MANY CUSPED HYPERBOLIC 3-MANIFOLDS DO NOT BOUND GEOMETRICALLY

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Abstract. In this note, we show that there exist cusped hyperbolic 3-manifolds that embed geodesically, but cannot bound geometrically. Thus, being a geometric boundary is a non-trivial property for such manifolds. Our result complements the work by Long and Reid on geometric boundaries of compact hyperbolic 4-manifolds, and by Kolpakov, Reid and Slavich on embedding arithmetic hyperbolic manifolds.

Key words: 3-manifold, 4-manifold, hyperbolic geometry, cobordism, geometric boundary.

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1. Introduction

In the sequel, all hyperbolic manifolds are assumed to be connected, orientable, complete, and of finite volume. We are interested in *cusped*, i.e. non-compact, manifolds.

A hyperbolic n-manifold M *bounds geometrically* if it is isometric to ∂W, for a hyperbolic $(n+1)$-manifold W with totally geodesic boundary, c.f. [18], and also [14, 16, 19, 20, 22, 28, 29] for further progress in this topic. A hyperbolic n-manifold M is said to *embed geodesically* if there exists a hyperbolic $(n+1)$-manifold N that contains a totally geodesic hypersurface isometric to M. We remark that many arithmetic hyperbolic 3-manifolds of simplest type embed geodesically by [15].

A geometrically bounding manifold embeds geodesically, but the converse is not necessarily true. Indeed, the Euler characteristic $\chi(M)$ of a geometrically bounding manifold M must be even. This can be seen by taking a hyperbolic $(n+1)$-manifold N with totally geodesic boundary $\partial N = M$, and doubling it along M in order to obtain a hyperbolic manifold DN. By the excision property, we have for the Euler characteristic that $\chi(DN) = 2\,\chi(N) - \chi(M)$. If n is odd, we have that $\chi(M) = 0$, while if n is even, then $\chi(DN) = 0$, and $\chi(M)$ is thus even. The fact that an odd-dimensional cusped hyperbolic manifold has $\chi = 0$ follows from Margulis' Lemma and the 1$^{\text{st}}$ Bieberbach Theorem.

Thus, the thrice-punctured sphere cannot bound geometrically. On the other hand, this manifold is arithmetic and of even dimension, so by [15] it embeds geodesically. This same discussion also applies when $n = 4, 6$, since the respective minimal-volume arithmetic manifolds constructed in [7, 27] have Euler characteristic $\chi = \pm 1$. Note that such an argument becomes vacuous if n is odd.

The aim of this note is providing examples of hyperbolic 3-manifolds that embed geodesically, but fail to bound geometrically, thereby explicitly showing that bounding is much more non-trivial to arrange for $n = 3$ too.

In particular, we show that several well-known cusped hyperbolic 3-manifolds cannot bound geometrically. Namely, we prove the following theorems, the first of which should be contrasted with [29], which shows that the figure-eight knot complement bounds geometrically.

Theorem 1.1. *The figure-eight knot sibling* 3*-manifold embeds geodesically but does not bound geometrically.*

The figure-eight knot complement and its "sibling" manifold are precisely the cusped hyperbolic 3-manifolds of smallest volume [3]. Both of them are known to be arithmetic [21] with invariant trace-field $\mathbb{Q}(\sqrt{-3})$. Our methods also show:

Theorem 1.2. *A single-cusped hyperbolic* 3*-manifold with invariant trace-field of odd degree does not bound geometrically.*

There are many such examples of single-cusped hyperbolic 3-manifolds, indeed even arising as knot complements in S^3. We record the following corollary of Theorem 1.2. This follows automatically from [11] which establishes that if K_m is the m-twist knot (see Figure 1) the degree of the invariant trace-field is given by $\mathrm{cr}(K_m)-2$ where $\mathrm{cr}(K_m)$ is the crossing number of K_m (here $m \neq -2,-1,0,1$). Note that the figure-eight knot is the 2-twist knot as depicted in Figure 1. If we assume that $m \geq 2$, then $\mathrm{cr}(K_m)-2=m$.

