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VIRTUALLY SPINNING HYPERBOLIC MANIFOLDS

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Abstract We give a new proof of a result of Sullivan establishing that all finite volume hyperbolic n-manifolds have a finite cover admitting a spin structure. In addition, in all dimensions ≥ 5 we give the first examples of finite volume hyperbolic n-manifolds that do not admit a spin structure.

Keywords: hyperbolic manifold, spin structure

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1. Introduction

Let M be an orientable Riemannian manifold and let $w_2(M)$ denote its second Stiefel-Whitney class. Then M admits a spin structure or said to be *spinnable* if $w_2(M) = 0$; we refer the reader to §2 and §3 for more details, definitions and background.

It is well-known that all compact orientable surfaces, and compact orientable 3-manifolds are spinnable, however, the situation in higher dimensions is a good deal more subtle. For example, a well-known consequence of the Hirzebruch Signature Theorem is that the signature of any closed orientable hyperbolic 4-manifold is 0, and so according to Rochlin's Theorem (see [9]) there is no obstruction to having a spin structure. On the other hand, whether a closed hyperbolic manifold of dimension 4 (or more) is spinnable, is much harder to establish.

However, on p. 553 of his paper [21], Sullivan notes that his previous work with Deligne [5], can be used to show that if M^n is a finite volume hyperbolic n-manifold, then M^n has a finite cover that is stably parallelizable, and hence spinnable (see Remark 2.2 below).

The aim of this note is to give a simple proof of Sullivan's virtually spinning result that seems not to have been noticed previously. Furthermore, in the setting of arithmetic hyperbolic manifolds of simplest type we will provide a sharper version of Sullivan's result in many cases (see §5 for definitions).

Theorem 1.1. *Assume that $n \geq 4$ and let $M^n = \mathbb{H}^n/\Gamma$ be a well-located arithmetic hyperbolic n-manifold of simplest type from the admissible quadratic form f defined over the totally real field k. Then M^n admits a finite cover of degree $C(k, f)$ that is spinnable. The constant $C(k, f)$ is an effectively computable constant depending on k and f.*

The question as to whether all orientable hyperbolic manifolds of finite volume in dimensions ≥ 4 are spinnable, seems to be open. To that end, in §8 we point out that in each dimension $n \geq 5$ there are infinitely many finite volume (non-compact) orientable hyperbolic n-manifolds that are not spinnable.

Added in proof: It has recently been shown in [15] that for all $n \geq 4$ there exist closed orientable hyperbolic n-manifolds that are not spinnable.

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2. Spin groups and spin structures

We assume $n \geq 4$ throughout this section. General references for what follows are [9] and [12]. We will restrict our discussion to spin structures associated with the Tangent Bundle of a Riemannian manifold, rather than an arbitrary vector bundle.

The spin group in dimension n will be denoted by Spin(n). Since $n \geq 4$, Spin(n) is the universal 2-fold covering group of the special orthogonal group SO(n) (with covering map j), and determines a short exact sequence:

$$1 \to \{\pm 1\} \to \mathrm{Spin(n)} \to \mathrm{SO(n)} \to 1.$$

Now let M be a connected orientable Riemannian manifold of dimension ≥ 4 with Tangent Bundle TM. Denote by $\mathrm{SO(TM)} \to \mathrm{M}$, the SO(n)-principal bundle of oriented orthonormal frames on TM. *M admits a spin structure or is spinnable* if there is a principal Spin(n)-bundle $\mathrm{Spin(TM)} \to \mathrm{M}$ together with a 2-fold covering map $\eta : \mathrm{Spin(TM)} \to \mathrm{SO(TM)}$ such that $\eta(pg) = \eta(p)j(g)$ for all $p \in \mathrm{Spin(TM)}$ and $g \in \mathrm{Spin(n)}$.

Not every orientable Riemannian manifold admits a spin structure, the obstruction to this is the second Stiefel-Whitney class. With M as above, we let $w_2(M) \in H^2(M, \mathbb{Z}/2\mathbb{Z})$ denote the second Stiefel-Whitney class of TM. We summarize what we need in the following proposition.

Proposition 2.1. *M admits a spin structure if and only if $w_2(M) = 0$.*

Remark 2.2. A smooth orientable manifold M^n is stably parallelizable if its tangent bundle is stably trivial; i.e. $TM \oplus E_1 = E_2$ where E_i are trivial vector bundles for $i = 1, 2$. When this is the case, since Stiefel-Whitney classes are invariants of the stable equivalence class of a vector bundle, it follows that $w_2(M) = 0$, and so this proves Sullivan's result mentioned in §1.

