# A combination theorem for Veech subgroups of the mapping class group

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September 30, 2004

**Abstract**

In this paper we prove a combination theorem for Veech subgroups of the mapping class group analogous to the first Klein-Maskit combination theorem for Kleinian groups in which two Fuchsian subgroups are amalgamated along a parabolic subgroup. As a corollary, we construct subgroups of the mapping class group (for all genus at least 2), which are isomorphic to non-abelian closed surface groups in which all but one conjugacy class (up to powers) is pseudo-Anosov.

## 1 Introduction

For $R$ a compact oriented surface, possibly with boundary, the *mapping class group* of $R$ is defined to be the group of isotopy classes of orientation preserving homeomorphisms

$$\mathrm{Mod}(R) = \mathrm{Homeo}^+(R)/\mathrm{Homeo}_0(R),$$

where, $\mathrm{Homeo}^+(R)$ is the group of orientation preserving homeomorphisms of $R$ and $\mathrm{Homeo}_0(R)$ are those homeomorphisms isotopic to the identity. We will write $\phi$ to denote a homeomorphism and its isotopy class, when no confusion can arise, or when the distinction is unimportant. An isotopy class of a homeomorphism is called an *automorphism* of $R$. We specify no boundary behavior when $R$ has boundary.

There have been many analogies made between $\mathrm{Mod}(R)$ and lattices in Lie groups. In particular, the study of the subgroup structure of $\mathrm{Mod}(R)$, both finite and infinite index, has many parallels in the theory of lattices. For example, the question of whether Property T holds for $\mathrm{Mod}(R)$ ($R$ a closed orientable surface of genus $\geq 3$), whether $\mathrm{Mod}(R)$ has a version of the Congruence Subgroup Property, or towards the other extreme, whether there are finite index subgroups of $\mathrm{Mod}(R)$ that surject onto $\mathbb{Z}$ (see for example [17], [18], and [10] for more on this).

The point of view of this paper is similar to [29] and [12] and is motivated by results and constructions in 3-manifold topology and Kleinian groups. A powerful tool in the theory of Kleinian groups are the Klein-Maskit combination theorems. Our purpose here is to take the first step in the development of combination theorems for subgroups of $\mathrm{Mod}(R)$.

Before we state our main theorem, we describe the motivating example from Kleinian groups. Suppose $G_1, G_2 < \mathrm{PSL}_2(\mathbb{R})$ are two finitely generated Fuchsian groups of finite co-area, and $G_0$ is

*This work was partially supported by an N.S.F. postdoctoral fellowship
†This work was partially supported by an N.S.F. grant

a maximal parabolic subgroup of both. If $h \in \mathrm{PSL}_2(\mathbb{C})$ is another parabolic centralizing $G_0$, then we may form the amalgamated product

$$G = G_1 *_{G_0} hG_2h^{-1}$$

The First Klein-Maskit Combination theorem tells us that if $h$ is "sufficiently complicated" (see §2.3), then the natural homomorphism of $G$ into $\mathrm{PSL}_2(\mathbb{C})$ is injective and every element is hyperbolic except those conjugate into an elliptic or parabolic subgroup of either factor. In the particular case that each $G_1$ and $G_2$ are torsion free and for $i = 1, 2$, $\mathbb{H}^2/G_i$ have one cusp, $G$ is isomorphic to a non-abelian closed surface subgroup of $\pi_1(M)$ with one accidental parabolic. This has been very useful in the construction of essential closed surfaces in cusped hyperbolic 3-manifolds (see [8] and [7]).

Our main theorem states that we can do essentially the same thing in $\mathrm{Mod}(R)$. The Fuchsian subgroups in this setting are the *Veech subgroups* of $\mathrm{Mod}(R)$. A hyperbolic element of $\mathrm{Mod}(R)$ is a pseudo-Anosov automorphism.

A special case of our main Theorem 6.1 is given by the following (see §2– §5 for definitions).

**Theorem 1.1** *Suppose $G(q_1)$, $G(q_2)$ are Veech subgroups of $\mathrm{Mod}(R)$ with $G_0$ a maximal parabolic subgroup of each. If $h \in \mathrm{Mod}(R)$ centralizes $G_0$ and is "sufficiently complicated", then*

$$G(q_1) *_{G_0} hG(q_2)h^{-1} \hookrightarrow \mathrm{Mod}(R)$$

*is injective. Moreover, any element not conjugate into a parabolic subgroup of either factor is pseudo-Anosov or a periodic element conjugate into a factor.*

One cusped Veech group lattices exist in most mapping class groups and in particular we have

**Corollary 1.2** *For every closed surface $R$ of genus $g \geq 2$, there exist subgroups of $\mathrm{Mod}(R)$ isomorphic to the fundamental group of a closed surface (of genus $2g$) for which all but one conjugacy class of elements (up to powers) is pseudo-Anosov.*

That surface subgroups exist in most mapping class groups is already known (see for example [14] for some discussion of this). One new feature about our construction is that "most" elements are pseudo-Anosov, unlike the other constructions.

These examples are of particular interest in connection with the following question which has implications into the existence of negatively curved 4-manifolds that fiber over a surface (see the discussion at the end of [14] and Question 12.3 of [1]).

**Question 1.3** *Do there exist subgroups of the mapping class group isomorphic to the fundamental group of a closed surface of genus at least 2 for which all non-identity elements are pseudo-Anosov?*

The rest of the paper is organized as follows. §2 gives the necessary definitions from combinatorial group theory and ends with a more detailed description of the Kleinian group example mentioned above. In §3 we recall some of the tools used in studying mapping class groups which we will need. Veech groups are defined by a Euclidean cone structure on a surface which we describe in §4. The first half of §5 gives a brief recap of the Nielsen-Thurston classification of surface automorphisms and provides the theorems on pseudo-Anosov automorphisms we will need. The end of this section describes Veech groups and provides examples.

Section 6 contains the statement of Theorem 6.1 (thus clarifying Theorem 1.1) and a proof of Corollary 1.2 assuming this. The proof of Theorem 6.1 is given in §9, with results and notation needed in the proof built up in §7 and 8. In §7 we exploit the structure of geodesic representatives of simple closed curves in the Euclidean cone metrics under consideration and use this to define sets which turn out to have properties analogous to those used in the Kleinian group example. Continuing to follow this example, we define in §8 those sets which will play the part of horoballs, called *horror-balls*, and show that they do indeed have the necessary properties. As well as proving Theorem 6.1 in §9, we give some further generalizations of Theorem 6.1 including a very simple version of the Second Klein-Maskit Combination theorem. We end with some concluding remarks and questions in §10.

# 2 Group theory

Since it will be useful in what follows we recall some basic facts about free products with amalgamations.

## 2.1 Free product with amalgamation

Given groups $G_0, G_1, ..., G_P$ and monomorphisms

$$\nu_i : G_0 \to G_i$$

for each $i = 1, ..., P$, the *free product of* $\{G_i\}_{i=1}^P$ *amalgamated over* $G_0$ is the group

$$G = G_1 *_{G_0} \cdots *_{G_0} G_P \tag{1}$$

defined by the universal property:

1. For each $i$, we have a monomorphism $\iota_i : G_i \to G$ with $\iota_i \circ \nu_i = \iota_{i'} \circ \nu_{i'}$ for every $i, i'$.
2. For any group $K$ and any collection of homomorphisms $\{\eta_i : G_i \to K\}$ with $\eta_i \circ \nu_i = \eta_{i'} \circ \nu_{i'}$ for each $i, i'$, there exists a unique homomorphism $\eta : G \to K$ such that $\eta \circ \iota_i = \eta_i$ for each $i$.

This allows us to unambiguously identify $G_0$ and each of the $G_i$'s as subgroups of $G$.

One can construct this group from the free product

$$G_1 * \cdots * G_P$$

as the quotient by the normal subgroup generated by all elements of the form $\nu_i(\phi) * \nu_{i'}(\phi^{-1})$ for $\phi \in G_0$, $i, i' \in \{1, ..., P\}$. In particular, this gives rise to normal forms for the elements of $G$. Namely, for every element $\phi \in G$ exactly one of the following holds:

1. $\phi$ is an element of $G_0$,
2. $\phi \in G_i \setminus G_0$ for some $i \in \{1, ..., P\}$.
3. $\phi = \phi_{i_1} \cdots \phi_{i_r}$ where $\phi_{i_j} \in G_{i_j} \setminus G_0$ with $i_j \in \{1, ..., P\}$ and $i_j \neq i_{j+1}$ for $j = 1, ..., r-1$.

## 2.2 Proper interactive $P$-tuples

Suppose we are now given subgroups $G_0, G_1, ..., G_P < \Gamma$ with $G_i \cap G_{i'} = G_0 < \Gamma$, for every $i, i' \in \{1, ..., P\}$ and $i \neq i'$. Using the inclusions of $G_0$ into each $G_i$ we form the amalgamated product $G$ as in (1). The inclusions of each $G_i$ into $\Gamma$ and property 2 of the amalgamated product implies the existence of a unique homomorphism $G \to \Gamma$ extending these inclusions.

If $\Gamma$ acts on a set $X$, then we say that a $P$-tuple of subsets $\Theta_1, ..., \Theta_P \subset X$ is a *proper interactive $P$-tuple* for $G_1, ..., G_P$ if

1. $\Theta_i \neq \emptyset$ for each $i$,

2. $\Theta_i \cap \Theta_{i'} = \emptyset$ for each $i \neq i'$,

3. $G_0$ leaves $\Theta_i$ invariant for each $i$,

4. for every $\phi_i \in G_i \setminus G_0$ we have $\phi_i(\Theta_{i'}) \subset \Theta_i$, for each $i' \neq i$, and

5. for every $i$, there exists $\theta_i \in \Theta_i$, such that for every $\phi_i \in G_i \setminus G_0$, $\theta_i \notin \phi_i(\Theta_{i'})$ for any $i' \neq i$.

The next proposition is proven using a standard "ping-pong" argument. The case of two subgroups $G_1, G_2$ amalgamated over $G_0$ is proved in [23].

**Proposition 2.1** *Suppose $G_0, G_1, ..., G_P, \Gamma, X$ are as above and $\Theta_1, ..., \Theta_P \subset X$ is a proper interactive $P$-tuple for $G_1, ..., G_P$. Then*

$$G = G_1 *_{G_0} G_2 *_{G_0} \cdots *_{G_0} G_P \hookrightarrow \Gamma$$

*is an embedding.*

*Proof.* Given $\phi \in G \setminus \{1\}$, we must show that the image of $\phi$ in $\Gamma$ is non-trivial. We do this by showing that, with respect to the given action of $\Gamma$, $\phi$ does not act as the identity. The only situation in question is when $\phi$ has normal form $\phi = \phi_{i_1} \cdots \phi_{i_r}$ of type 3 from §2.1.

