, e_3, e_4 \} \). Over the local field \( \mathbb{Q}_p \), the determinant \( \det(f') = -a^2S^2(S + m^2) \) is in \( (\mathbb{Q}_p)^2 \) since \( \left( \frac{-1}{p} \right) = 1 \) for \( p \equiv 1 \mod 4 \) (see Lemma 5.3(a)). As in the previous case, using Lemma 5.3(c), \( c_p(f') \) simplifies to

\[
c_p(f') = (-S, aS)p(-S, a)_p(aS, a)_p = (-S, a)_p(-S, a)_p = (-S, a)_p.
\]

Now since \( p|S, p \equiv 1 \mod 4 \) and \( \left( \frac{-2}{p} \right) = -1 \), Lemma 5.3(d) implies that \( (-S, a)_p = -1 = -(1, -1)_p \). Therefore \( f' \) is anisotropic over \( \mathbb{Q}_p \) and thus also over \( \mathbb{Q} \). Since different choices of \( m \) yield projectively inequivalent forms, the conclusion follows as before. \( \square \)

**Lemma 5.5.** Suppose \( f = \langle 1, 1, 1, aS, -a \rangle \) with \( S \equiv -1 \mod 4 \) and \( a \nmid S, a \equiv (-1)^{4+1} \mod 4 \) with and \( \left( \frac{-2}{p} \right) = -1 \) for all \( p \mid S \) as in Equation (2). Then \( f \) contains infinitely many projectively inequivalent anisotropic quadratic subforms over \( \mathbb{Q} \) of signature \((3,1)\).

**Proof.** Suppose first that \( s \) is even, so \( a \equiv -1 \mod 4 \). Pick \( \alpha > \beta > 0 \) such that \( aS(\alpha^2S - \beta^2) \equiv -1 \mod 8 \). Such \( \alpha, \beta \) always exist. To see this, if \( a \equiv -1 \mod 8 \) let \( \alpha = 1 \) and \( \beta = 0 \); if \( a \equiv 3 \mod 8 \) and \( S \equiv -1 \mod 8 \) let \( \alpha = 2 \) and \( \beta = 0 \); and if \( a \equiv 3 \mod 8 \) and \( S \equiv 3 \mod 8 \) let \( \alpha = 3 \) and \( \beta = 2 \). Note also that \( aS(\alpha^2S - \beta^2) > 0 \).

Set \( u = \beta e_3 + \alpha Se_4 \) so that \( f(u) = aS(\alpha^2S - \beta^2) < 0 \). Set \( m = -f(u) > 0 \). Then the diagonal form \( f' = \langle 1, 1, 1, -m \rangle \) is a subform of \( f \) with orthogonal basis \( \{ u, e_2, e_3, e_4 \} \).

Since \( m = -f(u) \equiv -1 \mod 8 \), it is not the sum of three squares and so \( f' \) is anisotropic. Since any other choice of \( \alpha, \beta \) congruent to the particular \( \alpha, \beta \) given as examples would still work, and different choices yield infinitely many projectively inequivalent forms, the conclusion follows as before.

Now suppose that \( s \) is odd, so \( a \equiv 1 \mod 4 \). Fix a prime \( p|S \) with \( p \equiv -1 \mod 4 \). By Lemma 5.3(c), there exists infinitely many primes \( q \equiv 1 \mod 4 \) such that \( \left( \frac{q}{p} \right) = -1 \). Since \( q \equiv 1 \mod 4 \) it can be written as a sum of two squares \( q = \alpha^2 + \beta^2 \). Let \( w_1 = \alpha e_1 + \beta e_2 \) so \( f(w_1) = q \). Consider the diagonal quadratic form \( g = \langle 1, q, -p^2 \rangle \).