Remark 2.3. If an oriented 4-manifold X is spinnable, then the intersection form of X is even. The converse holds if $H_1(X, \mathbb{Z})$ has no 2-torsion (see [9, Chapter II.4]).

3. Hyperbolic manifolds and spin structures

(a) *Some notation*

Let J_n be the diagonal matrix associated to the quadratic form $x_0^2 + x_1^2 + \ldots + x_{n-1}^2 - x_n^2$. We identify hyperbolic space $\mathbb{H}^n$ with $\{x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : J_n(x) = -1, x_n > 0\}$, and letting

$$\mathrm{O}(n,1) = \{X \in \mathrm{GL(n+1,\mathbb{R})} : \mathrm{X^t J_n X = J_n}\}.$$

we can identify $\mathrm{Isom}(\mathbb{H}^n)$ with the connected component of the identity of $\mathrm{O}(n,1)$, denoted by $\mathrm{O}_0(n,1)$. This is also the subgroup of $\mathrm{O}(n,1)$ preserving the hyperboloid $\{x \in \mathbb{R}^{n+1} : J_n = -1, x_{n+1} > 0\}$. Equivalently, $\mathrm{O}_0(n,1) = \mathrm{PO}(n,1)$ (the central quotient of $\mathrm{O}(n,1)$). With this notation, $\mathrm{Isom}^+(\mathbb{H}^n) = \mathrm{SO_0(n,1)}$, the index 2 subgroup in $\mathrm{O}_0(n,1)$ which is the connected component of the identity of $\mathrm{SO(n,1)}$.

Although we will not make explicit use of this, if $M = \mathbb{H}^n/\Gamma$ is a finite volume orientable hyperbolic n-manifold, then it is a well-known consequence of Mostow-Prasad Rigidity that we can conjugate

Γ to be a subgroup of $\mathrm{SO}_0(\mathrm{n},1)$ where the elements all have matrix entries in a real number field k (which we can take to be minimal).

(*b*) *Spin structures on hyperbolic manifolds*

The recent preprint [20] contains a particularly useful discussion of spin structures on hyperbolic manifolds, and we refer the reader there for a fuller discussion. We continue to assume that $n \geq 4$. As with SO(n), the group $\mathrm{SO}_0(\mathrm{n},1)$ has a universal 2-fold cover, which following [20], we denote by $\mathrm{Spin}^+(\mathrm{n},1)$ (with covering map ϕ) and as above there is an exact sequence:

$$1 \to \{\pm 1\} \to \mathrm{Spin}^+(\mathrm{n},1) \to \mathrm{SO}_0(\mathrm{n},1) \to 1.$$

Now if $M = \mathbb{H}^n/\Gamma$ is an orientable finite volume hyperbolic manifold, it is well-known that the SO(n)-principal bundle of oriented orthonormal frames on TM can be identified with $\Gamma\backslash\mathrm{SO}_0(\mathrm{n},1)$ (see [20, Section 2] for example). Via the exact sequence given above, we can construct an extension $\overline{\Gamma} < \mathrm{Spin}^+(\mathrm{n},1)$ with $\phi(\overline{\Gamma}) = \Gamma$. Note that $\overline{\Gamma}\backslash\mathrm{Spin}^+(\mathrm{n},1) \cong \Gamma\backslash\mathrm{SO}_0(\mathrm{n},1)$ since:

$$\Gamma\backslash\mathrm{SO}_0(\mathrm{n},1) \cong (\{\pm 1\}\backslash\overline{\Gamma})\backslash(\{\pm 1\}\backslash\mathrm{Spin}^+(\mathrm{n},1)) \cong \overline{\Gamma}\backslash\mathrm{Spin}^+(\mathrm{n},1).$$

The following is implicit in the proof of [20, Theorem 2.1], we include a brief sketch.