Fix any $i \neq i_r$. Then

$$\phi_{i_r}(\Theta_i) \subset \Theta_{i_r}$$

by property 4 of a proper interactive $P$-tuple. Similarly,

$$\phi_{i_{r-1}}(\phi_{i_r}(\Theta_i)) \subset \phi_{i_{r-1}}(\Theta_{i_r}) \subset \Theta_{i_{r-1}}$$

So, repeatedly applying 4 and inducting, we see see that

$$\phi(\Theta_i) \subset \Theta_{i_1}$$

Now if $i \neq i_1$, then we are done since $\phi$ does not act as the identity. If $i = i_1$ then $\phi(\Theta_i) \subset \Theta_i$. However, by property 5 above, $\phi(\Theta_i) \neq \Theta_i$ and hence $\phi$ is again not acting as the identity. □

## 2.3 Kleinian group example

We now describe the motivating example of the Klein-Maskit combination theorem in more detail. We will consider the action of $\mathrm{PSL}_2(\mathbb{C})$ on $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Continue to denote the Fuchsian groups $G_1, G_2 < \mathrm{PSL}_2(\mathbb{R})$. Each stabilizes $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and after conjugating if necessary, we may assume that the maximal parabolic subgroup of each is

$$G_0 = \mathrm{Stab}_{G_1}(\infty) = \mathrm{Stab}_{G_2}(\infty) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$$

The elements of $G_0$ act on $\mathbb{C}$ by translations parallel to the $\mathbb{R}$-axis.

Each $G_i$ also stabilizes upper and lower half planes, $\mathbb{U}$ and $\mathbb{L}$. Let $H$ denote the union of the two horoballs

$$H = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 1\} \cup \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) < -1\} \subset \mathbb{U} \cup \mathbb{L}$$

It is a well known consequence of the Jørgensen inequality that for every $\phi \in G_i \setminus G_0$, $i = 1, 2$, we have

$$\phi(H) \subset \Theta = \{z \in \mathbb{C} \,|\, |\mathrm{Im}(z)| \leq 1\} \tag{2}$$

Now let $h \in \mathrm{Stab}_{\mathrm{PSL}_2(\mathbb{C})}(\infty) \setminus \mathrm{PSL}_2(\mathbb{R})$ be any parabolic. This has the form

$$h = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$$

for some $\mu \in \mathbb{C} \setminus \mathbb{R}$, and acts on $\mathbb{C}$ by translations *transverse* to the $\mathbb{R}$-axis.

It follows that there exists $K > 0$ such that for any $k \geq K$, we have

$$h^k(\Theta) \subset H$$

(thus, "sufficiently complicated" means that it translates a large distance transverse to the $\mathbb{R}$-axis). Set $\Theta_1 = \Theta$ and $\Theta_2 = h^k(\Theta)$. This implies

$$\Theta_1 \subset h^k(H) \text{ and } \Theta_2 \subset H$$

We leave it as an exercise using this and (2) to verify that $\Theta_1, \Theta_2$ is a proper interactive pair for $G_1, h^k G_2 h^{-k}$.

# 3 Topology and combinatorics of surfaces

We will consider connected orientable surfaces of genus $g$ with $n$ punctures or boundary components and say that this has type $(g, n)$ (whenever we encounter a disconnected surface, we will work with its components). We will blur the distinction between a puncture and a boundary component whenever convenient. Two exceptions to this are (1) whenever we consider a complex structure on our surface all boundary components will be replaced by punctures, and (2) whenever we consider essential arcs on our surface (see §3.1), all punctures will be replaced by boundary components. We sometimes write $R_{g,n}$ to denote a surface of type $(g, n)$ and $R_g$ for a closed surface of genus $g$.

A surface with non-negative Euler characteristic or of type $(0, 3)$ is said to be of *excluded type*. In addition to surfaces of excluded type, those with type $(1, 1)$ or $(0, 4)$ are often too small for general arguments and definitions to be valid, and we refer to these as *sporadic*. We will only concern ourselves with surfaces of non-excluded type.

Given a punctured surface $R$ of type $(g, n)$, we denote the closed surface with the punctures "filled back in" by $\overline{R}$. We view this as a closed surface of genus $g$ with $n$ marked points.

## 3.1 Simple closed curves

We denote the set of isotopy classes of essential simple closed curves on $R$ by $\mathcal{C}^0(R)$. These are (isotopy classes of) simple closed curves which are homotopically essential in $R$ and not homotopic to any puncture or boundary component. A *multi-curve* is a finite union of essential simple closed curves which are pairwise disjoint and pairwise non-parallel. The isotopy classes of multi-curves will be denoted $\mathcal{C}(R)$. We sometimes view the elements of $\mathcal{C}(R)$ as finite subsets of $\mathcal{C}^0(R)$. We will generally confuse isotopy classes with representatives whenever convenient.

Given $A \in \mathcal{C}(R)$, we let $R \setminus N(A)$ denote the surface obtained by removing an open tubular neighborhood of a representative of $A$ from $R$. That is $R \setminus N(A)$ denotes $R$ *cut open along* $A$. A multi-curve $A$ is said to be *sparse* if every component of $R \setminus N(A)$ has non-excluded type.

We denote the set of isotopy classes of essential simple closed curves and essential arcs in $R$ by $\mathcal{A}^0(R)$. For us, an essential arc is an embedding of the pair $(I, \partial I) \to (R, \partial R)$ that cannot be homotoped (rel boundary) into a boundary component. Multi-arcs are defined analogously to multi-curves, and we denote the set of isotopy classes of multi-arcs by $\mathcal{A}(R)$.

Given $A, B \in \mathcal{A}(R)$ we denote the *geometric intersection number* of $A$ and $B$ by $i(A, B)$. This is the minimal number of transverse intersection points over all representatives of $A$ and $B$.

**Remark.** In our definitions we do *not* require that an isotopy fix the boundary pointwise.

## 3.2 Measured laminations

In what follows, we fix a complete hyperbolic metric of finite area on $R$ (with geodesic boundary).

### 3.2.1 Geodesic laminations

We denote the space of compact *geodesic laminations* on $R$ by $\mathcal{GL}_0(R)$, and we give this the Thurston topology. In this topology, a sequence of laminations $\{\mathcal{L}_m\}_{m=1}^{\infty}$ converges to a lamination $\mathcal{L}$ if and only if every leaf of $\mathcal{L}$ is a limit of leaves of $\mathcal{L}_m$ (this is weaker than the Hausdorff topology).

Any lamination with all compact leaves is a finite union of simple closed geodesics, and any such uniquely determines a multi-curve. Conversely, any multi-curve has a unique geodesic representative, thus we may identify $\mathcal{C}(R)$ (and hence also $\mathcal{C}^0(R)$) as a subset of $\mathcal{GL}_0(R)$.

### 3.2.2 Transverse measures

We denote the space of compactly supported *measured laminations* by $\mathcal{ML}_0(R)$. We view an element of $\mathcal{ML}_0(R)$ as a compact geodesic lamination along with an invariant transverse measure of full support. For any $\lambda \in \mathcal{ML}_0(R)$, let $|\lambda|$ denote the underlying geodesic lamination and $\lambda$ or $d\lambda$ the transverse measure. We have an action of $\mathbb{R}_+$ on $\mathcal{ML}_0(R)$ by scaling the transverse measure.

$\mathbb{R}_+ \times \mathcal{C}^0(R)$ injects into $\mathcal{ML}_0(R)$ by sending $t \cdot a$ to $t$ times the transverse counting measure on the geodesic representative of $a$. Given $\lambda$ and $\lambda'$ in $\mathcal{ML}_0(R)$, the total measure of the "product measure" $d\lambda \times d\lambda'$ is a natural homogeneous (with respect to the $\mathbb{R}_+$ action) extension of the intersection number function, which we also denote

$$i : \mathcal{ML}_0(R) \times \mathcal{ML}_0(R) \to \mathbb{R}$$

We endow $\mathcal{ML}_0(R)$ with the smallest topology for which $i$ is continuous. With this topology, $\mathbb{R}_+ \times \mathcal{C}^0(R)$ is dense.

Given a pair $\lambda_1, \lambda_2 \in \mathcal{ML}_0(R)$, with $|\lambda_1|$ transverse to $|\lambda_2|$, we say that $\lambda_1, \lambda_2$ *bind* $R$ if for every $a \in \mathcal{C}^0(R)$, we have $i(a, \lambda_1) + i(a, \lambda_2) > 0$. Equivalently, all complementary regions of $|\lambda_1| \cup |\lambda_2|$ are disks, punctured disks, or half-open annuli.

The quotient by $\mathbb{R}_+$

$$\mathbb{P} : \mathcal{ML}_0(R) \to \mathbb{P}\mathcal{ML}_0(R)$$

is the space of *projective measured laminations* on $R$. This is given the quotient topology and we denote $\mathbb{P}(\lambda)$ by $[\lambda]$, for any $\lambda \in \mathcal{ML}_0(R)$. The forgetful map to $\mathcal{GL}_0(R)$ obtained by forgetting the transverse measure factors through the projectivization.

An important point for us is the following result of Thurston [33] Proposition 8.10.3.

**Proposition 3.1** *The forgetful map*

$$\mathbb{P}\mathcal{ML}_0(R) \to \mathcal{GL}_0(R)$$

*is continuous.*

**Remark.** Although it appears that the spaces we have constructed depend on the choice of hyperbolic metric, any two hyperbolic metrics give rise to spaces which can be canonically identified.

### 3.2.3 Zero locus of a measured lamination

For any $\lambda \in \mathcal{ML}_0(R)$, $|\lambda|$ is a finite union of its components $|\lambda| = |\lambda_1| \cup \cdots \cup |\lambda_N|$. This decomposes $\lambda$ as a sum of sub-measured laminations $\lambda = \sum_{j=1}^{N} \lambda_j$. With this notation, we define

$$Z(\lambda) = \{\mu \in \mathcal{ML}_0(R) \,|\, i(\mu, \lambda) = 0\} \quad \text{and} \quad Z'(\lambda) = \bigcup_{j=1}^{N} Z(\lambda_j)$$

These sets depend only on $|\lambda|$. In particular, if $A$ is a multi-curve, we may define $Z(A)$ and $Z'(A)$ by arbitrarily prescribing a transverse measure of full support on the geodesic representative of $A$.

By an abuse of notation, we refer to the images of $Z(\lambda)$ and $Z'(\lambda)$ in $\mathbb{P}\mathcal{ML}_0(R)$ and $\mathcal{GL}_0(R)$ by the same names. Note that if $\mu \in Z(\lambda)$ and $\mu' \in \mathcal{ML}_0(R)$ has $|\mu| = |\mu'|$, then $\mu' \in Z(\lambda)$, and similarly for $Z'(\lambda)$. In particular, we may pass back and forth between the different sets representing $Z(\lambda)$ and $Z'(\lambda)$ by taking images and preimages without introducing any new points.

As a further abuse of notation, we will denote the intersection of $Z(A), Z'(A) \subset \mathcal{GL}_0(R)$ with $\mathcal{C}(R)$ by the same names. It will be clear from the context where these subsets are residing.

### 3.2.4 Singular foliations

The elements of $\mathcal{ML}_0(R)$ can also be realized as singular foliations on $S$ with invariant transverse measures of full support. The underlying singular foliation is unique up to isotopy and Whitehead equivalence. The corresponding geodesic lamination is obtained by straightening all leaves to geodesics. We will also write $|\lambda|$ to denote the underlying (isotopy/Whitehead equivalence class of) singular foliation. When it is clear from the context, we will not explicitly mention which of these we are referring to.

A foliation is *finite* if it has only compact leaves. Given $\lambda \in \mathcal{ML}_0(R)$, the foliation $|\lambda|$ is finite if and only if the geodesic lamination $|\lambda|$ represents a multi-curve. A finite foliation decomposes $R$ into a finite union of compact singular leaves and open annuli foliated by its nonsingular leaves. We refer to this as an *annular decomposition.* The union of the cores of the annuli will be called the *core* of the annular decomposition, and this is precisely the associated multi-curve for the finite foliation. An annular decomposition is *sparse* if its core is.