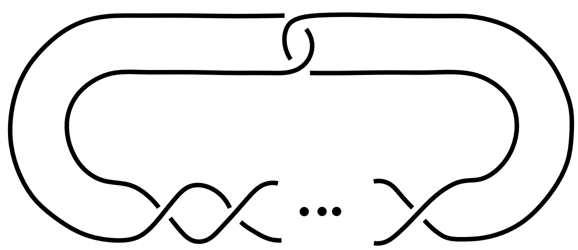

FIGURE 1. The twist knot K_m ($m>0$) in its alternating (and thus minimal) projection.

Corollary 1.3. *Let K_m be the m-twist with $m>1$ odd. Then $S^3 \smallsetminus K_m$ does not bound geometrically.*

Note that the figure-eight knot's sibling manifold does not satisfy Theorem 1.2, since as noted above its invariant trace-field has degree two. In addition, at present we do not know *a single example* of a finite volume hyperbolic 3-manifold with odd degree invariant trace-field that even embeds totally geodesically.

The proofs of Theorems 1.1 and 1.2 essentially follow from a simple observation which applies in all dimensions about the cusps of manifolds that bound geometrically. In the case of $n=3$, this implies that the cusp of a single-cusped hyperbolic 3-manifold that bounds geometrically must be "rectangular"(c.f. Proposition 2.8). We refer to Section 2 for the terminology. In the case of $n=4$ this gives another proof that the minimal volume hyperbolic 4-manifolds of [27] do not bound geometrically (see Section 2.4).

Remark 1.4. Nimershiem [26] proved that the cusp shapes of single-cusped hyperbolic 3-manifolds form a dense set in the moduli space of the 2-torus. Since the set of rectangular tori is nowhere dense, this lends credence to the claim that "most single-cusped hyperbolic 3-manifolds do not bound geometrically". Similar considerations were known to hold in the compact setting. Indeed, in [18] it is shown that if a closed hyperbolic 3-manifold bounds geometrically, then it has integral η-invariant. On the other hand, Meyerhoff [24] showed that a reduction modulo $\frac{1}{3}$ of the η-invariant of closed hyperbolic 3-manifolds takes values in a dense subset of the circle.

As a possible measure of geometric "complexity" of embedding geodesically a hyperbolic n-manifold M into a hyperbolic $(n+1)$-manifold N, or making M bound a hyperbolic $(n+1)$-manifold W geometrically, we introduce the following quantities:

$$\varepsilon\text{-vol}(M) = \min_N \mathrm{vol}(N), \quad \partial\text{-vol}(M) = \min_W \mathrm{vol}(W),$$

where "vol", here and below, means hyperbolic volume. It is easy to see that if M bounds, then $\varepsilon\text{-vol}(M) \leq 2 \cdot \partial\text{-vol}(M)$.

In [29], Slavich proved that the figure-eight knot complement has ∂-vol $= \frac{4\pi^2}{3}$, i.e. the minimum possible by the Gauß-Bonnet theorem. Concerning ε-vol, we adopt his technique to improve Theorem 1.1 as follows:

Theorem 1.5. *The figure-eight knot complement and its sibling manifold have* ε-vol $= \frac{4\pi^2}{3}$.

Note that if M embeds geodesically Miyamoto's work [25] implies that $\partial\text{-vol}(M) \geq d_{n+1}\,\mathrm{vol}(M)$ and $\varepsilon\text{-vol}(M) \geq 2\, d_{n+1}\,\mathrm{vol}(M)$, where d_n is the optimal horoball packing density in $\mathbb{H}^n$, c.f. [13, Table 3]. The former inequality holds also if M is disconnected, and the latter follows by cutting some (if any) N in which M embeds along the respective hyper-surface isometric to M, and considering the resulting manifold $N_{/\!/}M$ with boundary $M \sqcup (-M)$. The equality, in the case of bounding manifolds, can be attained only in dimensions $n = 2$ and 3.

The paper is organised as follows: in Section 2 we prove Theorems 1.1 and 1.2, in Section 3 we prove Theorem 1.5.

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2. Cusp sections of geometric boundaries

In this section, we provide a simple obstruction for a non-compact manifold to bound geometrically, formulated in Propositions 2.5 and 2.8, which we then use in order to prove Theorems 1.1 and 1.2.