Lemma 3.1. *We follow the notation established above. Let $M = \mathbb{H}^n/\Gamma$ be an orientable finite volume hyperbolic manifold with $n \geq 4$.*

1. *Suppose that there is a subgroup $H < \overline{\Gamma}$ of index 2 so that ϕ maps H isomorphically onto Γ (or equivalently, $H \cap \{\pm 1\} = 1$), then M is spinnable.*
2. *Let $D < \overline{\Gamma}$ be of finite index containing an index 2 subgroup D_0 such that $D_0 \cap \{\pm 1\} = 1$. Let $\Delta = \phi(D) < \Gamma$, then $\mathbb{H}^n/\Delta$ is a finite cover of M that is spinnable.*

Proof. From above we have an exact sequence:

$$1 \to \{\pm 1\} \to \overline{\Gamma} \to \Gamma \to 1.$$

If H is a subgroup of index 2 as claimed, then $H\backslash\mathrm{Spin}^+(\mathrm{n},1)$ is a principal Spin(n)-bundle that double covers $\overline{\Gamma}\backslash\mathrm{Spin}^+(\mathrm{n},1)$ which from the discussion above is $\cong \Gamma\backslash\mathrm{SO}_0(\mathrm{n},1)$. Moreover, as described in the proof of [20, Theorem 2.1], the right action of SO(n) lifts to the right action of Spin(n). That is to say M is spinnable. This proves the first part.

The second part follows from the first part on noting that $\mathbb{H}^n/\Delta$ is a finite cover of M. □

4. A proof of Sullivan's theorem

We prove the following.

Theorem 4.1 (Sullivan)**.** *Let $M^n = \mathbb{H}^n/\Gamma$ be a finite volume orientable hyperbolic n-manifold. Then M is virtually spinnable.*

The proof of Theorem 4.1 will follow from the next proposition (notation as in §b).

Proposition 4.2. *Let $M^n = \mathbb{H}^n/\Gamma$ be a finite volume orientable hyperbolic n-manifold and $\overline{\Gamma} < \mathrm{Spin}^+(\mathrm{n},1)$ with $\phi(\overline{\Gamma}) = \Gamma$. Then $\overline{\Gamma}$ is residually finite.*

Given this, Theorem 4.1 is proved as follows. Residual finiteness implies that there exists a finite quotient $\psi : \overline{\Gamma} \to Q$ so that ψ is injective on $\{\pm 1\}$. Let $D < \overline{\Gamma}$ be the subgroup of finite index given by $\psi^{-1}(\psi((\{\pm 1\})))$. Then D contains a subgroup D_0 of index 2 such that $D_0 \cap \{\pm 1\} = 1$. Let $\Delta = \phi(D)$, then Δ is a finite index subgroup of Γ and $\mathbb{H}^n/\Delta$ is spinnable by Lemma 3.1. □

The proof of Proposition 4.2 requires some additional material which is described below.

We follow [12] and [20, Section 3] which provides a very helpful detailed account of the general framework we describe below.

Let V be a m-dimensional vector space over $\mathbb{R}$ and q a non-degenerate quadratic form on V. The *Clifford algebra* $\mathcal{C}\ell(V,q)$ associated to (V,q) is the associative algebra with 1 obtained from the free tensor algebra on V by adding relations $v \otimes v = -q(v).1$ for each $v \in V$. Note that V embeds naturally into $\mathcal{C}\ell(V,q)$, and $\mathcal{C}\ell(V,q)$ has the structure of a real vector space of dimension 2^m with a basis $\mathcal{B}$ constructed naturally from V (we will not dwell on this). Following [20], let $P(V,q)$ denote the multiplicative group of $\mathcal{C}\ell(V,q)$ generated by all $v \in V$such that $q(v) \neq 0$. Then, the *Spin group* of (V,q) is the subgroup of $P(V,q)$ defined as:

$$\mathrm{Spin(V,q)} = \{\mathrm{v_1.\ldots v_k : v_i \in V, q(v_i) = \pm 1 \text{ for each i, and k even}}\}.$$

In the case when $q = J_n$ (from §2), $\mathcal{C}\ell(V,q)$ is denoted $\mathcal{C}\ell(n,1)$, the group $P(V,q)$ is denoted $P(n,1)$, and $\mathrm{Spin(V,q)} = \mathrm{Spin(n,1)}$. The group $\mathrm{Spin^+(n,1)}$ is the connected component of the identity in $\mathrm{Spin(n,1)}$.

Now the group $\mathrm{Spin(n,1)}$ acts on the vector space $\mathcal{C}\ell(n,1)$ by left multiplication (on the basis $\mathcal{B}$) thereby determining a *faithful* linear representation of $L : \mathrm{Spin(n,1)} \to \mathrm{GL(2^{n+1},\mathbb{R})}$.