**Claim 1:** \( g \) represents \( S \) over \( \mathbb{Q} \).

Assuming the claim for now, there is exists an integer solution \( x^2 + qy^2 - p^2z^2 = Sm^2 \). Let \( w_2 = xe_0 + \beta ye_1 - \alpha ye_2 \) so \( f(w_2) = Sm^2 + p^2z^2 \) and \( w_2 \) pairs trivially with \( w_1 \). Therefore, the diagonal form \( f' = \langle q, Sm^2 + p^2z^2, aS, -a \rangle \) is a subform of \( f \) with orthogonal basis \( \{ w_1, w_2, e_3, e_4 \} \). Let \( S' = S/p \). Then \( f(w_2) = Sm^2 + p^2z^2 = q(S'm^2 + p^2z^2) \), and since \( S'm^2 + p^2z^2 \equiv S'm^2 \mod p \), \( f(w_2) \equiv qS' \mod (\mathbb{Q}_p)^2 \). Considering \( f' \) over \( \mathbb{Q}_p \), since \( d(f') = -qa^2m^2S(S + p^2z^2) \) and since \( \left( \frac{q}{p} \right) = -1 \) implies \( -q \in (\mathbb{Q}_p)^2 \) (see Lemma 5.3(a)), we see that \( d(f') = -qa^2S(S + p^2z^2) \equiv qS'(S' + p^2z^2) \in (\mathbb{Q}_p)^2 \). Using Lemma 5.3(c) and (d), \( c_p(f') \) simplifies to

\[
c_p(f') = -(S' + p^2z^2, p)_p \cdot (-qS', p)_p \cdot (q, p)_p \cdot (-a, p)_p = -1 = -(1, -1)_p.
\]

Therefore \( f' \) is anisotropic over \( \mathbb{Q}_p \) and thus also over \( \mathbb{Q} \) (see Lemma 5.3(g)). As before, different choices of \( q \) yield projectively inequivalent forms, and the conclusion follows as before.
We now prove Claim 1. Since the determinant of $g$ is $-\sqrt{q}$ up to squares, by the Hasse principle it suffices to show that $g$ represents $S$ over $k = \mathbb{R}, \mathbb{Q}_2, \mathbb{Q}_p$. By [Se, IV.2.2, Corollary to Theorem 6] the ternary quadratic form $g$ represents $S$ if $S \neq -\sqrt{q}$ in $k^*/(k^*)^2$ or $(-1, q) = c_2(g)$. As $S > 0$ and $-\sqrt{q} < 0$, the first case holds over $\mathbb{R}$. As $q \nmid S$, the first case also holds over $\mathbb{Q}_2$. Over $\mathbb{Q}_2$ we have $c_2(g) = (q, -p^2)_2 = (q, -1)_2 = (-1, q)_2$ as required. Therefore $g$ represents $S$ over $\mathbb{Q}$ which proves the claim. □

6. Proof of Theorem 1.2

Throughout this section $W$ is as in the statement of Theorem 1.2 and $M \hookrightarrow W$ is a closed embedded orientable totally geodesic hyperbolic 3-manifold. We begin with a general lemma.

Lemma 6.1. Let $X$ be an orientable finite volume hyperbolic 4-manifold with $\chi(X) = 1$ containing a closed embedded orientable totally geodesic hyperbolic 3-manifold. Then $b_1(X) > 0$.

Proof. Let $M \hookrightarrow X$ be a closed embedded orientable totally geodesic hyperbolic 3-manifold. Suppose that $M$ separates, then $X$ is decomposed into two finite volume hyperbolic 4-manifolds with geodesic boundary, whose volumes are proportional to their (integral) Euler characteristic. However, $\chi(X) = 1$, and this is a contradiction. Duality now implies $b_1(X) > 0$. □

Referring to Section 8, since the manifolds labelled 16 – 22 have first betti number equal 0, we can apply Lemma 6.1 to derive a contradiction.

We now deal with the remaining 15 cases. Note from Section 8, each of these manifolds admits at least one orientable cross-section.

Lemma 6.2. $M$ is disjoint from all orientable cross-sections.