## 3.3 The arc complex

The set $\mathcal{A}^0(R)$ can be naturally identified with the vertex set of a simplicial complex $\mathcal{A}(R)$, called the *arc complex* of $R$. A set of distinct elements $a_0, ..., a_p \in \mathcal{A}^0$ spans a $p$-simplex if and only if the curves and arcs can be realized on the surface pairwise disjointly. Thus, the points of the complex $\mathcal{A}(R)$ (that is, its simplices) are precisely the multi-arcs. We are therefore justified in using $\mathcal{A}(R)$ to refer to either of these objects.

We denote the corresponding polyhedron $|\mathcal{A}|(R)$. This is not given the classical topology for the polyhedron of a simplicial complex, but rather the metric topology obtained by declaring each simplex to be a regular Euclidean simplex with all edge lengths equal to one. In particular, we obtain a metric on $\mathcal{A}^0(R)$ by viewing it as the set of vertices of the metric graph $|\mathcal{A}^1|(R)$. Whenever we refer to the distance between elements of $\mathcal{A}^0(R)$, it will be understood that it is meant with respect to this metrics.

### 3.4 Subsurface projections

A useful tool is the subsurface projection. If $S \subset R$ is a connected incompressible subsurface of non-excluded type, then we have the *subsurface projection*

$$\pi_S : \mathbb{PML}_0(R) \setminus Z(\partial S) \to \mathcal{A}^0(S)$$

obtained as follows. We fix a hyperbolic metric on $R$ and homotope the embedding of $S$ to have geodesic boundary (this may make the boundary no longer embedded, but the interior remains embedded). For $[\lambda] \in \mathbb{PML}_0(R) \setminus Z(\partial S)$, we consider the geodesic lamination $|\lambda|$ and take its intersection with $S$ (less any components of $\partial S$). The result is a disjoint union of essential arcs and a geodesic laminations in $S$ (we are using the fact that the transverse measure has full support to guarantee that $|\lambda|$ has no isolated leaves which are not simple closed curves). Since $[\lambda]$ is not in $Z(\partial S)$, some leaf of $|\lambda|$ intersects $\partial S$ non-trivially, and we take $\pi_S([\lambda])$ to be the isotopy class of one of these arcs. The collection of essential arcs and the geodesic lamination in $S$ obtained in this way depend only on the underlying geodesic lamination $|\lambda|$, and not the projective class of transverse measure. There is no canonical choice for this arc, however any two such arcs are disjoint and hence have distance one from each other in $\mathcal{A}^0(S)$.

The use of the hyperbolic structure in this construction is merely a convenient way of arranging that all intersection points of leaves of $|\lambda|$ with $\partial S$ are essential, and so the construction may be carried out assuming only this.

The subsurface projection is "coarsely continuous".

**Proposition 3.2** *If $\{[\lambda_m]\}_{m=1}^{\infty} \subset \mathbb{PML}_0(R) \setminus Z(\partial S)$ converges to $[\lambda] \in \mathbb{PML}_0(R) \setminus Z(\partial S)$ then there exists $N > 0$, such that for all $m \geq N$*

$$d(\pi_S([\lambda]), \pi_S([\lambda_m])) \leq 1$$

*Proof.* Assume the setup from the statement of the proposition, and let $l$ be any leaf of $|\lambda|$ which nontrivially intersects $Z(\partial S)$. If we let $l^0$ be an arc of $l \cap S$, then this is an allowable choice for $l^0 = \pi_S([\lambda])$. Since the forgetful map to $\mathcal{GL}_0(R)$ is continuous, there are leaves $l_m$ of $|\lambda_m|$ which converge to $l$. Thus there are arcs $l_m^0$ of $l_m \cap S$ which converge to $l^0$. In particular, there exists some $N > 0$ such that for all $m \geq N$, the arcs $l_m^0$ are isotopic in $S$ to $l^0$.

Clearly the same thing holds for any choice $l^0$ for $\pi_S([\lambda])$. So, since any choice of arc for $\Pi_S([\lambda_m])$ will be disjoint from the $l_m^0$ considered above, we obtain the required bound. □

**Remark.** As is noted there, Proposition 3.2 is the intuition behind the Bounded Geodesic Image Theorem of [27].

Since $\pi_S(\lambda)$ depends only on $|\lambda|$, it makes sense to define $\pi_S(A)$ for $A \in \mathcal{C}(R)\setminus Z(\partial S)$. However, we will adhere to the following

**Convention 3.3** *When considering multi-curves, the domain for the subsurface projections will be taken to be*

$$\mathcal{C}(R) \setminus Z'(\partial S)$$

*instead of $\mathcal{C}(R) \setminus Z(\partial S)$*

# 4 Quadratic differentials

The *Teichmüller space* of $R$, denoted $\mathcal{T}(R)$, is the space of isotopy classes complex structures of finite type on $R$. For any $X \in \mathcal{T}(R)$ let $A_2(X)$ denote the vector space of integrable holomorphic quadratic differentials on $R$ with respect to $X$. Varying $X$ over $\mathcal{T}(R)$ we obtain a vector bundle over $\mathcal{T}(R)$ with fiber over $X$ being $A_2(X)$. We denote the complement of the zero section of this bundle as $A_2\mathcal{T}(R)$. We refer to a point of $A_2\mathcal{T}(R)$ simply by the quadratic differential $q$, with the underlying complex structure $X \in \mathcal{T}(R)$ implicit.

## 4.1 Metric structure

Any point $q \in A_2\mathcal{T}(R)$ determines, and is determined by, a Euclidean cone metric on $\overline{R}$ with some additional structure which we now describe. There is an atlas of local coordinate charts $\{\zeta_j : U_j \to \mathbb{C}\}$ on the complement of the cone points with the following properties:

1. for each $j, k$ and on each component of $U_j \cap U_k$, there is a $w \in \mathbb{C}$ such that
$$\zeta_j = \pm\zeta_k + w$$
2. the metric on $U_j$ is the pullback of the Euclidean metric on $\mathbb{C}$ by $\zeta_j$.

It follows that cone points have cone angle $m\pi$ for some $m \in \mathbb{Z}_{\geq 1}$. We further require that at any cone point which is not a marked point, $m \geq 3$.

Any local coordinate as above is called a $q$-coordinate. We refer to the set of cone points and marked points as the singularities of $q$. Because of condition (1) on the transition functions, this structure on $\overline{R}$ is sometimes called a *half-translation structure*.

Any embedded geodesic segment in the $q$ metric connecting a pair of singular points and having no singular point in its interior is called a *saddle connection*. A simple closed geodesic which misses the singular locus will be called a *nonsingular closed geodesic*. There is a dichotomy for simple closed curves $a \in \mathcal{C}^0(R)$ with respect to $q$: either there is an annulus in $R$ foliated by nonsingular closed geodesics all in the isotopy class of $a$, or else there is a unique closed geodesic in $\overline{R}$ homotopic to $a$ made up of saddle connections.

The geodesic representative of $a \in \mathcal{C}^0(R)$ when $R$ has punctures requires a few additional comments. If we take a sequence of representatives of $a$ in $R$ for which the lengths converge to the infimum of all lengths of representatives, then a subsequence of these representatives will converge to a geodesic *in* $\overline{R}$, and this is the geodesic representative of $a$. In particular, arbitrarily close to the geodesic representative we can find simple closed curves in $R$ representing $a$.

## 4.2 $\mathcal{ML}_0(q)$

For any constant, nonzero 1-form $adx + bdy$ on $\mathbb{C}$ (so $a, b \in \mathbb{R}$), the measured foliation $|adx + bdy|$ is invariant under the transition functions for the atlas of $q$-coordinates. Therefore, it pulls back to a measured foliation on the complement of the cone points. This extends to a singular measured foliation on $R$ and $\overline{R}$.

The space of all such measured foliations is denoted $\mathcal{ML}_0(q)$ and is homeomorphic to $(\mathbb{R}^2 \setminus \{0\})/\pm I$. We view $\mathcal{ML}_0(q)$ as a subspace of $\mathcal{ML}_0(R)$ and we write $\mathbb{P}\mathcal{ML}_0(q) \subset \mathbb{P}\mathcal{ML}_0(R)$ to denote the projectivization.

We note that the leaves of every element of $\mathcal{ML}_0(q)$ are geodesic in the Euclidean cone metric, with a finite number of *critical leaves* containing one or two singular points of the metric as endpoints. When there are two singular points, this is just a saddle connection.

When $\lambda \in \mathcal{ML}_0(q)$ is a finite foliation, it defines an annular decomposition of $R$. We say that this is an *annular decomposition* for $q$.

**Lemma 4.1** *If $A_0, A_1$ are cores of two distinct annular decompositions for $q$, then $A_0 \notin Z'(A_1)$.*

*Proof.* For any two distinct $[\lambda_0], [\lambda_1] \in \mathbb{P}\mathcal{ML}_0(q)$, it is easy to see that $\lambda_0, \lambda_1$ bind $R$. In particular $A_0, A_1$ bind $R$. Every component $a \subset A_1$ must therefore intersect $A_0$ (otherwise $a$ would have $i(a, A_0) + i(a, A_1) = 0$). □

# 5 Mapping class group

## 5.1 Actions

$\mathrm{Mod}(R)$ acts on all the spaces associated to $R$ we have discussed so far: $\mathcal{A}^0(R)$, $\mathcal{C}(R)$, $\mathcal{ML}_0(R)$, $\mathbb{P}\mathcal{ML}_0(R)$, and $\mathcal{GL}_0(R)$. These actions respect the various natural maps between the spaces (e.g. the inclusion $\mathcal{C}^0(R) \subset \mathcal{ML}_0(R)$, and the projection $\mathcal{ML}(R) \to \mathbb{P}\mathcal{ML}(R)$). These actions also preserves the intersection number $i$ (on the spaces where it is defined) as well as the metric on $\mathcal{A}^0(R)$.

## 5.2 Nielsen-Thurston classification

An automorphism $\phi \in \mathrm{Mod}(R) \setminus \{1\}$ is said to be *reducible* if there exists a multi-curve $A \in \mathcal{C}(R)$ invariant by $\phi$. In this case, write $S = R \setminus N(A)$ and denote the components of $S$ by $S_1, ..., S_M$. We thus have an induced automorphism

$$\tilde{\phi} : S \to S$$

Some power $\tilde{\phi}^r$, preserves the components of this surface and hence gives rise to automorphisms

$$\tilde{\phi}^r_j : S_j \to S_j$$

The multi-curve $A$ is called a *reducing system* for $\phi$ and the maps $\tilde{\phi}^r_j$ are called the *components* of $\tilde{\phi}^r$ (or of $\phi^r$). The process just described for a reducible automorphism is called a *reduction* of $\phi$ along $A$.

An example of a reducible automorphism is a Dehn twist $T_a$ along some simple closed curve $a \in \mathcal{C}^0(R)$. More generally, given a multi-curve $A$, a composition of Dehn twists about the components of $A$ defines a reducible automorphism called a *multi-twist*. If all the Dehn twists are positive, it is called a *positive multi-twist.*

An element $\phi \in \mathrm{Mod}(R)$ of infinite order is called *pseudo-Anosov* if there exists a pair $\lambda_s, \lambda_u \in \mathcal{ML}_0(R)$ called the *stable* and *unstable* laminations, respectively, which bind $R$ and such that $\{[\lambda_s], [\lambda_u]\}$ is invariant by $\phi$.