2.1. Cusps with boundary.

In this section, we analyse the ends of a hyperbolic manifold with totally geodesic boundary. We basically follow [10, 2.10.C].

Definition 2.1. An $(n+1)$-dimensional *cusp with boundary* is a Riemannian warped product $C = F \times_f (0, +\infty)$, where F is a compact connected flat n-manifold with totally geodesic boundary and $f(r) = e^{-r}$.

This means that $F \times (0, +\infty)$ is endowed with the Riemannian metric $e^{-2r} g + dr^2$, where g is the flat metric on F.

Definition 2.2. A *section* of a cusp C as above is a level set $F \times \{r_0\} \subset C$.

Note that all sections of C are homothetic.

Definition 2.3. The *shape* of a cusp C is the similarity class of a section.

The following fact is well known:

Proposition 2.4. [10, 2.10.D] *Let W be a complete, finite-volume hyperbolic $(n+1)$-manifold with (possibly empty) totally geodesic boundary. There is a compact subset of W whose complement is isometric to a disjoint union of cusps with (possibly empty) boundary.*

Given W as above, we call *boundary cusps* the cusps of W with non-empty boundary. Each cusp of the hyperbolic n-manifold ∂W is a boundary component of a boundary cusp of W.

2.2. **Cusps of bounding manifolds.** We now furnish an obstruction for a hyperbolic manifold to bound geometrically.

Proposition 2.5. *If a cusped hyperbolic n-manifold M bounds geometrically, then the cusps of M that do not admit a fixed-point-free orientation-reversing isometric involution are isometric in pairs.*

In particular, the number of such cusps has to be even, possibly zero. The proof will follow easily from a simple lemma about flat manifolds with totally geodesic boundary:

Lemma 2.6. *Let F be a compact connected orientable flat n-manifold with non-empty totally geodesic boundary, such that F is not isometric to a product with an interval. Then, ∂F is connected and has a fixed-point-free orientation-reversing isometric involution.*

Proof. The Riemannian universal cover of any constant-curvature manifold with totally geodesic boundary embeds isometrically into the model space as the intersection of some half-spaces with pairwise disjoint boundaries. In the flat case, the number of such half-spaces can only be 0, 1, or 2. The first case is excluded because $\partial F \neq \emptyset$, while the second one is excluded because F is compact.

Thus, the universal cover $\widetilde{F}$ of F is isometric to a strip $\mathbb{R}^{n-1} \times I \subset \mathbb{R}^n$, where $I = [-a, a]$, and F is isometric to a quotient of this strip by a discrete group of Euclidean isometries acting on it. Every isometry of the strip $\widetilde{F}$ must preserve the I-fibration and the 0-section $\mathbb{R}^{n-1} \times \{0\}$. This implies that there is an I-bundle $\pi\colon F \to B$ whose 0-section $B_0 \subset F$ is a totally geodesic hypersurface. Note that $\pi\big|_{\partial F}$ is a Riemannian double covering.

By hypothesis the bundle π is non-trivial, so B_0 is a one-sided hypersurface inside F, and B is non-orientable. Thus, $\pi\big|_{\partial F}$ is the orientation double cover of B, and ∂F has a fixed-point-free orientation-reversing involution. $\square$

Remark 2.7. It follows from the proof of Lemma 2.6 that if $n = 3$ and ∂F is connected, then F is diffeomorphic to $K \tilde{\times} I$, which is the orientable manifold arising as a twisted I-bundle over the Klein bottle.

We are ready to prove Proposition 2.5.

Proof of Proposition 2.5. Let $M = \partial W$ for a hyperbolic $(n+1)$-manifold W with totally geodesic boundary, and let $C' \subset M$ be a cusp of M with section F'. Then, $F' \subset \partial F$ is a boundary component of a section F of a boundary cusp C of W. If F' has no fixed-point-free orientation-reversing involution, by Lemma 2.6 we have $F \cong F' \times I$, and F' is isometric to a section of another cusp of M. $\square$

2.3. **Rectangular tori.** In this section, we give a more precise characterisation of cusp shapes of geometrically bounding manifolds in the case $n = 3$, and then prove Theorems 1.1 and 1.2.