The proof of Proposition 4.2 is now complete since $L(\overline{\Gamma})$ is a finitely generated subgroup of $\mathrm{GL(2^{n+1},\mathbb{R})}$ and hence is residually finite by Malcev's theorem. □

Remark 4.3. Note that what is really important in Proposition 4.2 is that $\overline{\Gamma}$ is finitely generated. However, it is important that the extensions considered above are of arithmetic groups in $\mathrm{SO_0(n,1)}$, since [17] constructs arithmetic groups in $\mathrm{SL(n,\mathbb{R}) \times SL(n,\mathbb{R})}$ that have extensions by $\mathbb{Z}/2\mathbb{Z}$ that are not residually finite.

5. Arithmetic hyperbolic manifolds of simplest type

(*a*) *Quadratic forms and arithmetic lattices*

Let k be a totally real number field of degree d over $\mathbb{Q}$ equipped with a fixed embedding into $\mathbb{R}$ which we refer to as the identity embedding, and denote the ring of integers of k by R_k. Let V be an $(n+1)$-dimensional vector space over k equipped with a non-degenerate quadratic form f defined over k which has signature $(n,1)$ at the identity embedding, and signature $(n+1,0)$ at the remaining $d-1$ embeddings. Given this, the quadratic form f is equivalent over $\mathbb{R}$ to the quadratic form $x_0^2 + x_1^2 + \ldots + x_{n-1}^2 - x_n^2$, and for any galois embedding $\sigma : k \to \mathbb{R}$, the quadratic form f^σ (obtained by applying σ to each entry of f) is equivalent over $\mathbb{R}$ to $x_0^2 + x_1^2 + \ldots + x_{n-1}^2 + x_n^2$. We call such a quadratic form *admissable.*

Let F be the symmetric matrix associated to the quadratic form f and let $\mathrm{O}(f)$ (resp. SO(f)) denote the linear algebraic groups defined over k described as:

$$\mathrm{O}(f) = \{X \in \mathrm{GL(n+1,\mathbb{C}) : X^tFX = F}\} \text{ and } \mathrm{SO(f) = \{X \in SL(n+1,\mathbb{C}) : X^tFX = F\}}.$$

For a subring $L \subset \mathbb{C}$, we denote the L-points of $\mathrm{O}(f)$ (resp. SO(f)) by $\mathrm{O}(f,L)$ (resp. SO(f, L)). An *arithmetic subgroup* of $\mathrm{O}(f)$ (resp. SO(f)) is a subgroup $\Gamma < \mathrm{O}(f)$ commensurable with $\mathrm{O}(f,R_k)$

(resp. $\mathrm{SO(f, R_k)}$. Note that an arithmetic subgroup of $\mathrm{SO(f)}$ is an arithmetic subgroup of $\mathrm{O}(f)$, and an arithmetic subgroup $\Gamma < \mathrm{O}(f)$ determines an arithmetic subgroup $\Gamma \cap \mathrm{SO(f)}$ in $\mathrm{SO(f)}$.

(*b*) *Constructing arithmetic groups*

To pass to arithmetic subgroups of $\mathrm{O}_0(n,1)$ and $\mathrm{SO_0(n,1)}$, we first note from §5a, that, given an admissable quadratic form defined over k of signature $(n,1)$, there exists $T \in \mathrm{GL(n+1,\mathbb{R})}$ such that $T\mathrm{O}(f,\mathbb{R})T^{-1} = \mathrm{O}(n,1)$.

A subgroup $\Gamma < \mathrm{O}_0(n,1)$ is called *arithmetic of simplest type* if Γ is commensurable with the image in $\mathrm{O}_0(n,1)$ of an arithmetic subgroup of $\mathrm{O}(f)$ (under the conjugation map described above). An arithmetic hyperbolic n-manifold $M = \mathbb{H}^n/\Gamma$ is called *arithmetic of simplest type* if Γ is. The same set-up using special orthogonal groups constructs orientation-preserving arithmetic groups of simplest type (and orientable arithmetic hyperbolic n-manifolds of simplest type).

It is known (see [22]) that when n is even all arithmetic hyperbolic manifolds are of simplest type. Furthermore, when n is even, the algebraic groups $\mathrm{SO(f)}$ are centerless, and it follows from a result of Borel [2] that any arithmetic subgroup of $\mathrm{SO(f,\mathbb{R})}$ commensurable with $\mathrm{SO(f,R_k)}$ is contained in $\mathrm{SO(f,k)}$.