**Theorem 5.1 (Nielsen-Thurston)** *Given any element $\phi \in \mathrm{Mod}(R)$, one of the following holds*

1. *$\phi$ has finite order,*
2. *$\phi$ is reducible, or*
3. *$\phi$ is pseudo-Anosov.*

If $\phi$ is reducible, there is a unique reducing system $A$ with the property that, after passing to an appropriate power $\phi^r$, each of the components $\tilde{\phi}^r_j$ are either pseudo-Anosov or the identity, and any sub-multi-curve of $A$ fails to have this property. We refer to this multi-curve as the *canonical reducing system* for $\phi$ and those components of (a power of) $\phi$ which are pseudo-Anosov are called the *pseudo-Anosov components.*

Whenever we perform a reduction, it will always be assumed to be done along the canonical reducing system. We say that $\phi$ is *pure* if it does not permute the components of $S = R \setminus N(A)$ or the components of $A$ and each component of $\phi$ is either pseudo-Anosov or the identity. Any reducible automorphism has a power which is pure.

## 5.3 Identifying the type of an automorphism

Given $\phi \in \mathrm{Mod}(R)$, one can decide which of the three types from Theorem 5.1 $\phi$ falls into by considering the action on $\mathbb{P}\mathcal{ML}_0(R)$.

In case 1 of Theorem 5.1, there is nothing to say (some power of $\phi$ acts as the identity). In case 3, the two points $[\lambda_s]$ and $[\lambda_u]$ are attracting and repelling fixed points for the action, respectively. In particular for every $[\lambda] \in \mathbb{P}\mathcal{ML}_0(R) \setminus \{[\lambda_s], [\lambda_u]\}$, we have

$$\phi^n([\lambda]) \to [\lambda_s] \text{ and } \phi^{-n}([\lambda]) \to [\lambda_u]$$

as $n \to \infty$.

For simplicity, we only describe the reducible case when $\phi$ is pure. Let $A$ denote a canonical reducing system for $\phi$. Let $\lambda_{s,1}, ..., \lambda_{s,M}, \lambda_{u,1}, ..., \lambda_{u,M}$ denote the stable and unstable measured laminations for the pseudo-Anosov components of $\phi$, which we view as elements of $\mathcal{ML}_0(R)$. From this data we obtain two laminations

$$\psi_s = A \cup |\lambda_{s,1}| \cup \cdots \cup |\lambda_{s,M}| \text{ and } \psi_u = A \cup |\lambda_{u,1}| \cup \cdots \cup |\lambda_{u,M}|$$

These define subsets $\Psi_s$ and $\Psi_u$ of $\mathcal{ML}_0(R)$ consisting of all measured laminations with measures supported on (non-empty) sublaminations of $\psi_s$ and $\psi_u$, respectively. We will need the following result of Ivanov and McCarthy, see [18] Theorem A1.

**Theorem 5.2** *With the notation as above, for every $\lambda \in \mathcal{ML}_0(R) \setminus Z(\Psi_s) \cup Z(\Psi_u)$*

$$\lim_{m\to\infty} \phi^m([\lambda]) \in \mathbb{P}\Psi_s \text{ and } \lim_{m\to-\infty} \phi^m([\lambda]) \in \mathbb{P}\Psi_u$$

Note that if $\lambda \in Z(\Psi_s) \cap Z(\Psi_u)$, then $\phi$ fixes $\lambda$.

## 5.4 Pseudo-Anosov automorphisms and $\mathcal{A}^0(R)$

The following theorem of Masur and Minsky is a strengthening of the fact that a pseudo-Anosov automorphism cannot fix any essential simple closed curve or arc.

**Theorem 5.3 (Masur-Minsky)** *There exists $c > 0$ (depending only on $R$), such that for any pseudo-Anosov automorphism $\phi \in \mathrm{Mod}(R)$ the action of $\phi$ on $\mathcal{A}^0(R)$ satisfies*

$$d(\phi^k(a), a) \geq c|k|$$

*for every $a \in \mathcal{A}^0(R)$.*

**Remark.** Although this is stated for the *curve complex* instead of the arc complex, the two spaces are quasi-isometric by a $\mathrm{Mod}(R)$-equivariant quasi-isometry (in the sporadic cases, the curve complex is replaced by the Farey graph). This easily implies the version we need.

### 5.5 Veech groups

Let $\mathrm{Aff}^+(q)$ denote the subgroup of $\mathrm{Mod}(R)$ consisting of all automorphisms with representatives which are affine with respect to the $q$-metric. We will refer to any subgroup $G(q) < \mathrm{Aff}^+(q)$ as a *Veech group* for $q$. Note this is non-standard notation, but is convenient for our purposes.

Taking derivatives in $q$-coordinates (with respect to the standard basis of $\mathbb{C} \cong \mathbb{R}^2$) we obtain a homomorphism

$$D : \mathrm{Aff}^+(q) \to \mathrm{PSL}_2(\mathbb{R})$$

This gives us the short exact sequence shown below, and in which $\mathrm{PSL}(q)$ is the image group of the homomorphism $D$ above, and $\mathrm{Aut}(q)$ is the subgroup of $\mathrm{Aff}^+(q)$ preserving $q$.

$$1 \to \mathrm{Aut}(q) \to \mathrm{Aff}^+(q) \to \mathrm{PSL}(q) \to 1 \tag{3}$$

The next theorem is well known. It can be derived from Bers proof of Theorem 5.1, as is done by Kra in [21]. A proof is also given by Veech in [34] and Thurston in [32].

**Theorem 5.4** *$D$ is a discrete representation with finite kernel. For $\phi \in \mathrm{Aff}^+(q) \setminus \{1\}$:*

1. *$\phi$ has finite order if and only if $D\phi$ is elliptic or $D\phi = 1$,*
2. *$\phi$ has a power that is a positive multi-twist if and only if $D\phi$ is parabolic, and*
3. *$\phi$ is pseudo-Anosov if and only if $D\phi$ is hyperbolic.*

The proof boils down to the following elementary observation. Given $\phi \in \mathrm{Aff}^+(q)$, either $D\phi$ preserves a quadratic form (case 1), has a single eigenspace (case 2), or a pair of distinct eigenspaces (case 3). Then, either $\phi$ preserves a metric affine equivalent to the $q$-metric (case 1) or a (pair of) foliation(s) everywhere tangent to the eigenspace(s) (cases 2 and 3).

In case 2, the invariant foliation is finite and so defines an annular decomposition with core $A_0$. Some power of $\phi$ is a positive multi-twist about $A_0$. The stabilizer of $A_0$ in $\mathrm{Aff}^+(q)$ contains $\langle\phi\rangle$ with finite index. For any Veech group $G(q) < \mathrm{Aff}^+(q)$ we denote the stabilizer of $A_0$ in $G(q)$ by $G_0(q)$.

We say that a subgroup of $\mathrm{Aff}^+(R)$ has *elliptic* or *parabolic type* if its image under $D$ is an elementary subgroup with elliptic or parabolic type, respectively. In particular, $G_0(q)$ is of parabolic type.

### 5.6 Examples

The following construction is due to Veech. Given $g \geq 2$, let $\Delta_g$ be the non-convex polygon obtained as the union of two regular $2g+1$-gons in the Euclidean plane which meet along an edge and have disjoint interiors (see Figure 1 for the case $g = 2$).

Let $R_g$ denote the surface obtained by gluing opposite sides of $\Delta_g$ by translations. This is easily seen to be a closed surface of genus $g$. The Euclidean metric on the interior of $\Delta_g$ is the restriction of a Euclidean cone metric on $R_g$, and we can find local coordinates defining a quadratic differential on $R_g$ compatible with this metric (as in 4.1). We denote this quadratic differential $\xi_g \in A_2\mathcal{T}(R_g)$.

**Theorem 5.5 (Veech)** $\mathrm{PSL}(\xi_g)$ *is a lattice isomorphic to a triangle group of type* $(2, 2g+1, \infty)$. *The single primitive parabolic conjugacy class is the image by $D$ of the conjugacy class of a positive multi-twist in $\mathrm{Aff}^+(\xi_g)$ by $D$ about a sparse multi-curve $A_0 \in \mathcal{C}(R_g)$.*

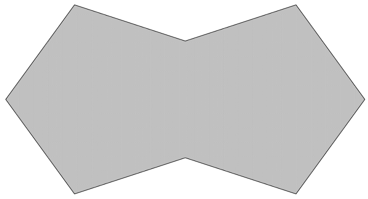

Figure 1: $\Delta_2$

See [34] for a proof.

$\mathrm{PSL}(\xi_g)$ is generated by a pair of elliptic elements $\gamma_0$ and $\gamma_1$ with orders 2 and $2g+1$, respectively. Moreover, these can be chosen so that $\gamma_0\gamma_1$ generates the parabolic subgroup.

There is an epimorphism

$$\nu : \mathrm{PSL}(\xi_g) \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/(2g+1)\mathbb{Z}$$

with $\nu(\gamma_0)$ and $\nu(\gamma_1)$ generating the first and second factors, respectively. It follows that $\ker(\nu)$ is a free group. Moreover, since $\nu(\gamma_0\gamma_1)$ generates the range, we see that $\ker(\nu)$ has exactly one conjugacy class of parabolics. In fact, $\ker(\nu) \cong \pi_1(S_{g,1})$ with the parabolic $(\gamma_0\gamma_1)^{2(2g+1)}$ representing a peripheral element, where $S_{g,1}$ is a surface of type $(g,1)$.

The sequence (3) restricts to a short exact sequence

$$1 \to D^{-1}(\ker(\nu)) \cap \mathrm{Aut}(\xi_g) \to D^{-1}(\ker(\nu)) \to \ker(\nu) \to 1$$

which splits since $\ker(\nu)$ is free. Denoting the image of $\ker(\nu)$ under the splitting isomorphism by $G(\xi_g) < \mathrm{Aff}^+(\xi_g)$ we have

**Corollary 5.6** *$G(\xi_g) < \mathrm{Mod}(R_g)$ is isomorphic to $\pi_1(S_{g,0,1})$ with every non-peripheral conjugacy class represented by a pseudo-Anosov automorphism and the peripheral subgroup generated by a positive multi-twist about the sparse multi-curve $A_0 \in \mathcal{C}(R_g)$.*

*Proof.* This follows from Theorem 5.5 and the comments above by appealing to Theorem 5.4. □

# 6 The combination theorem

*For the remainder of this paper, we let $R$ denote a non-sporadic surface of type $(g,n)$.*

We first define the groups to which our theorem applies. Let $q_1, ...., q_P \in A_2\mathcal{T}(R)$ and suppose that $A_0 \in \mathcal{C}(R)$ is the core of sparse annular decomposition for each of $q_1, ..., q_P$. Write $S = R \setminus N(A_0)$ and let $h \in \mathrm{Mod}(R)$.

We say that $h, G(q_1), ..., G(q_P)$ are *compatible* along $A_0$ if

1. $G_0 = G_0(q_1) = ... = G_0(q_P)$,
2. $h$ centralizes $G_0$ and,
3. $h$ is pure and pseudo-Anosov on all components of $S$.

Our main theorem is the following.