Proof. By Lemma 3.1, these cross-sections are all isometric to one of the complements of the links $L$ stated in Lemma 3.1. Suppose that $M$ meets one of the cross-sections, then $M$ must meet $S^3 \setminus L$ in a closed orientable embedded totally geodesic surface. However, this is impossible by Lemma 3.2. □

Since $M$ is disjoint from any orientable cross section, and $W$ is a regular cover of $\mathbb{H}^4/\Gamma(2)$, using the isometries of $W$ induced from the reflections in the co-ordinate hyperplanes we get at least 2 disjoint copies of $M$ embedded in $W$ and disjoint from the orientable cross-section. If $M$ is disjoint from all cross sections (orientable or not), then we get 16 disjoint copies of $M$. Now the minimal volume of a closed hyperbolic 3-manifold is that of the Weeks manifold and is approximately 0.9427... [GMM]. Using Corollary 4.2(1) we see that $\text{Vol}(W) \geq 16\sqrt{\chi(0.94)}$, which a computation shows is approximately 22.8. On the other hand, since $\chi(W) = 1$, $\text{Vol}(W) = \frac{4\pi^2}{3}$ which is approximately 13.2, a contradiction.

Thus we can assume that $M$ meets one of the non-orientable cross-sections. It follows that $M$ contains an embedded totally geodesic surface $S$.

If $S$ is orientable and separates $M$, then by [KM] the 3-dimensional hyperbolic volume of $M$ is at least 12.8 and using Corollary 4.2(1), the two disjoint copies of $M$ provide tubular neighbourhoods of volume $\geq 14.9204...$, which again exceeds $\text{Vol}(W)$.

If $S$ is either orientable and non-separating in $M$ or is non-orientable, then applying [KM] directly, or to a double cover of $M$ shows that the 3-dimensional hyperbolic volume of $M$ is at
least 6.4. Once again, an application of Corollary 4.2(1) shows that the two disjoint copies of $M$ provide tubular neighbourhoods of volume $9.78864 \ldots$. In addition, the orientable cusped cross-section (of volume approximately 7.3) provides a tubular neighbourhood of volume $5.30825 \ldots$ which by Lemma 4.3 then provides an estimate for $\text{Vol}(W) \geq 15$, which again exceeds 13.2. This completes the proof.

**Remark 6.3.** If $X$ is any cusped orientable arithmetic hyperbolic 4-manifold with $\chi(X) = 1$, and $M$ is a closed embedded orientable totally geodesic hyperbolic 3-manifold in $X$, then Corollary 4.2(1) provides a bound on the the 3-dimensional hyperbolic volume of $M$. A calculation shows that this bound can be taken to be 35. Hence there are only finitely many possibilities for $M$ (being arithmetic of bounded volume). However, this still presents too many possibilities to try to deal with. What was crucially exploited in the proof of Theorem 1.2 was existence of additional symmetries in the setting of these integral congruence two hyperbolic 4-manifolds, to provide additional disjoint copies of $M$ thereby giving more volume in disjoint collar neighbourhoods. Without these additional symmetries the problem seems a deal harder.

**Remark 6.4.** Using the equivalence of the quadratic forms $J$ and $J_7 = x_0^2 + x_1^2 + x_2^2 + 7x_3^2 - 7x_4^2$ given in the proof of Proposition 5.1, one can construct explicit manifolds commensurable with any of the hyperbolic 4-manifolds considered in the proof of Theorem 1.2 containing a closed embedded orientable totally geodesic hyperbolic 3-manifold.