In addition, we will say that an arithmetic hyperbolic manifold $M = \mathbb{H}^n/\Gamma$ of simplest type is *well-located* if Γ is conjugate to a subgroup of some $\mathrm{SO(f,R_k)}$ for some admissible quadratic form f as above.

(*c*) *Arithmetic subgroups of* $\mathrm{Spin^+(n,1)}$

Now we can repeat much of the discussion in §4 on replacing $\mathbb{R}$ by a totally real field k and take for q an admissible quadratic from f. In particular, following the construction of §4, we can construct a Clifford algebra $\mathcal{C}\ell(V,k,q)$ with a basis $\mathcal{B}$ and define an algebraic group $\mathrm{Spin(f)}$ defined over k, with k-points $\mathrm{Spin(f,k)}$ and an arithmetic subgroup $\mathrm{Spin(f,R_k)}$. Note that $\mathrm{Spin(f,k)} \to \mathrm{SO(f,k)}$ with kernel $\{\pm 1\}$ and similarly $\mathrm{Spin(f,R_k)} \to \mathrm{SO(f,R_k)}$.

In addition, on taking the vector space $V_{\mathbb{R}} = V \otimes_k \mathbb{R}$ we can construct arithmetic subgroups of $\mathrm{Spin(n,1)}$, since by admissibility of f we have that $\mathrm{Spin(f,\mathbb{R})}$ is conjugate to $\mathrm{Spin(n,1)}$, and so $\mathrm{Spin(f,R_k)}$ can be conjugated into $\mathrm{Spin(n,1)}$.

Again referring to the construction in §4, via the basis $\mathcal{B}$ the following proposition can be proved.

Proposition 5.1. *Let f be an admissible quadratic form of signature $(n,1)$ defined over the totally real field k. The group $\mathrm{Spin(f,R_k)}$ admits a faithful representation into $\mathrm{GL(2^{n+1},R_k)}$.*

6. Proof of Theorem 1.1

We assume throughout that $n \geq 4$, and f will be an admissible quadratic form of signature $(n,1)$ defined over the totally real field k. For $I \subset R_k$ an ideal, we denote the norm of I by NI. Denote the principal ideal $4R_k$ by I_4 and define the constant, $N_k = \min\{\mathrm{N}\mathcal{P}, \mathrm{NI_4}\}$ where $\mathcal{P}$ runs over all prime ideals $\mathcal{P}$ with odd residue class degree. Note that if $[k:\mathbb{Q}] = d$ then $NI_4 \leq 4^d$. Let I denote the ideal of norm N_k.

The key proposition is the following (using the notation established).

Proposition 6.1. *Let $M = \mathbb{H}^n/\Gamma$ be an arithmetic hyperbolic manifold of simplest type that is well-located. Then M has a finite cover of degree $\leq |\mathrm{GL(2^{n+1},R_k/I)}|/2$ that is spinnable.*

Proof. We can assume that M is not spinnable, otherwise we are done. Following the notation above, we have an extension $\overline{\Gamma}$ of Γ by $\{\pm 1\}$. By Proposition 4.2 and the sharper version Proposition

5.1 we have the following. Let I denote the ideal of norm N_k as in the statement and consider the homomorphism $\psi : \overline{\Gamma} \to \mathrm{GL}(2^{n+1}, \mathrm{R_k}/\mathrm{I})$ given by restricting the reduction homomorphism $\mathrm{GL}(2^{n+1}, \mathrm{R_k}) \to \mathrm{GL}(2^{n+1}, \mathrm{R_k}/\mathrm{I})$. By choice of ideal I (i.e. either it is prime of odd residue class degree or the principal ideal generated by 4) we see that $\{\pm 1\}$ injects under ψ. Hence, $K = \ker \psi$ excludes -1 and has index bounded as claimed.

Let D denote the preimage of $\{\pm 1\}$ under ψ, so that $[D : K] = 2$. Let $\Delta = \phi(D)$, and let N denote the cover of M given by $\mathbb{H}^n/\Delta$. Then N is a cover of degree $\le |\mathrm{GL}(2^{n+1}, \mathrm{R_k}/\mathrm{I})|/2$ and by construction is spinnable by Lemma 3.1. □