**Theorem 6.1** *Suppose $h, G(q_1), ..., G(q_P)$ are compatible along the sparse multi-curve $A_0$. Then there exists $K_2, ..., K_P \geq 0$ such that for $k_i \geq K_i + k_{i-1}$, $i = 2, ..., P$, and $k_1 = 0$,*

$$G(q_1) *_{G_0} h^{k_2} G(q_2) h^{-k_2} *_{G_0} \cdots *_{G_0} h^{k_P} G(q_P) h^{-k_P} \hookrightarrow \mathrm{Mod}(R)$$

*is an embedding. Moreover, every element not conjugate into an elliptic or parabolic subgroup of any $h^{k_i} G(q_i) h^{-k_i}$ is pseudo-Anosov.*

This obviously implies Theorem 1.1, where "sufficiently complicated" simply means a sufficiently high power of a pure automorphism in the centralizer of $G_0$ being pseudo-Anosov on all components of $S$. Appealing to the examples described in Section 5.6, we have the following.

**Corollary 1.2** *For every closed surface $R$ of genus $g \geq 2$, there exist subgroups of* $\mathrm{Mod}(R)$ *isomorphic to the fundamental group of a closed surface (of genus $2g$) for which all but one conjugacy class of elements (up to powers) is pseudo-Anosov.*

*Proof.* Let $G(q_1) = G(q_2) = G(\xi_g)$ from Corollary 5.6. Since $A_0$ is sparse, there exists an automorphism $h \in \mathrm{Mod}(R)$ which is pseudo-Anosov on all component of $S$ (in fact, $S$ is connected in all of these examples). $G_0$ is generated by a multi-twist about $A_0$, and so $h$ certainly centralizes $G_0$. It follows that $h, G(q_1), G(q_2)$ are compatible along $A_0$. Note that $G_0$ is the only conjugacy class of parabolic subgroups in $G(\xi_g)$.

Theorem 6.1 therefore implies that for sufficiently large $k$

$$G(\xi_g) *_{G_0} h^k G(\xi_g) h^{-k}$$

embeds in $\mathrm{Mod}(R)$ and every element not conjugate into the cyclic subgroup $G_0$ is pseudo-Anosov. On the other hand

$$G(\xi_g) *_{G_0} h^k G(\xi_g) h^{-k} \cong \pi_1(S_{g,1}) *_{\mathbb{Z}} \pi_1(S_{g,1}) \cong \pi_1(S_{2g,0})$$

where the $\mathbb{Z}$ in the amalgamated product is the peripheral subgroup of each. □

The proof of Theorem 6.1 is given in §9. This will require some preliminary notation and results which occupy the next few sections.

# 7 More on geodesics

For the remainder of this section, fix $q \in A_2\mathcal{T}(R)$.

## 7.1 Bad singularities

For any multi-curve $A$, we define $\sigma(A)$ to be the $q$-geodesic representative of $A$. This is a union of saddle connections and non-singular closed geodesics with pairwise disjoint interiors. We make this unique by requiring all components of $A$ which are homotopic to nonsingular geodesics to be represented by a geodesic in the interior of the corresponding annulus, equidistant from both boundary components.

Since $\sigma(A)$ is a union of geodesics, if we follow an arc of any component as it enters and exits a singularity $p$, we see that it must make an angle at least $\pi$ on both sides, unless $p$ is a puncture point. In this case it must make an angle at least $\pi$ on the side opposite the puncture.

At each singularity $p$ of $\sigma(A)$, we consider the ends of those saddle connections meeting $p$. By an end we simply mean a component of a saddle connection intersected with a small neighborhood of $p$. We cyclically order these ends as we encounter them by encircling $p$ in a counter-clockwise direction. Let us write these in order as $b_1, ..., b_r$ (see Figure 2). Between each consecutive pair $b_i$ and $b_{i+1}$ (mod $r$), we have the angle $\theta_i$. We say that the singularity $p$ is *bad* if $\theta_i < \pi$ for each $i = 1, ..., r$.

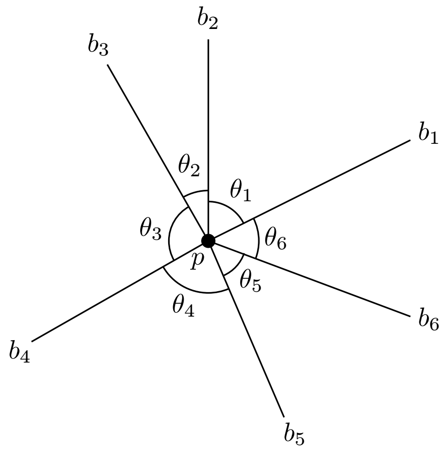


Figure 2: Saddle connection ends

**Lemma 7.1** *Given $A \in \mathcal{C}(R)$, $\sigma(A)$ contains no bad singularities.*

*Proof.* This is an outer-most arc argument. Suppose $\sigma(A)$ contains a bad singularity. Take a small neighborhood of $p$ and of $b_1, ..., b_r$ and by small homotopy we assume that $A$ is embedded in the union of these neighborhoods (see Figure 3). The neighborhood of $p$ is a disk and the neighborhoods of the $b_i$'s define intervals $I_i$ on the boundary of this disk. The intersection of $A$ with the disk is a collection of arcs with endpoints on the intervals.

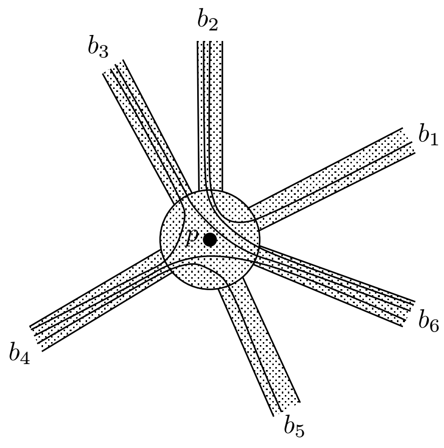


Figure 3: Neighborhood of a singularity and its saddle connection ends

There are finitely many arcs, and so we may consider one which is outer-most. The endpoints of this arc are in adjacent intervals $I_i, I_{i+1}$. Therefore the arc lies in a component of $A$ whose geodesic representative enters $p$ along $b_i$ and exits along $b_{i+1}$. Because $\theta_i < \pi$, this makes an angle less than $\pi$. Therefore $\sigma(A)$ cannot be geodesic, providing a contradiction, unless $p$ is a puncture cut off by the arc. In this case, we choose a different outer-most arc (or if this is the only arc, we think of it as outer-most on the other side). As we have already cut off the puncture with the first arc, the new outer-most arc cannot cut it off, and we arrive at the same contradiction. □

## 7.2 Subsurface projections and spines

Given any annular decomposition of $R$ for $q$ with core $A_0$, let $\sigma_0$ denote the union of singular leaves parallel to $A_0$. Equivalently, this is the union of the boundary components of maximal annuli in the annular decomposition. Call $\sigma_0$ a *spine* parallel to $A_0$.

Fix a spine $\sigma_0$ parallel to a sparse core $A_0$ and let $\sigma_{0,1}, ..., \sigma_{0,N}$ denote the components of $\sigma_0$. For each $j = 1, ..., N$, define

$$\Theta(q)_j = \{A \in \mathcal{C}(R) \setminus Z'(A_0) \,|\, \sigma(A) \not\supset \sigma_{0,j}\}$$

and set

$$\Theta(q) = \bigcap_{j=1}^{N} \Theta(q)_j$$

Let $S = R \setminus N(A_0)$ with components $S_1, ..., S_N$. We may order these components so that $\sigma_{0,j} \subset S_j$. For each $j = 1, ..., N$, take $\delta_j$ to be a geodesic arc in $S_j$ with endpoints on $\partial S_j$ and intersecting the interior of a single saddle connection, $b_j$, of $\sigma_{0,j}$ (see Figure 4).

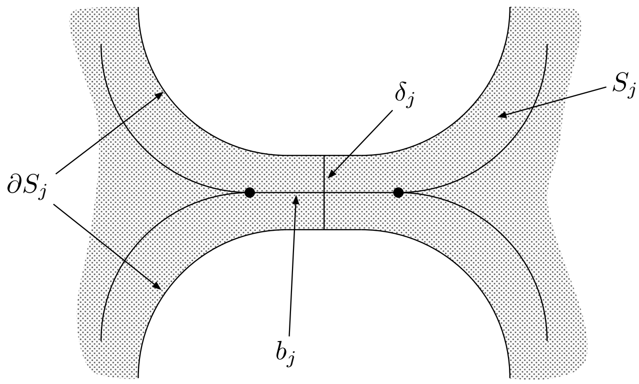


Figure 4: The arc $\delta_j$

**Proposition 7.2** *For each $j = 1, ..., N$,*

$$\pi_{S_j}(\Theta(q)) \subset \pi_{S_j}(\Theta(q)_j) \subset B(\delta_j, 2)$$

*Proof.* The first inclusion is obvious, so all we need prove is the second. Before we begin, we note that for any sufficiently small $\epsilon > 0$, we can take $S_j$ to be the $\epsilon$-neighborhood of $\sigma_{0,j}$.

There is a linear deformation retraction $H_t$, $t \in [0, 1]$, of $S_j$ onto $\sigma_{0,j}$ which slides points along a perpendicular geodesic arc from $\partial S_j$ to $\sigma_{0,j}$. We assume that $H_t$ is an embedding for $t \in [0, 1)$.

Moreover, $\delta_j$ may be chosen to be $H_1^{-1}(p)$ for any $p \in \mathrm{int}(b_j)$ (the interior of $b_j$), and this is isotopic to any other choice of $\delta_j$.

We fix some $A \in \Theta(q)_j$. By definition, $\sigma_{0,j} \not\subset \sigma(A)$. Suppose first that $b_j \not\subset \sigma(A)$. We claim that for an appropriate choice of $p \in \mathrm{int}(b_j)$, $\delta_j = H_1^{-1}(p)$ is disjoint from $\sigma(A)$.

To prove the claim, we note that because $b_j \not\subset \sigma(A)$, we can find $p \in \mathrm{int}(b_j)$ and $\epsilon' > 0$ such that

$$d_q(p, \sigma(A)) > 2\epsilon'$$

Now, take any $0 < \epsilon \leq \epsilon'$ which defines $S_j$ as the $\epsilon$-neighborhood of $\sigma_{0,j}$. Let $\gamma$ be an arc of $b_j$ centered at $p$ of radius no more than $\epsilon$. Then $H_1^{-1}(\gamma)$ contains $\delta_j$ and is contained in the $2\epsilon'$-neighborhood of $p$. In particular, this set (and hence $\delta_j$) is disjoint from $\sigma(A)$ as claimed.

Thus, if $b_j \not\subset \sigma_{0,j}$, then $\delta_j$ is disjoint from $\sigma(A)$ and in particular

$$d(\delta_j, \pi_{S_j}(A)) \leq 1$$

for any allowable choice of projection $\pi_{S_j}(A)$.

So to prove the proposition, we assume $b_j \subset \sigma(A)$. Then by hypothesis there is another branch $b'_j$ of $\sigma_{0,j}$ such that $b'_j \not\subset \sigma(A)$. Let $\delta'_j$ be an analogous arc for $b'_j$. The previous argument implies that $d(\delta'_j, \pi_{S_j}(A)) \leq 1$ for any allowable choice of projection. $\delta_j$ and $\delta'_j$ are disjoint however, and so $d(\delta_j, \pi_{S_j}(A)) \leq 2$. □

### 7.3 $G(q)$ action on geodesics

$G(q)$ acts on the set of saddle connections and non-singular closed geodesics since it is acting by affine homeomorphisms. Taking geodesic representatives is obviously equivariant with respect to this action and the action on $\mathcal{C}(R)$. That is

$$\sigma(\phi(A)) = \phi(\sigma(A)) \tag{4}$$

for any $A \in \mathcal{C}(R)$ and any $\phi \in G(q)$.

