373K Algebra I, Homework 3

From Artin

Chapter 2 (pp. 69–73): 4.6, 5.3, 5.6, 6.6, 6.7, 8.1, 8.3, 8.4, 8.6

Others:

1. Complete the proof that D_4 has order 8 and the the set of quaternions Q (from class) is a group of orrder 8 that is not isomorphic to D_4.

2. Let

$$\mathrm{Aut(G)} = \{\sigma : \mathrm{G} \to \mathrm{G} : \sigma \text{ is an isomorphism}\}.$$

(i) Prove that Aut(G) is a group (with the group multiplication being composition). This is the *automorphism group* of G.

(ii) Suppose that $G = \mathbf{Z}/5\mathbf{Z}$. What is Aut(G)?

3. Let G be a group and $H, K < G$ with $|H|$ and $|K|$ being coprime. Prove that $H \cap K = 1$.

4. Compute the orders of elements in $\mathbf{Z}/12\mathbf{Z}$.

5. For p a prime, set $U_p = (\mathbf{Z}/p\mathbf{Z}) \setminus \{[0]\}$. Define $\cdot$ on U_p by $[a] \cdot [b] = [a.b]$. Prove that $\cdot$ is well-defined and that $(U_p, \cdot)$ is a group. Is U_p cyclic?

6. Classify groups of order 6 up to isomorphism.

7. Construct the lattice of subgroups (together with indices) of the quaternion group Q.