Graph Theory Mini-course

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Abstract

Intuitively speaking, a graph is a collection of dots and lines joining some of these dots. Many problems in the real world can be modeled in graph theoretic terms. Thus, the study of the properties of these objects is of the utmost importance. In this mini-course, we shall not attempt to give a thorough discussion of graphs. We will instead focus on planar graphs. Our goal shall be to prove the five color theorem. This mini-course was aimed at first-years, and although some of the definitions presented are a bit technical, only a good understanding of the Well-Ordering Principle was assumed.

1 The Birth of Graph Theory

In 1736, Euler wrote a paper about the famous 'Bridges of Konigsberg' problem. This marked the beginning of graph theory. The question Euler set out to answer was whether there existed or not a walk that went through each bridge in the town of Konigsberg once *and only once* (Figure 1(a)).

Figure 1: A schematic representation of the town of Konigsberg (a) and its graph-theoretic representation (b).

To attack the problem, we let each landmass be represented by a vertex and each bridge by an undirected edge. We now want a walk that goes through all edges once and only once (Figure 1(b)).

Principle What comes in, must go out.

The number of edges incident on a vertex is known as the *valency* of that vertex. Our principle may be restated as follows: it is necessary that every vertex have even valency so that you can walk in and out of it. Alternatively, *two* vertices can have odd valency; these vertices would be the starting and ending points of our walk. With this principle in hand we see that the walk through the bridges of Konigsberg is impossible; it's graph has four vertices with odd valency.

2 Definitions

There is no standard terminology in graph theory. Nevertheless, we need to define some vocabulary in precise mathematical language before we can do mathematics. We'll adopt conventions used in both Trudeau [2] and Taylor & Garnier [1]. In graph theory, we tend to have natural definitions; a 'circuit' means exactly what our intuition and daily experience tells you it means. Therefore, if some of the following mathematical jargon doesn't make much sense, skim through it the first time and come back to it later when you meet it again.

To define what a graph is, we'll want to specify its set of vertices and its set of edges. Then we'll want to say which edges join which vertices in precise mathematical terms.

Definition. *An* undirected graph *consists of:*

(i) a finite, non-empty set V of vertices,

(ii) a finite set E of edges and

(iii) a function $\delta : E \to \mathcal{P}(V)$ such that for all $e \in E$, $\delta(e)$ is a one or two element subset of $\mathcal{P}(V)$[1]

Look at the example below.

Figure 2: An example of an undirected graph

In this graph, edge e_2 joins vertices v_1 and v_2 together. Thus, we have $\delta(e_1) = \{v_1, v_2\}$.

[1] $\mathcal{P}(V)$ denotes the power set of V, that is, the set of all subsets of V. For example, if $V = \{v_1, v_2, v_3\}$ then $\mathcal{P}(V) = \{\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}, \{v_1, v_2, v_3\}\}$.

Some graphs look 'alike.' The precise term for this is an isomorphism.

Definition. *Two graphs, $\Gamma = (V, E)$ and $\Gamma' = (V', E')$, are said to be isomorphic if there exists a one-to-one and onto map $\phi : V \to V'$ such that $\{v_1, v_2\} \in E \Rightarrow \{\phi(v_1), \phi(v_2)\} \in E'$ (Figure 3).*

Figure 3: Graph isomorphism. Graph Γ is isomorphic to graph Γ'. The vertex correspondence is given by $X \mapsto X'$ for any $X \in V_\Gamma$ and $X' \in V_{\Gamma'}$.

2.1 Different kinds of graphs

Definition. *A* simple graph *is a graph that has no loops and no multiple edges.*

Definition. *A* complete graph *is a simple graph in which every pair of distinct vertices is joined by an edge. We will denote the complete graph with n vertices by K_n (Figure 4).*

Figure 4: K_5, the complete grpah on 5 vertices

Definition. *A* bipartite graph Γ *has a vertex set V that can be partitioned, i.e., $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$, and such that every edge joins a vertex from V_1 to one from V_2 (Figure 5(a)).*

Definition. *A* complete bipartite graph *is a bipartite graph such that every vertex of V_1 is joined to every vertex of V_2 by a unique edge. If the set V_1 has m elements and the set V_2 has n elements, then the respective complete bipartite graph is denoted by $K_{m,n}$ (Figure 5(b)).*

Figure 5: A bipartite graph (a) and the complete bipartite graph $K_{3,3}$ (b).