This implies

**Lemma 7.3** *$\Theta(q)$ is invariant under $G_0(q)$.*

*Proof.* $\sigma_0$ is invariant under $G_0(q)$ although some elements may permute the components. Let $A \in \mathcal{C}(R)$ and $\phi \in G_0(q)$ and let $\phi(\sigma_{0,j}) = \sigma_{0,j'}$. Then we have

$$\sigma_{0,j'} = \phi(\sigma_{0,j}) \subset \phi(\sigma(A) = \sigma(\phi(A)) \Leftrightarrow \sigma_{0,j} \subset \sigma(A)$$

Since $A_0$ is also invariant under $G_0(q)$, so is $Z'(A_0)$. The lemma now follows from the definition of $\Theta(q)$. □

## 8 Horror-balls

We wish to define the analog of the horoballs used in the Kleinian group example of §2.3. A *horror-ball* will depend on $q$ and a sparse core $A_0$ of an annular decomposition for $q$.

We define the *horror-ball* for $q$ and $A_0$ by the equation

$$H(q) = \bigcap_{j=1}^{N} \pi_{S_j}^{-1}\left(\mathcal{A}^0(S_j) \setminus B(\delta_j, 2)\right) \subset \mathcal{C}(R) \setminus Z'(A_0)$$

where $\delta_j$ is as in §7.2 and Proposition 7.2 (the dependance on $A_0$ is implicit in the notation $H(q)$).

**Proposition 8.1** *For every $j = 1, ..., N$, we have*

$$H(q) \cap \Theta(q)_j = \emptyset$$

*(in particular, $H(q) \cap \Theta(q) = \emptyset$), and for every $\phi \in G(q) \setminus G_0$*

$$\phi(H(q)) \subset \Theta(q)$$

*Proof.* By Proposition 7.2, for every $j = 1, ..., N$, we have

$$\left(\mathcal{A}^0(S_j) \setminus B(\delta_j, 2)\right) \cap \pi_{S_j}(\Theta(q)_j) = \emptyset$$

Therefore

$$\begin{aligned} H(q) \cap \Theta(q)_j &= \left(\bigcap_{k=1}^N \pi_{S_k}^{-1}\left(\mathcal{A}^0(S_k) \setminus B(\delta_k, 2)\right)\right) \cap \Theta(q)_j \\ &\subset \pi_{S_j}^{-1}\left(\mathcal{A}^0(S_j) \setminus B(\delta_j, 2)\right) \cap \pi_{S_j}^{-1}\left(\pi_{S_j}(\Theta(q)_j)\right) \\ &= \pi_{S_j}^{-1}\left(\left(\mathcal{A}^0(S_j) \setminus B(\delta_j, 2)\right) \cap \pi_{S_j}(\Theta(q)_j)\right) \\ &= \emptyset \end{aligned}$$

which proves the first assertion.

To prove the second, we suppose that $A \in H(q)$ and $\phi \in G(q) \setminus G_0$ and show that $\phi(A) \in \Theta(q)$. By the first part, we know $A \notin \Theta(q)_j$ for any $j = 1, ..., N$ and so $\sigma_0 \subset \sigma(A)$. Since

$$\phi(\sigma(A)) = \sigma(\phi(A))$$

by (4), we see that $\phi(\sigma_0) \subset \sigma(\phi(A))$. Since $\sigma_0$ is a spine, so is $\phi(\sigma_0)$. Now, suppose $\sigma_{0,j} \subset \sigma(\phi(A))$ for some $j$. $\sigma_{0,j}$ is a component of the spine $\sigma_0$ and so $\sigma(\phi(A))$ contains the $\sigma_{0,j} \cup \phi(\sigma_0)$. This has a bad vertex (see Figure 5) and so contradicts Lemma 7.1. Hence $\sigma_{0,j} \not\subset \sigma(\phi(A))$ for any $j = 1, ..., N$.

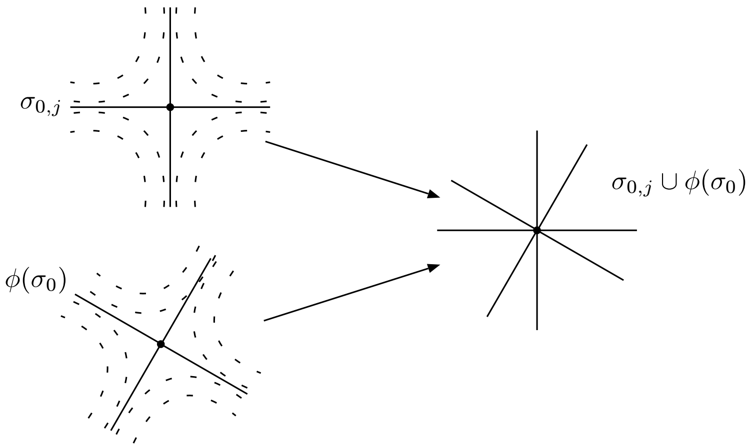


Figure 5: Local picture near the singularity of $\sigma_{0,j} \cup \phi(\sigma_0)$

To prove that $\phi(A) \in \Theta(q)$, all that remains is to show that $\phi(A) \notin Z'(A_0)$. Again, we note that $\phi(\sigma_0)$ is a spine and it therefore (nontrivially and transversely) intersects any nonsingular geodesic which its associated core, $\phi(A_0)$, intersects. However, $\phi(A_0)$ and $A_0$ are distinct cores and so Lemma 4.1 guarantees $\phi(A_0)$ intersects every component of $A_0$. Therefore, $\phi(A)$ intersects every component of $A_0$ and thus $\phi(A) \notin Z'(A_0)$. $\square$

## 8.1 Spacing out

We assume the hypotheses on $h, G(q_1), ..., G(q_P)$ and $A_0$ from the statement of Theorem 6.1 in what follows. Let $\Theta(q_i)_j$ for $j = 1, ..., N$ and $i = 1, ..., P$ be defined as in §7.2 with respect to our given multi-curve $A_0 \in \mathcal{C}(R)$. Let $h_j \in \text{Mod}(S_j)$ for each $j = 1, ..., N$ be the $S_j$-component of $h$, which is pseudo-Anosov by assumption. As a convenience, we use the flexibility in the definition of $\pi_{S_j}$ (see §3.4) to guarantee that it is equivariant with respect to $\langle h \rangle$. That is, for each $j = 1, ..., N$, each $k \in \mathbb{Z}$, and each $A \in \mathcal{C}(R) \setminus Z'(A_0)$

$$\pi_{S_j}(h^k(A)) = h_j^k(\pi_{S_j}(A))$$

This is accomplished by defining $\pi_{S_j}$ on $\langle h \rangle$-orbit representatives, then extending so that it is equivariant (it is easy to check that this is an allowable definition for $\pi_{S_j}$).

In the statement of Theorem 6.1 we require the existence of certain numbers $K_2, ..., K_P$. The following lemma will provide us with these numbers.

**Lemma 8.2** *Suppose $h, G(q_1), ..., G(q_P)$ are compatible along the sparse multi-curve $A_0$. Then there exists $K_2, ..., K_P \geq 0$ such that for $k_i \geq K_i + k_{i-1}$, $i = 2, ..., P$, and $k_1 = 0$*

$$d(h_j^{k_i}(\pi_{S_j}(\Theta(q_i))), h_j^{k_{i'}}(\pi_{S_j}(\Theta(q_{i'})))) \geq 5$$

*for every $i, i' \in \{1, ..., P\}$ with $i \neq i'$.*

*Proof.* Proposition 7.2 implies that for each $j = 1, ..., N$ and each $i = 1, ..., P$, $\pi_{S_j}(\Theta(q_i))$ has diameter no more than 4. In particular, by Theorem 5.3 and the triangle inequality, for each $i = 2, ..., P$ there are only finitely many integers $k_i$ for which

$$d(h_j^{k_i}(\pi_{S_j}(\Theta(q_i))), \pi_{S_j}(\Theta(q_{i'}))) < 5$$

for some $i' \neq i$. Therefore, for each $i = 2, ..., P$, there exists $K_i \geq 0$ such that for every $k_i \geq K_i$ and every $i' \neq i$,

$$d(h_j^{k_i}(\pi_{S_j}(\Theta(q_i))), \pi_{S_j}(\Theta(q_{i'}))) \geq 5 \tag{5}$$

Now set $k_1 = 0$ and let $k_i \geq K_i + k_{i-1}$. Suppose $i > i'$. By induction on $i - i'$, we see that

$$k_i \geq \left( \sum_{l=i'+1}^{i} K_l \right) + k_{i'}$$

In particular, $k_i - k_{i'} \geq K_i$.

Since $h_j^{-k_{i'}}$ acts by isometries on $\mathcal{A}^0(S_j)$, we have

$$d(h_j^{k_i}(\pi_{S_j}(\Theta(q_i))), h_j^{k_{i'}}(\pi_{S_j}(\Theta(q_{i'})))) = d(h_j^{k_i - k_{i'}}(\pi_{S_j}(\Theta(q_i))), \pi_{S_j}(\Theta(q_{i'}))) \geq 5$$

by (5), as required. □

# 9 Proof of Theorem 6.1

We can now prove

**Theorem 6.1** *Suppose $h, G(q_1), ..., G(q_P)$ are compatible along the sparse multi-curve $A_0$. Then there exists $K_2, ..., K_P \geq 0$ such that for $k_i \geq K_i + k_{i-1}$, $i = 2, ..., P$, and $k_1 = 0$,*

$$G(q_1) *_{G_0} h^{k_2} G(q_2) h^{-k_2} *_{G_0} \cdots *_{G_0} h^{k_P} G(q_P) h^{-k_P} \hookrightarrow \mathrm{Mod}(R)$$

*is an embedding. Moreover, every element not conjugate into an elliptic or parabolic subgroup of any $h^{k_i} G(q_i) h^{-k_i}$ is pseudo-Anosov.*

*Proof.* Let $K_2, ..., K_P$ be as in Lemma 8.2, $k_1 = 0$, and $k_i \geq K_i + k_{i-1}$ for each $i = 2, ..., P$. To simplify the notation, we replace each $G(q_i)$ by its conjugate $h^{k_i} G(q_i) h^{-k_i}$. By the $\langle h \rangle$-equivariance of $\pi_{S_j}$ (see §8.1), this has the effect of replacing each $\pi_{S_j}(\Theta(q_i))$ by

$$\pi_{S_j}(h^{k_i}(\Theta(q_i))) = h_j^{k_i}(\pi_{S_j}(\Theta(q_i)))$$

Therefore, by Lemma 8.2, we have

$$d(\pi_{S_j}(\Theta(q_i)), \pi_{S_j}(\Theta(q_{i'}))) \geq 5 \tag{6}$$

We first claim that $(\Theta(q_1), ..., \Theta(q_P))$ is a proper interactive $P$-tuple (see §2.2). Clearly each $\Theta(q_i)$ is nonempty and Lemma 7.3 implies each of these sets is invariant under $G_0$. By (6) we see that for each $i \neq i'$ and each $j = 1, ..., N$

$$\pi_{S_j}(\Theta(q_{i'})) \subset \mathcal{A}^0(S_j) \setminus B(\delta_{i,j}, 2)$$

where $\delta_{i,j}$ is defined for $S_j$ in terms of $q_i$ as in §7.2 and Proposition 7.2 so that

$$\Theta(q_{i'}) \subset \bigcap_{j=1}^{N} \pi_{S_j}^{-1}(\pi_{S_j}(\Theta(q_{i'}))) \subset H(q_i) \tag{7}$$

Thus, by the first part of Proposition 8.1 (or by (6) directly), we see that

$$\Theta(q_i) \cap \Theta(q_{i'}) = \emptyset$$

for each $i \neq i'$.