A cusp of a hyperbolic 3-manifold has section a flat 2-torus. Recall that a flat torus $T = \mathbb{R}^2/\Gamma$ has a fixed-point-free orientation-reversing isometric involution if and only if a conjugate of the lattice Γ is generated by two vectors that span a rectangle or a rhombus. We call such flat tori respectively *rectangular* or *rhombic*.

In the usual fundamental domain

$$\mathcal{D} = \{|z| \geq 1,\ |\mathrm{Re}(z)| \leq 1/2,\ \mathrm{Im}(z) > 0\} \subset \mathbb{C}$$

for the moduli space of tori (c.f. for instance [8, §12.2] and [23, §4.2]), the rectangular and rhombic ones correspond to the curves $\mathcal{D} \cap \{\mathrm{Re}(z) = 0\}$ and $\mathcal{D} \cap \{|z| = 1 \text{ or } |\mathrm{Re}(z)| = 1/2\}$, respectively. Thus, we call a cusp of a hyperbolic 3-manifold *rectangular* or *rhombic* depending on the shape of its section.

With these definitions in hand, we can now improve Proposition 2.5:

Proposition 2.8. *If a cusped hyperbolic 3-manifold M bounds geometrically, then the non-rectangular cusps of M are isometric in pairs.*

Proof. Suppose again $M = \partial W$, and let $C \cong F \times_f (0, +\infty)$ be a boundary cusp of W with connected boundary. By Remark 2.7, F is diffeomorphic to $K \tilde{\times} I$ and ∂F is a flat torus. We now show that ∂F is rectangular. To that end, we have

$$F \cong \left(\mathbb{R}^2 \times [-a, a]\right)/\Gamma,$$

where we can assume that the group $\Gamma < \mathrm{Isom}(\mathbb{R}^2 \times [-a, a]) < \mathrm{Isom}(\mathbb{R}^3)$ is generated by a parallel translation T_x along $(2b, 0, 0)$, a translation T_y along $(0, c, 0)$, and a roto-translation $R_x \colon (x, y, z) \mapsto (x+b, -y, -z)$ (c.f. for instance [31, Theorem 3.5.5, 2nd item]). In particular, we have

$$\partial F \cong \left(\mathbb{R}^2 \times \{a\}\right)/\langle T_x, T_y\rangle,$$

and the lattice $\langle T_x, T_y\rangle$ is generated by two vectors spanning a rectangle. □

We can now prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1: Let $\mathbb{H}^3/\Gamma$ be the figure-eight's sibling manifold. Up to conjugation, Γ is an index 12 subgroup in $PSL_2(\mathcal{O}_3)$, c.f. [21, §13.7.1(vi)]. Thus, Γ can be embedded in $SO(q, \mathbb{Z})$ for a quadratic form of signature $(3, 1)$, c.f. [4, §3] and [6, 12]. The argument given in [15, §9.1] now applies to show that the sibling manifold embeds geodesically.

However, from [30], one sees that the modulus of the cusp is $\omega = \frac{-1+\sqrt{-3}}{2}$, and hence is not rectangular. Thus, by Proposition 2.8, the figure-eight knot's sibling manifold does not bound geometrically. □

Remark 2.9. The cusp shape of the figure-eight knot sibling can also be computed using SnapPy [5]. By setting `M = Manifold('m003')` in SnapPy (where `'m003'` is the entry for the figure-eight knot's sibling manifold in the Callahan-Hildebrand-Weeks census [2]), and issuing the command `M.cusp_info(0).modulus`, one sees that the cusp section of the sibling manifold is not rectangular (a numerical estimate suffices).

Proof of Theorem 1.2: Let T be the cusp section of a single-cusped hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ whose invariant trace field has odd degree. We shall show that the odd degree assumption precludes T from being rectangular. Thus, assume to the contrary that T is rectangular. Let K be the trace-field of Γ, and k its invariant trace field; i.e. the trace field of the group $\Gamma^{(2)} = \langle g^2 \mid g \in \Gamma \rangle$.