For example, if $\Gamma(49) < O^+(J, \mathbb{Z})$ denotes principal congruence subgroup of level 49, then it can be checked that the equivalence described above conjugates $\Gamma(49)$ into a subgroup of the principal congruence subgroup $\Gamma(7) < O^+(J_7, \mathbb{Z})$. The subform $x_0^2 + x_1^2 + x_2^2 - 7x_4^2$ defines a cocompact subgroup of $O^+(J_7, \mathbb{Z})$ acting on a hyperbolic 3-space $H$. Using the reflection in $\mathbb{H}^4$ through $H$ and arguing as in [Mil], it can be shown that $\mathbb{H}^4/\Gamma(7)$ contains a closed embedded orientable totally geodesic hyperbolic 3-manifold, and hence so does the quotient of $\mathbb{H}^4$ by $\Gamma(49) < O^+(J, \mathbb{Z})$. The Euler characteristic of $\mathbb{H}^4/\Gamma(49)$ is enormous, exceeding 700,000.

7. **Proof of Theorem 1.3**

Let $W$ be as in the statement of Theorem 1.3, and suppose that $M$ is a closed embedded totally geodesic hyperbolic 3-manifold in $W$. Since $M \subset W \subset S^4$, $M$ is orientable and separates $W$. This will allow us to use the improved version of the formula for the volume of a tubular neighbourhood given in Corollary 4.2(2).

Let $N$ be the non-orientable manifold 1011 in the census of [RT] with $W \to N$ the orientable double cover, and $L$ denote the link $S^3_0$. Note that again by the construction of $N$ in [RT], the manifold $N$ is a regular cover of $\mathbb{H}^4/\Gamma(2)$ with covering group $K$.

**Lemma 7.1.** In the case of the manifold $N$, each of the four cross sections is isometric to $S^3 \setminus L$.

**Proof.** In [RT, Table 3] the manifold 1011 is given by the code 14FF28 which represents the side pairing 11114444FFFFF22228888 for the 24 sides of the ideal 24-cell $Q^3$. In the notation of [RT], the four cross sections have $k_1k_5k_9$ codes 714, 274, 172, 147, which correspond to the side pairings $r_1k_i$ for the 12 sides of the polytope $Q^3$ where $r_1$ is the reflection on side $i$ and $k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = k_7 = k_8 = k_9 = k_{10} = k_{11} = k_{12}$. Since $r_1$ is a reflection, the side pairing $r_1k_i$ is orientation preserving if and only if the
corresponding \( k_i \) is orientation reversing. But this happens only if \( k_i \in \{1, 2, 4, 7\} \) since then it corresponds to the diagonal matrices with \( 1 \leftrightarrow \text{diag}(-1, 1, 1, 1), 2 \leftrightarrow \text{diag}(1, -1, 1, 1), 4 \leftrightarrow \text{diag}(1, 1, -1, 1), 7 \leftrightarrow \text{diag}(-1, -1, -1, 1) \). Therefore, all four cross sections of \( N \) are orientable.

In [RT, Table 1] we see that the code 147 corresponds to the integral congruence two 3-manifold \( M_3^2 \) of [RT] and which is isometric to the link complement \( S^3 \setminus L \) [RT, Page 108]. The other three codes 714, 274, 172 do not appear in [RT, Table 1]. However, it can be checked that they are equivalent up to symmetries of \( Q^3 \) to \( 147 \). □

**Lemma 7.2.**

(1) \( W \) is a regular cover of \( \mathbb{H}^4/\Gamma(2) \).

(2) The lift of any cross section of \( N \) to \( W \) consists of two embedded totally geodesic copies of \( S^3 \setminus L \).

(3) \( M \) is disjoint from all such lifts.

**Proof.** For (1) we note that \( W \) is the orientable double cover of \( N \), as such it is a characteristic cover of \( N \). Now \( N \) is a regular cover of \( \mathbb{H}^4/\Gamma(2) \), hence \( W \) is a regular cover of \( \mathbb{H}^4/\Gamma(2) \).

For (2), we have from Lemma 7.1 that all cross sections of \( N \) are isometric to the link complement \( S^3 \setminus L \). Being orientable, these must lift to two copies in the orientable double cover.

For (3), we argue as in the proof of Lemma 6.2. □

The first part of Theorem 1.3 now follows from Lemma 7.2(2).