Now, the second part of Proposition 8.1 combined with (7) shows that for every $i$ and each $\phi \in G(q_i) \setminus G_0$

$$\phi(\Theta(q_{i'})) \subset \phi(H(q_i)) \subset \Theta(q_i) \tag{8}$$

for every $i' \neq i$. Therefore $\Theta(q_1), ..., \Theta(q_P)$ satisfies properties 1–4 in the definition of proper interactive $P$-tuple.

To prove that this also satisfies property 5, for every $i$ choose any $\phi_i \in G(q_i) \setminus G_0$. We claim that

$$\phi_i(A_0) \in \Theta(q_i) \tag{9}$$

This is because $\phi_i(A_0)$ is a core of an annular decomposition different than $A_0$, so $\sigma(\phi_i(A_0))$ (with respect to $q_i$) contains no saddle connections and therefore no component of $\sigma_0$. Moreover, by Lemma 4.1, $\phi_i(A_0) \notin Z'(A_0)$. Define $\theta_i = \phi_i(A_0)$.

For every $i' \neq i$ and any $\tilde{\phi} \in G(q_i) \setminus G_0$, we have

$$\theta_i \notin \tilde{\phi}(\Theta(q_{i'}))$$

For, as we saw in the proof of Proposition 8.1, given any $A \in \tilde{\phi}(\Theta(q_{i'}))$, $\sigma(A)$ contains a spine (with respect to $q_i$), namely, $\tilde{\phi}(\sigma_0)$. However, $\theta_i$ contains no spine. We have thus proved that

$\Theta(q_1), ..., \Theta(q_P)$ also satisfies property 5 and hence is a proper interactive $P$-tuple. Proposition 2.1 implies the required injectivity.

We now prove the statement about pseudo-Anosov elements. By Theorem 5.4 we know that in each factor, any element not in an elliptic or parabolic subgroup is pseudo-Anosov. It suffices therefore to consider an element $\phi$ which is not conjugate into any factor. We assume that $\phi$ is reducible and arrive at a contradiction. By the structure theory of amalgamated free products, since $\phi$ is not conjugate into a factor, $\phi$ must have infinite order. Thus Theorem 5.1 implies that $\phi$ is pseudo-Anosov.

After conjugating, we may write

$$\phi = \phi_{i_1}\phi_{i_2}\cdots\phi_{i_r}$$

with $\phi_{i_k} \in G(q_{i_k}) \setminus G_0$ and $i_k \neq i_{k+1} \pmod r$ for each $k = 1, ..., r$. By passing to an appropriate power if necessary, we may also assume that $\phi$ is pure. In what follows, we let $A$, $\lambda_{s,1}, \ldots, \lambda_{s,M}$, $\lambda_{u,1}, \ldots, \lambda_{u,M}$, $\psi_s$, $\psi_u$, $\Psi_s$, and $\Psi_u$ be as in §5.3.

We consider the image of $A_0$ under $\phi$ and its iterates. First, by (9), $\phi_{i_r}(A_0) \in \Theta(q_{i_r})$. It follows from (8) that $\phi_{i_{r-1}}(\phi_{i_r}(A_0)) \in \Theta(q_{i_{r-1}})$. Applying the rest of the word for $\phi$, by repeatedly appealing to (8) and induction, we see

$$\phi(A_0) \in \Theta(q_{i_1})$$

Similarly,

$$\phi^{-1}(A_0) \in \Theta(q_{i_r})$$

Iterating $\phi$ and $\phi^{-1}$ and applying (8) and induction we find that

$$\phi^m(A_0) \in \Theta(q_{i_1}) \text{ and } \phi^{-m}(A_0) \in \Theta(q_{i_r}) \tag{10}$$

for every $m \geq 1$.

So, combining (6) and (10) we see that for every $j = 1, ..., N$ and $m \geq 1$

$$d(\pi_{S_j}(\phi^m(A_0)), \pi_{S_j}(\phi^{-m}(A_0))) \geq 5 \tag{11}$$

Now consider the limits of these multi-curves in $\mathbb{P}\mathcal{ML}_0(R)$

$$[\lambda_0^s] = \lim_{m\to\infty} \phi^m(A_0) \text{ and } [\lambda_0^u] = \lim_{m\to\infty} \phi^{-m}(A_0)$$

($A_0$ is assigned the projective class of transverse counting measure, although any transverse measure of full support will do). Theorem 5.2 implies that if $A_0$ nontrivially intersects $\psi_s$, then $[\lambda_0^s] \in \mathbb{P}\Psi_s(\phi)$ and $[\lambda_0^u] \in \mathbb{P}\Psi_u(\phi)$ (because $A_0$ is a multi-curve it intersects $\psi_s$ if and only if it intersects $\psi_u$). However, because $\phi(A_0) \in \Theta(q_{i_1})$ and $A_0 \notin \Theta(q_{i_1})$, the comment following Theorem 5.2 implies that $A_0$ nontrivially intersects $\psi_s$. In particular, we find $[\lambda_0^s], [\lambda_0^u] \in \mathbb{P}\mathcal{ML}_0(R) \setminus Z(\partial S_j)$ for some $j$.

By Proposition 3.2, the triangle inequality, and (11), we see that for sufficiently large $m$,

$$\begin{aligned} d(\pi_{S_j}([\lambda_0^s]), \pi_{S_j}([\lambda_0^u])) \quad &\geq \quad d(\pi_{S_j}(\phi^m(A_0)), \pi_{S_j}(\phi^{-m}(A_0))) \\ &\qquad - d(\pi_{S_j}(\phi^m(A_0)), \pi_{S_j}([\lambda_0^s])) - d(\pi_{S_j}([\lambda_0^u]), \pi_{S_j}(\phi^{-m}(A_0))) \\ &\geq \quad 5 - 2 \quad = \quad 3 \end{aligned}$$

Therefore $\lambda_0^s, \lambda_0^u$ bind when restricted to $S_j$. Said differently, the intersections of $|\lambda_0^s|$ and $|\lambda_0^u|$ with $S_j$ nontrivially intersect every essential simple closed curve and essential arc in $S_j$.

Suppose $|\lambda_0^s|$ misses some component of $A_0$. The connectivity of $R$ implies that, among all components of $A_0$ which $|\lambda_0^s|$ fails to intersect, there is (at least) one which is the boundary of a component $S_j$ of $S$ for which $\lambda_0^s, \lambda_0^u$ bind when restricted (as in the previous paragraph). Call this component $a$.

**Claim.** $a$ is also a component of $A$.

*Proof of claim.* Because $\lambda_0^s, \lambda_0^u$ bind when restricted to $S_j$, and since they miss $a$, the boundary of the complementary region of $|\lambda_0^s| \cup |\lambda_0^u|$ must have a component parallel to $a$ (see Figure 6).

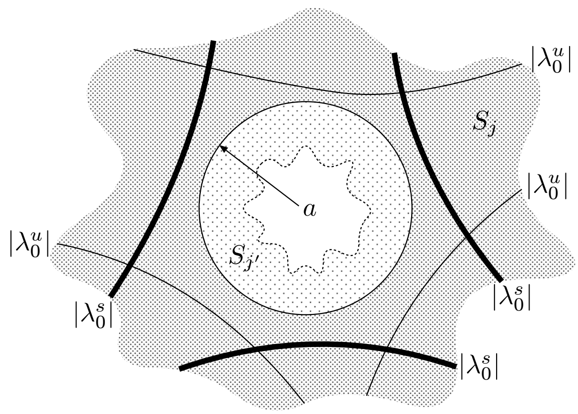


Figure 6: $|\lambda_0^s| \cup |\lambda_0^u|$ near $a$.

The components of $|\lambda_0^s| \cup |\lambda_0^u|$ are components of $\psi_s \cup \psi_u$. Any such component is either a component of $A$ or a set of the form $|\lambda_{s,j}| \cup |\lambda_{u,j}|$ for $j = 1, ..., M$. So, the component of the boundary of the complementary region above is a component of the boundary of the complementary region for some $|\lambda_{s,j}| \cup |\lambda_{u,j}|$. However, all such boundary components which are essential in $R$ are parallel to the boundaries of subsurfaces on which $\phi$ is pseudo-Anosov. Therefore, $a$ is such a boundary, and hence a component of $A$ (the boundary of a component on which $\phi$ is pseudo-Anosov is necessarily in the canonical reducing system). This proves the claim.

Since $\phi(A_0)$ transversely intersects *every* component of $A_0$ (it lies in the complement of $Z'(A_0)$), it follows that $\phi(a) \neq a$. However, this contradicts the fact that $a$ is a reducing curve and was thus assumed to be fixed (by the purity assumption). This proves that $\phi$ is pseudo-Anosov and completes the proof the theorem. $\square$

## 9.1 Other combinations

The arguments in the proof of Theorem 6.1 carry over verbatim to more general settings. For example, one can form the amalgamated product of countably many groups $\{G(q_i)\}_{i=1}^{\infty}$ conjugated by (appropriate powers of) distinct pure automorphisms $\{h(i)\}_{i=1}^{\infty}$ each centralizing $G_0$. However, the statement seems sufficiently messy that we have only stated it in the finite case.

One example of the countable case which seems interesting enough to mention explicitly is when $G(q_i) = G(q)$ for some fixed $q$. In this case, we have

**Theorem 9.1** *Suppose $h, G(q)$ are compatible along $A_0$. Then there exists $k > 0$ such that*

$$G = \cdots h^{-2k} G(q) h^{2k} *_{G_0} h^{-k} G(q) h^{k} *_{G_0} G(q) *_{G_0} h^{k} G(q) h^{-k} *_{G_0} h^{2k} G(q) h^{-2k} \cdots \hookrightarrow \mathrm{Mod}(R)$$

*is an embedding and any element not conjugate into an elliptic or parabolic subgroup of $G(q)$ is pseudo-Anosov.*

*Proof.* All that is needed is the analog of Lemma 8.2 to show that there exists $k > 0$ so that

$$d(h^{rk}(\pi_{S_j}(\Theta(q))), h^{r'k}(\pi_{S_j}(\Theta(q)))) \geq 5$$

for all $r, r' \in \mathbb{Z}$ and $r \neq r'$. This follows from Proposition 7.2, Theorem 5.3, and the fact that $h_j$ acts on $\mathcal{A}^0(S_j)$ by isometries. The rest of the proof is identical to the proof of Theorem 6.1.□

We now consider the trivial HNN extension

$$G(q) *_{G_0}$$

(we are extending by $\mathbb{Z}$ over $G_0$ and the stable letter, $\tau$, acts trivially on $G_0$). We write

$$\eta : G(q) *_{G_0} \to \mathbb{Z}$$

for the epimorphism obtained by sending $G(q)$ to 0 and $\tau$ to 1.