Definition. *A graph is said to be* connected *if one can find a 'path' joining any two vertices of the graph.*

Definition. *A* tree *is a connected graph which contains no 'circuits'. (Figure 6)*

Figure 6: Examples of trees

Definition. *A graph whose vertices are points in the plane and whose edges are lines or arcs that only meet at vertices is called a* plane graph*.*

Definition. *A graph is called* planar *if it is isomorphic to a plane graph.*

3 Euler's Formula for Planar Graphs

When a planar graph is drawn without any edge-crossings, it divides the plane into regions called faces. Euler discovered a truly beautiful relationship between the number of edges (e), vertices (v) and faces(f) of a planar graph.

Theorem 3.1 (Euler's formula). *Let* Γ *be a simple connected planar graph. Then*

$$f - e + v = 2. \tag{1}$$

To prove this theorem, we'll need the following lemma:

Lemma 3.1.1. *Let Γ be a tree. Then $e = v - 1$.*

Proof. Let $\mathbf{S} = \{v \in \mathbf{N} \mid \exists \text{ a tree } \Gamma \text{ for which } e \neq v - 1\}$. Suppose $S \neq \emptyset$. By the Well-Ordering principle, $\mathbf{S}$ has a smallest element. Call this element v_1, and let Γ_1 be the graph associated to v_1.

Remove an edge from Γ_1. This will produce two trees, Γ_2 and Γ_3 with v_2 and v_3 vertices respectively. Since both v_2 and v_3 are smaller than v_1, we have $e_2 = v_2 - 1$ and $e_3 = v_3 - 1$. Adding these two equations we obtain $e_2 + e_3 = v_2 + v_3 - 2$. However, $e_2 + e_3 = e_1 - 1$ and $v_2 + v_3 = v_1$ by construction. Thus $e_1 = v_1 - 1$. Hence v_1 is not in $\mathbf{S}$, a contradiction. Our assumption that $\mathbf{S}$ is non-empty must be wrong. □

Remark. *Trees satisfy Euler's formula. This is easily seen from our lemma and the fact that $f = 1$ for trees.*

Proof of Euler's formula. Let $\mathbf{S} = \{e \in \mathbf{N} \mid \exists \text{ a graph } \Gamma \text{ for which } f - e + v \neq 2\}$. Suppose $S \neq \emptyset$. By the Well-Ordering principle, $\mathbf{S}$ has a smallest element. Call this element e_1, and let Γ_1 be the graph associated to e_1.

We know Γ_1 is not a tree because trees satisfy Euler's formula. This means Γ_1 has a circuit. Remove an edge from this circuit. This will produce a smaller graph Γ_1' with $f_1 - 1$ faces and $e_1 - 1$ edges. This graph must satisfy Euler's formula since e_1 is the smallest element of $\mathbf{S}$. Hence

$$\begin{aligned}(f_1 - 1) - (e_1 - 1) + v_1 &= 2 \\ \Rightarrow f_1 - e_1 + v_1 &= 2\end{aligned}$$

This means e_1 cannot be a member of $\mathbf{S}$. Again, our assumption that $\mathbf{S}$ is not empty must be wrong. □

4 Applications of Euler's formula

A few careful attempts at drawing K_5, the complete graph on five vertices, should convince you that this graph is not planar. But how can you *prove* that this is the case? Euler's formula gives us the key.

Theorem 4.1. *Let Γ be a simple planar graph with at least three vertices and two edges. Then*

$$\frac{3}{2} f \leq e \leq 3v - 6 \tag{2}$$

Proof. We follow Trudeau [2] in our proof. We'll need two cases.

Case 1 Γ has a face bounded by less than 3 edges. This forces Γ to look the tree in Figure 6(b) plus a few isolated vertices. In this case, it is easy to check (2) is satisfied.

