1. Del Pezzo surfaces

(1) Use the Riemann-Roch theorem for surfaces, together with Castelnuovo’s rationality criterion to show that if $X$ is a nice surface over a field $k$ with ample anticanonical sheaf, then $X$ is geometrically rational.

(2) Let $k$ be an algebraically closed field. Recall that a finite set of $k$-points $S \subset \mathbb{P}^2_k$ is said to be in general position if

- no three points are colinear,
- no six points lie on a conic, and
- no eight points lie on a singular cubic with a singularity at one of the points.

Let $S \subset \mathbb{P}^2_k$ be a finite set of $k$-points, and consider the blow-up $X := \text{Bl}_S \mathbb{P}^2_k$. Show that $-K_X$ is ample if and only if $S$ is in general position.

(3) Let $k = \mathbb{F}_p(t)$. Consider the closed subscheme of $\mathbb{P}^2_k = \text{Proj} k[x,y,z]$ given by $S = V(x^p - tz, y)$. Show that $\text{Bl}_S \mathbb{P}^2_k$ is not smooth.

(4) Let $k$ be an algebraically closed field. Recall that an exceptional curve on a nice surface $X$ is an irreducible curve such that $(C,C) = (C,K_X) = -1$. Let $S = \{P_1, \ldots, P_r\}$ be a finite set of distinct $k$-points in $\mathbb{P}^2_k$ in general position and let $X = \text{Bl}_S \mathbb{P}^2_k$.

(a) Show that the number of exceptional curves of $X$ is finite and depends on $d = 9 - r$ as follows:

<table>
<thead>
<tr>
<th>$d$</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td># of exceptional curves</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>16</td>
<td>27</td>
<td>56</td>
<td>240</td>
</tr>
</tbody>
</table>

(b) Let $R_r$ be the set of roots of $X$, i.e,

$$R_r := \{v \in \text{Pic} X : (v,K_X) = 0, (v,v) = -2\}.$$ 

Show that $R_r$ is finite and depends on $d = 9 - r$ as follows:

<table>
<thead>
<tr>
<th>$d$</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td># of $R_r$</td>
<td>8</td>
<td>20</td>
<td>40</td>
<td>72</td>
<td>126</td>
<td>240</td>
</tr>
</tbody>
</table>

(c) Verify that $R_r$ satisfies the axioms of a root system.
(5) Let \( S = \{ P_1, \ldots, P_r \} \) be a finite set of distinct \( k \)-points in \( \mathbb{P}^2_k = \text{Proj} \ k[x_0, x_1, x_2] \) in general position and let \( X = \text{Bl}_S \mathbb{P}^2_k \). Let \( \mathcal{I} \subseteq \mathcal{O}_{\mathbb{P}^2_k} \) be the coherent ideal sheaf associated to the scheme \( S \) with its reduced-induced subscheme structure.

Show there is an isomorphism of graded \( k \)-algebras

\[
R(X, \omega_X^{-1}) \cong \bigoplus_{m \geq 0} H^0(\mathbb{P}^2_k, \mathcal{I}^m(3m)).
\]

The vector space \( H^0(\mathbb{P}^2_k, \mathcal{I}^m(3m)) \) is the set of homogenous degree \( 3m \) polynomials in \( k[x_0, x_1, x_2] \) that have \( m \)-fold vanishing at each \( P_i \).

(6) (Use a computer algebra system for this exercise) We make tacit use of the previous exercise to compute an equation for a cubic surface given 6 points in general position on the plane.

Let \( k = \mathbb{F}_5 \) (or your favorite finite field—this will make the computations instantaneous and the coefficients of the expressions involved manageable).

(a) Write down six \( k \)-points \( P_1, \ldots, P_6 \) of \( \mathbb{P}^2_k = \text{Proj} \ k[x_0, x_1, x_2] \) in general position.

(b) Compute a basis for the vector space of cubic polynomials in \( k[x_0, x_1, x_2] \) that vanish along \( P_i \) (\( i = 1, \ldots, 6 \)) with multiplicity 1. This vector space is 4-dimensional. Call the 4 elements of your basis \( x, y, z \) and \( w \).

(c) Show there is a dependence relation amongst the monomials of degree 3 in \( x, y, z \) and \( w \). This relation gives a cubic surface in \( \text{Proj} \ k[x, y, z, w] \) isomorphic to \( \text{Bl}_\{P_1, \ldots, P_6\} \mathbb{P}^2_k \).

(7) (Use a computer algebra system for this exercise) Let \( X \) be the del Pezzo surface of degree 1 over \( \mathbb{F}_7 \) given by

\[
w^2 = z^3 + 2x^6 + 2y^6
\]

in \( \mathbb{P}(1, 1, 2, 3) = \text{Proj} \mathbb{F}_7[x, y, z, w] \). Let \( F_7 \in \text{Gal}(\overline{\mathbb{F}_7}/\mathbb{F}_7) \) be the Frobenius map \( x \mapsto x^7 \). Let

\[
\phi_X : \text{Gal}(\overline{\mathbb{F}_7}/\mathbb{F}_7) \to O(K_X^{\perp})
\]

be the Galois representation introduced in lecture. Use the Lefschetz trace formula for surfaces to prove that the trace of \( \phi_X(F_7) \) is negative. Conclude that \( X \) cannot be \( \mathbb{F}_7 \)-isomorphic to a blow-up of \( \mathbb{P}^2_{\mathbb{F}_7} \) at points in general position.

(8) Let \( k \) be an algebraically closed field. Describe all the automorphisms of the weighted projective space \( \mathbb{P}_k(1, 1, 2, 3) \).