There is a homomorphism

$$\delta : G(q) *_{G_0} \to \mathrm{Mod}(R)$$

extending the inclusion of $G(q)$ obtained by sending $\tau$ to $h^k$. Then the kernel of $\eta$ is isomorphic to the group $G$ from Theorem 9.1 in such a way that the restriction of $\delta$ is the inclusion of $G$ into $\mathrm{Mod}(R)$.

**Corollary 9.2** *With the notation as above,*

$$\delta : G(q) *_{G_0} \hookrightarrow \mathrm{Mod}(R)$$

*is an embedding. Moreover, any element not conjugate into an elliptic or parabolic subgroup or into $\langle h^k \rangle$ is pseudo-Anosov.*

*Proof.* We have the short exact sequence

$$1 \to G \to G(q) *_{G_0} \to \mathbb{Z} \to 1$$

which splits by the homomorphism sending 1 to $\tau$. By construction, the splitting homomorphism composed with $\delta$ is injective with image $\langle h^k \rangle$.

Any element in $\phi \in G(q) *_{G_0} \setminus \{1\}$ is a product $\tau^n \gamma$ where $\gamma \in G$ and $n \in \mathbb{Z}$. If $\delta(\phi) = 1$ then $n \neq 0$ and $\gamma \neq 1$ since $\delta$ restricted to each of $G$ and $\langle \tau \rangle$ is injective. Moreover

$$\gamma = \delta(\tau^{-n} \tau^n \gamma) = \delta(\tau^{-n} \phi) = \delta(\tau^{-n}) = h^{-nk}$$

This is impossible since the element $\gamma \in G$ is either pseudo-Anosov, a root of a positive multi-twist, or of finite order by Theorem 9.1 and Theorem 5.4, whereas $h^{-nk}$ is reducible with all pseudo-Anosov components by hypothesis.

The proof of the pseudo-Anosov statement follows by an argument similar to that in the proof of Theorem 6.1 where we conjugate our element to have the form

$$\phi = h^{nk} \phi_{i_1} \cdots \phi_{i_r}$$

with $\phi_{i_j} \in h^{i_j k} G(q) h^{-i_j k} \backslash G_0$, $i_j \neq i_{j+1} \pmod r$, and $r$ minimal. For then, $\phi^m(A_0) \in h^{nk+i_1 k}(\Theta(q))$ and $\phi^{-m}(A_0) \in h^{i_r k}(\Theta(q))$ for all $m \geq 1$. These have projections which are a distance at least 5 apart, unless $n + i_1 = i_r$. However, in this case $\phi$ is conjugate to

$$h^{nk} \phi_{i_2} \cdots \phi_{i_{r-1}} (\phi_{i_r} h^{nk} \phi_{i_1} h^{-nk})$$

and $\phi_{i_r} h^{nk} \phi_{i_1} h^{-nk} \in h^{i_r k} G(q) h^{-i_r k}$ which contradicts the minimality of $r$. We may now proceed as in Theorem 6.1. $\square$

# 10 Concluding remarks

## 10.1

A more geometric description of Veech groups is the following. Any $q \in \mathcal{A}_2\mathcal{T}(R)$ defines a holomorphic totally geodesic embedding of the hyperbolic plane

$$f_q : \mathbb{H}^2 \to \mathbb{H}_q \subset \mathcal{T}(R)$$

The stabilizer $\mathrm{Stab}_{\mathrm{Mod}(R)}(\mathbb{H}_q)$ acts on $\mathbb{H}_q$ and this action can be conjugated back to $\mathbb{H}^2$, via $f_q$, thus defining a homomorphism to $\mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{Aut}(\mathbb{H}^2)$. It turns out that $\mathrm{Stab}_{\mathrm{Mod}(R)}(\mathbb{H}_q) = \mathrm{Aff}^+(q)$ and with the appropriate identifications, the homomorphism to $\mathrm{PSL}_2(\mathbb{R})$ is given by $D$. In the special case that $\mathbb{H}^2/\mathrm{PSL}(q)$ has finite area, this quotient is called a Teichmüller curve and this immerses into the moduli space.

This sheds some additional light on the analogy with Kleinian groups. In particular, when the Fuchsian subgroups $G_1, G_2$ lie in a fixed Kleinian group $\Gamma$, these define totally geodesic surfaces in the associated hyperbolic 3-manifold $\mathbb{H}^3/\Gamma$. If $h$ is taken to lie in $\Gamma$ (and is sufficiently complicated), then the amalgamated product injects into $\Gamma$. In this case, one can view the construction as truncating one of the cusps of each of the totally geodesic surfaces and connecting the exposed boundary components with an annulus whose co-core is determined by $h$. Similarly, we can view our construction as truncating one of the cusps of the Teichmüller curves and connect the exposed boundary components by an annulus in the moduli space whose co-core is determined by $h$. We remark that our theorem (as in the Kleinian group case) does not require $\mathrm{PSL}(q)$ to have finite area.

To further the analogy with hyperbolic spaces, we recall that Thurston's compactification of $\mathcal{T}(R)$ is obtained by adding $\mathbb{P}\mathcal{ML}_0(R)$ at infinity to obtain

$$\overline{\mathcal{T}}(R) = \mathcal{T}(R) \cup \mathbb{P}\mathcal{ML}_0(R) \cong B^{6g-6}$$

where $B^{6g-6}$ is the closed ball of dimension $6g-6$ and $\mathbb{P}\mathcal{ML}_0(R)$ is identified with the boundary. Moreover, $\mathbb{P}\mathcal{ML}_0(R)$ has a natural piecewise projective structure and the action of $\mathrm{Mod}(R)$ on $\mathcal{T}(R)$ and $\mathbb{P}\mathcal{ML}_0(R)$ fit together to give a well defined action on $\overline{\mathcal{T}}(R)$ which is holomorphic on the interior and piecewise projective on the boundary.

There is a natural identification of $\mathbb{P}\mathcal{ML}_0(q)$ with the boundary at infinity $\partial_\infty \mathbb{H}^2$. In this way, the inclusion of $\mathbb{P}\mathcal{ML}_0(q)$ into $\mathbb{P}\mathcal{ML}_0(R)$ can be thought of as an extension $\partial_\infty f_q$ of $f_q$ to infinity. Indeed, the natural projective structure on $\mathbb{RP}^1 = \partial_\infty \mathbb{H}^2 = \mathbb{P}\mathcal{ML}_0(q)$ makes $\partial_\infty f_q$ into a piecewise projective embedding, equivariant with respect to the $\mathrm{Aff}^+(q)$ action.

Moreover, $\partial_\infty f_q$ sends the limit set $\Lambda(\mathrm{PSL}(q)) \subset \partial_\infty \mathbb{H}^2 = \mathbb{P}\mathcal{ML}_0(q)$ homeomorphically and $\mathrm{Aff}^+(q)$-equivariantly to the limit set $\Lambda(\mathrm{Aff}^+(q)) \subset \mathbb{P}\mathcal{ML}_0(R)$ as defined by McCarthy and Papadopoulos [29].

The map

$$\overline{f}_q = f_q \cup \partial_\infty f_q : \overline{\mathbb{H}}^2 = \mathbb{H}^2 \cup \mathbb{P}\mathcal{ML}_0(q) \to \overline{\mathcal{T}}(R)$$

is continuous for every $p \in \mathbb{H}^2$ and almost every $p \in \mathbb{P}\mathcal{ML}_0(q)$ by a theorem of Masur [25]. However, Masur's theorem implies that this is in general not continuous at *every* point of $\mathbb{P}\mathcal{ML}_0(q)$.

This now leads us to the following natural question, the analog of which is true in the setting of Kleinian groups.

**Question 10.1** *Let $G \cong \pi_1(S_{2g}) \to \mathrm{Mod}(R_g)$ be the embedding given by Corollary 1.2. Consider $\partial_\infty(G)$ which can be canonically identified with the circle at infinity of the universal cover $\widetilde{S}_{2g} \cong \mathbb{H}^2$ of $S_{2g}$. Does there exist a continuous $G$-equivariant map*

$$\partial_\infty(G) \to \mathbb{P}\mathcal{ML}_0(R)$$

It is not hard to see that if such a map exists, it must be unique. In fact, the proposed map is already defined on a subset of $\partial_\infty(G)$, namely, the limit points of the conjugates of the Veech groups in the amalgamation.

## 10.2

More generally, it is known that many classes of lattices in Lie groups cannot inject into $\mathrm{Mod}(R)$. Indeed if the lattices are superrigid the image of such a lattice in $\mathrm{Mod}(R)$ is necessarily finite (see [11] and [37]). From this it follows (cf. Theorem 2 of [37]), that the only lattices that can admit a faithful representation (or even an infinite representation) into $\mathrm{Mod}(R)$ are lattices in $\mathrm{SO}(m,1)$, $m \geq 2$ or $\mathrm{SU}(q,1)$, $q \geq 1$. Indeed, since solvable subgroups of $\mathrm{Mod}(R)$ are virtually abelian [4], this observation also excludes non-cocompact lattices of $\mathrm{SU}(q,1)$ for $q \geq 2$ from injecting.

In light of these comments and the results of this paper, it seems interesting to ask whether such injections can arise for (cocompact) lattices in $\mathrm{SO}(m,1)$, $m \geq 3$ or $\mathrm{SU}(q,1)$, $q \geq 2$. There are simple obstructions to injecting certain of these lattices, or indeed for any group. Namely if a finitely generated group $G$ admits an injection into $\mathrm{Mod}(R)$ for some $R$, the $\mathrm{vcd}(G) \leq \mathrm{vcd}(\mathrm{Mod}(R))$ (see [5]). If the vcd of a group $G$ satisfies the above inequality, then we call $G$ *admissable for $R$* or simply *admissable.* It follows that for lattices in $\mathrm{SO}(m,1)$, $m \geq 3$ or $\mathrm{SU}(q,1)$, $q \geq 2$ and for a fixed $R$, the values of $m$ and $q$ above are bounded.

The vcd's of the groups $\mathrm{Mod}(R) = \mathrm{Mod}(R_{g,n})$ are well-known (see [17]). If $R$ is not sporadic, the vcd of $\mathrm{Mod}(R)$ is $4g-5$ if $n=0$, $4g-4+n$ if $n>0$ and $g \geq 1$ and $n-3$ if $n \geq 5$.

Motivated by this discussion we pose.

**Question 10.2** *Suppose $R_{g,n}$ is not sporadic. Let $\Gamma$ be an admissable lattice in* $\mathrm{SO}(m,1)$*, $m \geq 3$ or* $\mathrm{SU}(q,1)$*, $q \geq 2$. Does $\Gamma$ inject in* $\mathrm{Mod}(R)$*? If such an injection exists does there exist a continuous $\Gamma$-equivariant map*

$$\partial_\infty(\Gamma) \to \mathbb{P}\mathcal{ML}_0(R)?$$

More generally, for a fixed $R$ and hence fixed vcd, one can ask the above question for negatively curved groups.

Thus we pose the following question.

**Question 10.3** *Suppose $R_{g,n}$ is not sporadic. Which 1-ended admissable hyperbolic groups $G$ inject in* $\mathrm{Mod}(R)$*? If such an injection exists does there exist a continuous $G$-equivariant map*

$$\partial_\infty(G) \to \mathbb{P}\mathcal{ML}_0(R)?$$

For the sporadic surfaces, the associated mapping class groups are virtually free, hence no 1-ended group embeds.

Work of a similar vein is done in [12], where convex cocompact subgroups of $\mathrm{Mod}(R)$ are investigated.

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