By [21, Theorem 4.2.3], we may assume that up to conjugation $\Gamma \subset PSL_2(K)$, and moreover $\pi_1(T) = \langle a, b \rangle$, with

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

and in addition, that there exists an element $x \in \Gamma$ such that

$$x = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

The complex number s above is the modulus parameter for the torus T.

Since $a^2, b^2, x^2 \in \Gamma^{(2)}$, it follows that $\mathrm{tr}(a^2x^2) \in k$, hence $t \in k$. Furthermore, $\mathrm{tr}(b^2x^2) = 2+4st \in k$. Thus the cusp parameter s of M belongs to k, which implies that s has odd degree over $\mathbb{Q}$. Hence $F = \mathbb{Q}(s) \subset k$ is a sub-field of k, having odd degree. By Proposition 2.8, T is a rectangular torus, and thus its modulus belongs to the imaginary axis $i\mathbb{R}$. However, if $s = ir$, for some $r \in \mathbb{R}$, then F is preserved by complex conjugation, implying that the real sub-field $F \cap \mathbb{R}$ has degree 2 in F, contradicting the fact that the degree of F is odd. This completes the proof. □

2.4. **Minimal volume hyperbolic 4-manifolds.** The Ratcliffe-Tschantz census [27] contains most of the known cusped hyperbolic 4-manifolds of minimal volume. All of these manifolds are arithmetic. In particular, by [15] they all embed geodesically.

By [27, Table 2], each of the 22 Ratcliffe-Tschantz orientable 4-manifolds has an odd number of cusps with section diffeomorphic to the so-called *Hantzsche-Wendt manifold* (denoted by F in [27] and by $\mathcal{G}_6$ in [31]). This flat 3-manifold has no fixed-point-free orientation-reversing self-homeomorphism (c.f. [31, Theorem 3.5.9] and also [32]). Thus, by Proposition 2.5, none of the manifolds from [27, Table 2] bounds geometrically.[1]

3. Embedding the figure-eight's sibling

Although Theorem 1.1 shows that the figure eight knot sibling embeds in a hyperbolic 4-manifold, it gives little control on the topology of the 4-manifold. It is the purpose of this section to prove Theorem 1.5 using an approach due to Slavich [29], and which will afford additional control. We start with some necessary definitions.

Definition 3.1. A 4-dimensional *triangulation* $\mathcal{T}$ is a pair $(\{\Delta_i\}_{i=1}^{2k}, \{g_j\}_{j=1}^{5k})$, where k is a positive integer, the Δ_i's are copies of the standard 4-dimensional simplex, and the g_j's are simplicial pairings between all the $10k$ facets of the Δ_i's.

Definition 3.2. A triangulation $\mathcal{T}$ is *orientable* if it is possible to choose an orientation for each Δ_i so that all the g_j's are orientation-reversing (c.f. also [17, Definition 4.2]).

Definition 3.3. A 4-dimensional triangulation $\mathcal{T}$ is 6-*valent* if all cycles of 2-faces in $\mathcal{T}$ have length exactly 6.

With each cycle c of 2-faces in $\mathcal{T}$ there is a naturally associated return map r_c from a 2-simplex to itself. In order to obtain it, one has to follow the simplicial pairings from one 4-simplex to the following one, until the cycle closes up.

Our proof will make essential use of the following fact.

[1] One can also arrive at this conclusion by the Euler characteristic argument mentioned in Section 1.

Proposition 3.4 (Proposition 3.9 in [29]). *Let M be a hyperbolic 3-manifold obtained by glueing through isometries the sides of some copies of the hyperbolic ideal regular tetrahedron. If this glueing can be realised as the link of a vertex in a 6-valent orientable 4-dimensional triangulation $\mathcal{T}$ with trivial return maps, then M embeds geodesically. Moreover,*

$$\varepsilon\text{-vol}(M) \leq \frac{4\pi^2}{3} \cdot \frac{2k}{3},$$

where $2k$ is the number of 4-simplices in $\mathcal{T}$.

Sketch of proof. By replacing each 4-simplex of $\mathcal{T}$ with an ideal hyperbolic rectified 4-simplex, one gets a hyperbolic 4-manifold W with totally geodesic boundary ∂W tessellated by ideal regular tetrahedra. The link of each vertex of $\mathcal{T}$ gives the tessellation into tetrahedra of a boundary component of W. The manifold M embeds geodesically in the double of W. Finally, the volume of the ideal rectified 4-simplex is $\frac{2\pi^2}{9}$ [17]. □

We are ready to prove Theorem 1.5.

