ARITHMETIC OF DEL PEZZO AND K3 SURFACES: EXERCISES I

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1. Del Pezzo surfaces

(1) Use the Riemann-Roch theorem for surfaces, together with Castelnuovo's rationality criterion to show that if X is a nice surface over a field k with ample anticanonical sheaf, then X is geometrically rational.

(2) Let k be an algebraically closed field. Recall that a finite set of k-points $S \subset \mathbb{P}^2_k$ is said to be in general position if

- no three points are colinear,
- no six points lie on a conic, and
- no eight points lie on a singular cubic with a singularity at one of the points.

Let $S \subset \mathbb{P}^2_k$ be a finite set of k-points, and consider the blow-up $X := \mathrm{Bl}_S \mathbb{P}^2_k$. Show that $-K_X$ is ample if and only if S is in general position.

(3) Let $k = \mathbb{F}_p(t)$. Consider the closed subscheme of $\mathbb{P}^2_k = \operatorname{Proj} k[x, y, z]$ given by

$$S = V(x^p - tz, y).$$

Show that $\mathrm{Bl}_S \mathbb{P}^2_k$ is not smooth.

(4) Let k be an algebraically closed field. Recall that an exceptional curve on a nice surface X is an irreducible curve such that $(C, C) = (C, K_X) = -1$. Let $S = \{P_1, \dots, P_r\}$ be a finite set of distinct k-points in $\mathbb{P}^2_k$ in general position and let $X = \mathrm{Bl}_S \mathbb{P}^2_k$.

(a) Show that the number of exceptional curves of X is finite and depends on $d = 9 - r$ as follows:

d	7	6	5	4	3	2	1
# of exceptional curves	3	6	10	16	27	56	240

(b) Let R_r be the set of roots of X, i.e,

$$R_r := \{v \in \operatorname{Pic} \overline{X} : (v, K_X) = 0, (v, v) = -2\}.$$

Show that R_r is finite and depends on $d = 9 - r$ as follows:

d	6	5	4	3	2	1
$\#R_r$	8	20	40	72	126	240

(c) Verify that R_r satisfies the axioms of a root system.

(5) Let $S = \{P_1, \dots, P_r\}$ be a finite set of distinct k-points in $\mathbb{P}^2_k = \operatorname{Proj} k[x_0, x_1, x_2]$ in general position and let $X = \operatorname{Bl}_S \mathbb{P}^2_k$. Let $\mathscr{I} \subseteq \mathscr{O}_{\mathbb{P}^2_k}$ be the coherent ideal sheaf associated to the scheme S with its reduced-induced subscheme structure.

Show there is an isomorphism of graded k-algebras

$$R(X, \omega_X^{-1}) \cong \bigoplus_{m \geq 0} H^0(\mathbb{P}^2_k, \mathscr{I}^m(3m)).$$

The vector space $H^0(\mathbb{P}^2_k, \mathscr{I}^m(3m))$ is the set of homogenous degree $3m$ polynomials in $k[x_0, x_1, x_2]$ that have m-fold vanishing at each P_i.

(6) (Use a computer algebra system for this exercise) We make tacit use of the previous exercise to compute an equation for a cubic surface given 6 points in general position on the plane.

Let $k = \mathbb{F}_5$ (or your favorite finite field—this will make the computations instantaneous and the coefficients of the expressions involved manageable).

(a) Write down six k-points $P_1, \dots, P_6$ of $\mathbb{P}^2 = \operatorname{Proj} k[x_0, x_1, x_2]$ in general position.
(b) Compute a basis for the vector space of cubic polynomials in $k[x_0, x_1, x_2]$ that vanish along P_i ($i = 1, \dots, 6$) with multiplicity 1. This vector space is 4-dimensional. Call the 4 elements of your basis x, y, z and w.
(c) Show there is a dependence relation amongst the monomials of degree 3 in x, y, z and w. This relation gives a cubic surface in $\operatorname{Proj} k[x, y, z, w]$ isomorphic to $\operatorname{Bl}_{\{P_1, \dots, P_6\}} \mathbb{P}^2_k$.

(7) (Use a computer algebra system for this exercise) Let X be the del Pezzo surface of degree 1 over $\mathbb{F}_7$ given by

$$w^2 = z^3 + 2x^6 + 2y^6$$

in $\mathbb{P}(1, 1, 2, 3) = \operatorname{Proj} \mathbb{F}_7[x, y, z, w]$. Let $F_7 \in \operatorname{Gal}(\overline{\mathbb{F}}_7/\mathbb{F}_7)$ be the Frobenius map $x \mapsto x^7$. Let

$$\phi_X \colon \operatorname{Gal}(\overline{\mathbb{F}}_7/\mathbb{F}_7) \to O(K_X^{\perp})$$

be the Galois representation introduced in lecture. Use the Lefschetz trace formula for surfaces to prove that the trace of $\phi_X(F_7)$ is negative. Conclude that X cannot be $\mathbb{F}_7$-isomorphic to a blow-up of $\mathbb{P}^2_{\mathbb{F}_7}$ at points in general position.

(8) Let k be an algebraically closed field. Describe all the automorphisms of the weighted projective space $\mathbb{P}_k(1, 1, 2, 3)$.