For the second part, note that from Lemma 7.2(3) and (1), since \( M \) is disjoint from all lifts of the cross sections, and \( W \) is a regular cover of \( \mathbb{H}^4/\Gamma(2) \), using the isometries of \( W \) induced from the reflections in the co-ordinate hyperplanes we get 16 disjoint copies of \( M \), all embedded and separating in \( W \). As in the proof of Theorem 1.2, we use the minimal volume of a closed hyperbolic 3-manifold is approximately 0.9427, and then using the improved tubular neighbourhood function in Corollary 4.2(2) we see that \( \text{Vol}(W) \geq 16V(0.94) \), which is approximately 28.9. On the other hand, since \( \chi(W) = 2, \text{Vol}(W) = \frac{8\pi^2}{3} \) which is approximately 26.3, a contradiction.

**Remark 7.3.** In [IRT], three additional examples of link complements of 2-tori were found in manifolds homeomorphic to \( S^4 \). These arise as the orientable double covers of the non-orientable manifolds in the census of [RT] with numbers 71, 1091 and 1092. One can check that of these three examples, 1092 contains two orientable cross-sections whilst 71 and 1091 only have one. Using the methods of proof of Theorem 1.2 and 1.3, it can be shown that the link complement of 2-tori covering 1092 also does not contain any closed embedded totally geodesic 3-manifolds.

Note that for the example \( W \) of [I], it is shown in [I2] that the link complement is in \( S^4 \) with the standard smooth structure.

**Remark 7.4.** In [Sar] an investigation of finite volume hyperbolic link complements of 2-tori and Klein bottles in other smooth, closed, simply connected 4-manifolds was initiated. Amongst other things, this work provided restrictions on the simply connected manifolds that can admit such link complements; namely they can only be homeomorphic to \( S^4 \), \#(\( S^2 \times S^2 \)) or \#(\( \mathbb{C}P^2 \# \mathbb{C}P^2 \)), with \( r > 0 \). Furthermore, using the examples of [I], examples of
link complements of 2-tori in \(\#_r(S^2 \times S^2)\) for \(r\) even were exhibited in [Sar] (these cover the manifold \(W\) above). Other examples of link complements of 2-tori and Klein bottles in closed simply connected manifolds are also given in [IRT].

**Remark 7.5.** It is known that every closed orientable 3-manifold embeds in \(\#_r(S^2 \times S^2)\), for some \(r > 0\) (see [Ki0, Chapter VII, Theorem 4]). On the other hand, we do not know whether the link complements of 2-tori in \(\#_r(S^2 \times S^2)\) that cover \(W\) mentioned in Remark 7.4 contain a closed embedded totally geodesic hyperbolic 3-manifold.

Motivated by the results of this paper, these remarks and recent work on embedding (arithmetic) hyperbolic manifolds as co-dimension one totally geodesic submanifolds (see [KRS] and references therein) we pose the following questions:

**Question 7.6.** Is there a cusped, orientable, finite volume hyperbolic 4-manifold \(W\) with \(\chi(W) = 1\) (or 2) which contains a closed embedded orientable totally geodesic hyperbolic 3-manifold? If not what is the minimal Euler characteristic of such a hyperbolic 4-manifold?

**Question 7.7.** Do any of the link complements of 2-tori in \(\#_r(S^2 \times S^2)\) that cover \(W\) mentioned in Remark 7.4 contain a closed embedded orientable totally geodesic hyperbolic 3-manifold?

**Question 7.8.** Does there exist a finite volume hyperbolic link complement of 2-tori and Klein bottles in \(\#_r(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})\), for some \(r > 0\)?
8. The orientable integral congruence two hyperbolic 4-manifolds

The following table is composed from data in [RT0, Table 2] and [RT, Table 2].

Table 1. The 22 orientable integral congruence two hyperbolic 4-manifolds of [RT]

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References


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