Proof of Theorem 1.5: We will adopt the usual ideal triangulations of the figure-eight knot complement and its sibling manifold by regular hyperbolic tetrahedra. Each of them consists of two tetrahedra, A and B, with the following glueing maps between their 2-faces.

For the figure-eight knot complement, depicted in Figure 2–(i), we set:

$$\begin{array}{ccc} A & & B \\ (1,2,3) & \leftrightarrow & (3,2,1) \\ (1,2,4) & \leftrightarrow & (1,4,2) \\ (1,3,4) & \leftrightarrow & (3,4,2) \\ (2,3,4) & \leftrightarrow & (4,1,3), \end{array} \tag{1}$$

and for the figure-eight sibling manifold, depicted in Figure 2–(ii), we set:

$$\begin{array}{ccc} A & & B \\ (1,2,3) & \leftrightarrow & (4,1,2) \\ (1,2,4) & \leftrightarrow & (3,4,1) \\ (1,3,4) & \leftrightarrow & (1,3,2) \\ (2,3,4) & \leftrightarrow & (2,4,3). \end{array} \tag{2}$$

Let Y be the cone over the 3-dimensional triangulation in Figure 2–(i), and X, Z be two copies of the cone over the triangulation in Figure 2–(ii).

This means that each of X, Y and Z consists of two 4-simplices A' an B' whose facets are identified as follows:

$$Y : \left\{ \begin{array}{ccc} A' & & B' \\ (1,2,3,5) & \leftrightarrow & (3,2,1,5) \\ (1,2,4,5) & \leftrightarrow & (1,4,2,5) \\ (1,3,4,5) & \leftrightarrow & (3,4,2,5) \\ (2,3,4,5) & \leftrightarrow & (4,1,3,5), \end{array} \right. \tag{3}$$

and

$$X, Z : \left\{ \begin{array}{ccc} A' & & B' \\ (1,2,3,5) & \leftrightarrow & (4,1,2,5) \\ (1,2,4,5) & \leftrightarrow & (3,4,1,5) \\ (1,3,4,5) & \leftrightarrow & (1,3,2,5) \\ (2,3,4,5) & \leftrightarrow & (2,4,3,5). \end{array} \right. \tag{4}$$

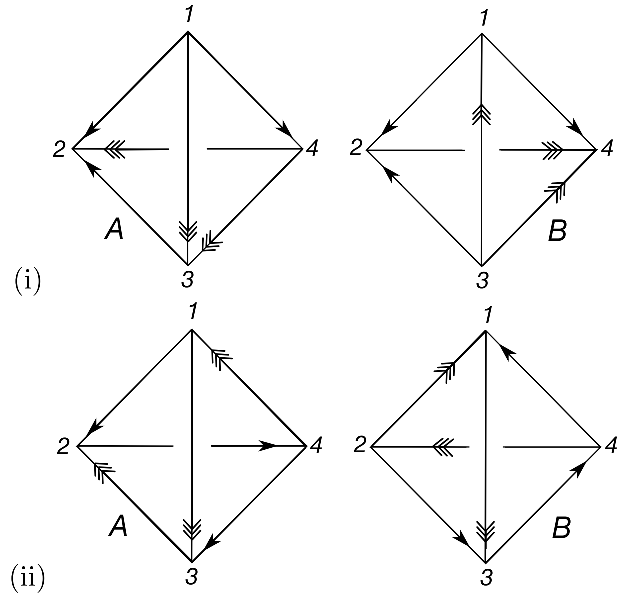

FIGURE 2. Ideal triangulations: (i) figure-eight knot complement; (ii) figure-eight sibling manifold.