Case 2 Every face of Γ is bounded by at least three edges. Then

$$\begin{aligned}
3 &\leq \text{the number of edges bounding face1} \\
3 &\leq \text{the number of edges bounding face2} \\
&\vdots \\
3 &\leq \text{the number of edges bounding face}f
\end{aligned}$$

If we add both sides of the inequalities, we get $3f$ on the left hand side, and some number d on the right hand side. What can we say about d? Since Γ is planar, every edge borders at most two faces. Hence $d \leq 2e$. This proves the left inequality in (2) since $3f \leq d \leq 2e \Rightarrow \frac{3}{2}f \leq e$.

Now rewrite the last inequality as $f \leq \frac{2}{3}e$ and add $-e+v$ to both sides. By Euler's formula we have

$$\begin{aligned}
f - e + v &\leq \frac{2}{3}e - e + v \\
\Rightarrow 2 &\leq \frac{2}{3}e - e + v \\
\Rightarrow e &\leq 3v - 6. \quad \square
\end{aligned}$$

$\square$

Corollary 4.1.1. *K_5 is not planar.*

Proof. Suppose K_5 is planar. It is also simple, so it satisfies (2). But K_5 has ten edges and five vertices. Thus $10 \leq 9$. This is a contradiction. $\square$

The next goal is to show that the complete bipartite graph $K_{3,3}$ is also not planar. This solves, or rather, proves there is no solution to the famous 'utilities problem,' where you want to connect three utilities (say electricity, telephone and water) to three different houses without the cables or pipes of any utility passing over the cables or pipes of any other. This is equivalent to the statement that $K_{3,3}$ is planar. Our approach is very similar to the proof of theorem 4.1. The inequality (2) is not good enough to prove $K_{3,3}$ is not planar, but if we add an extra hypothesis to theorem 4.1, then we'll obtain an inequality fine enough for our purpose.

Theorem 4.2. *Let Γ be a simple connected planar graph without 3-cycles (i.e., no 'triangles'). Then*

$$2f \leq e \leq 2v - 4 \tag{3}$$

Proof. Like last time, we'll divide the problem into two cases:

Case 1 Γ has a face bounded by less than 4 edges. This forces Γ to look like one of the graphs of Figure 6(c) & (d) plus a few isolated vertices. In each case, it is easy to check (2) is satisfied.

Case 2 Every face of Γ is bounded by at least four edges. Then

$$\begin{aligned}
4 &\leq \text{the number of edges bounding face1} \\
4 &\leq \text{the number of edges bounding face2} \\
&\vdots \\
4 &\leq \text{the number of edges bounding face} f
\end{aligned}$$

If we add both sides of the inequalities, we get $4f$ on the left hand side, and some number d on the right hand side. Again, we'll have $d \leq 2e$. This proves the left inequality in (3) since $4f \leq d \leq 2e \Rightarrow 2f \leq e$.

Now rewrite the last inequality as $f \leq \frac{1}{2}e$ and add $-e+v$ to both sides. By Euler's formula we have

$$\begin{aligned}
f - e + v &\leq \frac{1}{2}e - e + v \\
\Rightarrow 2 &\leq \frac{1}{2}e - e + v \\
\Rightarrow e &\leq 2v - 4.
\end{aligned}$$

□

Corollary 4.2.1. *$K_{3,3}$ is not planar.*

Proof. Suppose $K_{3,3}$ is planar. It is also simple, and it has no 3-cycles, so it satisfies (3). But $K_{3,3}$ has nine edges and six vertices. Thus $9 \leq 8$. This is a contradiction. □

4.1 The Five-Color Theorem

You might have heard of the *four*-color theorem. Roughly speaking, it states that any map can be colored using only four colors in such a way that no neighboring countries share the same color. This theorem was proved in 1977 by Appel and Haken. The 'proof', however, requires tremendous amount of super-computer power and is not feasibly verifyable by hand. Thus, we will have to settle on the five-color theorem for this summer.

As usual, we need to carefully define our terms and our goal in precise mathematical language.