Remark 6.2. By definition any arithmetic subgroup $\Gamma < \mathrm{SO}(\mathrm{f}, \mathbb{R})$ contains a well-located subgroup of finite index. As discussed in §5b, when n is even, all arithmetic subgroups commensurable with $\mathrm{SO}(\mathrm{f}, \mathrm{R_k})$ are contained in $\mathrm{SO}(\mathrm{f}, \mathrm{k})$, and when n is odd, the group $\Gamma^{(2)}$ is contained in $\mathrm{SO}(\mathrm{f}, \mathrm{k})$ (see the proof of [6, Lemma 10]). Now $[\Gamma : \Gamma^{(2)}] = |H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})|$, and so to get a sharper version of Sullivan's result (i.e. bounding the index) in the general arithmetic setting reduces to consideration of $\Gamma < \mathrm{SO}(\mathrm{f}, \mathrm{k})$ and getting an effectively computable constant bounding $[\Gamma : \Gamma \cap \mathrm{SO}(\mathrm{f}, \mathrm{R_k})]$.

7. Some examples

Example 1: As is shown in [7], the Davis manifold D [4] is a well located arithmetic hyperbolic 4-manifold. This was proved to be spinnable in [19] by showing that the intersection form of D is even (c.f. Remark 2.3).

Example 2: The closed orientable hyperbolic 4-manifold X of euler characteristic 16 obtained by Conder and Maclachlan in [3] is not known to be spinnable. Although the homology groups are computed in [3], its intersection form is not known at present.

It is known to be arithmetic and well located (see [3] and [8]) with quadratic form defined over the field $\mathbb{Q}(\sqrt{5})$. Since 2 and 3 are inert in $\mathbb{Q}(\sqrt{5})$, the ideal I of Propostion 6.1 is the ideal $< \sqrt{5} >$. Hence Proposition 6.1 provides a spinnable cover of X of degree at most $|\mathrm{GL}(2^5, \mathbb{F}_5)|/2 = \frac{1}{2}(5^{32} - 1)(5^{32} - 5)\ldots(5^{32} - 5^{31})$. Note $5^{32} = 23283064365386962890625$.

As is evident in this example, the method given by Propostion 6.1 for producing a spinnable cover gives a gigantic bound for the degree of a spinnable cover!

On the other hand, since D (of Example 1) and X are commensurable [8], it seems likely that a smaller degree spinnable cover of X can be constructed using the fact that the Davis manifold is spinnable and this can be seen to pass to finite sheeted covering spaces.

Remark 7.1. It follows from [1, Theorem 1] that if a closed hyperbolic 4-manifold X has even intersection form then there is a finite cyclic cover that is spinnable. This finite cover is a power of 2 controlled by the the size of the two-torsion in $H_1(X, \mathbb{Z})$.

8. Non-spinnable hyperbolic manifolds

As advertised in §1, we now produce non-spinnable examples of finite volume orientable hyperbolic manifolds in all dimensions ≥ 5.

Theorem 8.1. *For every $n \ge 5$ there exists infinitely many non-spinnable orientable finite volume hyperbolic n-manifolds.*

Proof. As shown in [11] (see also [18]) there exists non-spinnable orientable flat 4-manifolds (although only 3 of the 27 orientable flat, 4-manifolds are not spinnable). Let Y be one of these non-spinnable orientable flat 4-manifolds. By [14] and the improvement in [16], every flat 4-manifold occurs as some cusp cross-section of a possibly (indeed likely) multi-cusped arithmetic hyperbolic 5-manifold. Hence Y can be arranged as a cusp cross-section of an arithmetic hyperbolic 5-manifold X. X cannot be spin, since it is well-known that a spin structure induces a spin structure on a boundary component (see [12, Chapter II, Proposition 2.15]).

To get higher dimensional examples, the manifold $Y_n = Y \times S^1 \times S^1 \times \ldots \times S^1$ (with $n-4$ copies of S^1) is a flat n-manifold. Hence we can repeat the above construction to produce a multi-cusped arithmetic hyperbolic $n+1$-manifold X_n for which Y_n is a cusp cross-section. To see that Y_n is not spinnable, note that $Y \subset Y_5$ as a codimension one orientable submanifold of an orientable manifold, and so an application of [12, Chapter II, Proposition 2.15] implies that Y_5 is not spinnable since Y is not spinnable. Repeating this argument and proceeding by induction shows that Y_n is not spinnable for all $n \geq 4$. Hence, it follows as above, that X_n is not spinnable.

To get infinitely many in each dimension we note that the fundamental groups $\pi_1(Y_n)$ are separable subgroups of $\pi_1(X_n)$ (see [14] for example), and hence for each n, we can construct infinitely many finite sheeted covers for which Y_n occurs as a cusp cross-section. □

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