Observe that each of X, Y and Z has two remaining facets A and B with vertices $\{1, 2, 3, 4\}$ unidentified. We shall build a 4-dimensional triangulation $\mathcal{T}$ by pairing these free facets of X, Y and Z as depicted in Figure 3. The map σ_{XY} will be used in order to identify facet

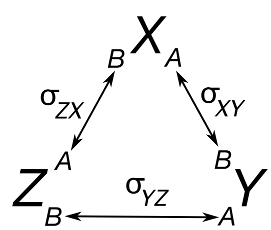

FIGURE 3. Identifying the free facets of X, Y, and Z.

A of X to B of Y, and an analogous notation σ_{YZ} and σ_{ZX} is adopted for the remaining maps. We set:

$$\begin{aligned} \sigma_{XY} &: (1,2,3,4) \to (3,1,4,2) \\ \sigma_{YZ} &: (1,2,3,4) \to (3,4,2,1) \\ \sigma_{ZX} &: (1,2,3,4) \to (2,4,1,3). \end{aligned} \tag{5}$$

We now check that the resulting 4-dimensional triangulation $\mathcal{T}$ satisfies the conditions of Proposition 3.4. First of all, $\mathcal{T}$ is orientable because the pairing maps in (3), (4) and (5) are identified with odd permutations in the symmetric group $\mathfrak{S}_5$. Let us finally check the condition on the cycles of 2-faces.

By using the glueing equations (3), (4) and (5), together with the diagram in Figure 3, we can compute the cycles of the 2-faces with no vertex labelled 5:

$$\begin{aligned} X_A : (1,2,3) \to Y_B : (3,1,4) \to Y_A : (4,3,2) \to Z_B : (1,2,4) \to \\ \to Z_A : (2,3,1) \to X_B : (4,1,2) \to X_A : (1,2,3), \end{aligned}$$

$$X_A : (1,2,4) \to Y_B : (3,1,2) \to Y_A : (1,3,2) \to Z_B : (3,2,4) \to$$
$$\to Z_A : (4,2,3) \to X_B : (3,4,1) \to X_A : (1,2,4),$$

$$X_A : (1,3,4) \to Y_B : (3,4,2) \to Y_A : (1,3,4) \to Z_B : (3,2,1) \to$$
$$\to Z_A : (3,4,1) \to X_B : (1,3,2) \to X_A : (1,3,4),$$

$$X_A : (2,3,4) \to Y_B : (1,4,2) \to Y_A : (1,2,4) \to Z_B : (3,4,1) \to$$
$$\to Z_A : (1,2,4) \to X_B : (2,4,3) \to X_A : (2,3,4).$$

All such cycles have length 6 and trivial return maps. The same conclusion holds for the cycles of 2-faces containing the vertex 5, since they correspond to the glueing of edges of the simplices A and B in the manifold triangulations from Figure 2.

By using Regina [1] we can conclude that $\mathcal{T}$ has 4 vertices.[2] Two of their links are isomorphic to the sibling manifold triangulation, one to the figure-eight triangulation, and the remaining fourth link is isomorphic to the triangulation of the manifold $O =$ `otet24_00260` from the census [9] of tetrahedral manifolds. Thus, as described in the proof of Proposition 3.4, we have

$$\partial W \cong K \sqcup L \sqcup L \sqcup O,$$

where K is the figure-eight knot complement, and L is its sibling manifold.

The figure-eight knot complement K is the orientation double-cover of the non-orientable Gieseking manifold, while O is the orientation double-cover of the non-orientable manifold `ntet12_00019` from [9] (as one can verify by SnapPy). Thus, we can quotient the O and K boundary components of W, in order to obtain a hyperbolic manifold W' with two boundary components, each isometric to L. By identifying the L boundary components of W', we obtain a hyperbolic 4-manifold N of volume $\frac{4\pi^2}{3}$, in which the sibling manifold L embeds geodesically.

Similarly, Slavich produced a hyperbolic 4-manifold W'' with totally geodesic boundary

$$\partial W'' \cong K \sqcup K \sqcup K \sqcup O'$$

with $\mathrm{vol}(W'') = \frac{4\pi^2}{3}$, where O' is another tetrahedral 3-manifold with an orientation-reversing fixed-point-free involution [29, Remark 4.4]. To conclude the proof for the figure-eight knot complement, we glue together two K-components of $\partial W''$ via an isometry, and quotient the remaining boundary components as before.

□

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