Definition. *Let Γ be a graph. We shall denote by $\chi(\Gamma)$ the minimum number of colors needed for the vertices of Γ in such a way that no two vertices joined by an edge are colored the same.*

Recall that the valency of a vertex is the number of edges that are incident on that vertex. We shall denote the valency of v by $\sigma(v)$. In what follows, V_Γ will be used to denote the vertex set of Γ.

Theorem 4.3 (Five-Color Theorem). *Let Γ be a simple planar graph. Then $\chi(\Gamma) \leq 5$*

In proving the five-color theorem, we shall follow Trudeau's exposition [2]. Before we begin the actual proof, we need two lemmas.

Lemma 4.3.1 (The Handshaking lemma). *For a simple graph Γ, it is true that*

$$\sum_{v \in V_\Gamma} \sigma(v) = 2e. \tag{4}$$

Proof. The left hand side of (4) is the sum of the valencies of $v \in V_\Gamma$. This counts all the edges of the graph. But since each edge is incident on two vertices, we have overcounted the edges by a factor of two. Hence the equality. □

Lemma 4.3.2. *Let Γ be a simple planar graph. Assume also that G is connected. Then there is a vertex $v \in V_\Gamma$ such that $\sigma(v) \leq 5$, i.e., there is a vertex of Γ that has at most five incident edges.*

Proof. If Γ has less than three vertices, then it must be either K_1 or K_2. In either case the lemma is true. Suppose Γ has at least three vertices, and that $\sigma(v) \geq 6$ for all $v \in V_\Gamma$. Summing over all the vertices, we have

$$\sum_{v \in V_\Gamma} \sigma(v) \geq 6v \tag{5}$$

But we know by the hand-shaking lemma that the left side of (5) is equal to $2e$. Thus $2e \geq 6v \Rightarrow e \geq 3v$. Since Γ is simple and planar, by theorem 4.1 $e \leq 3v - 6$. We conlude that $3v \leq 3v - 6$, which is impossible. Thus our assumption that $\sigma(v) \geq 6$ for all $v \in V_\Gamma$ must be false. □

Proof of the five-color theorem. Let $\mathbf{S} = \{v \in \mathbf{N} \,|\, \exists$ a graph Γ with v vertices such that $\chi(\Gamma) > 5\}$. Assume $\mathbf{S}$ is not empty. Then by the Well-Ordering Principle $\mathbf{S}$ has a least element. Let v_1 be this least element and let Γ be the graph associated with v_1. By the lemma we just proved, Γ has a vertex A with $\sigma(A) \leq 5$. Consider the graph $\Gamma - A$, i.e., erase from Γ the vertex A and all edges incident on it. This new graph has $v_1 - 1$ vertices. Therefore $\chi(\Gamma - A) \leq 5$ since v_1 is the least element of $\mathbf{S}$.

We'll need to consider three cases:

Case 1 $\chi(\Gamma - A) < 5$. Then we can give A the fifth color and we're done.

Case 2 $\chi(\Gamma - A) = 5$ and $\sigma(A) < 5$. In this case, A is joined to at most four other vertices. Again, we can give A the fifth color in the worst case, and $\chi(\Gamma)$ will be 5.

Case 3 $\chi(\Gamma - A) = 5$ and $\sigma(A) = 5$. This is the hardest case. A is joined to five other vertices, so each of these vertices could be colored differently. We must show that it is possible to re-color $\Gamma - A$ without altering $\chi(\Gamma - A)$.

Let P, Q, R, S, T be the five vertices, arranged in clockwise order, that are joined to A (Figure 7). Assume that they are painted with different colors. We'll use numbers to represent colors; thus we'll say P has color **1**, Q has color **2**, and so on.

subcase (i) Suppose there does not exist a walk between P and R that consists *only* of colors **1** and **3**. Figure 7 shows the relevant part of the graph Γ in this situation: only the walks composed entirely by colors **1** and **3** that touch P or R have been shown. If we change P's color from **1** to **3** and recolor all the relevant paths that touch P, we can give A the color **1**. We'll still have $\chi(\Gamma - A) = 5$ and A will only have neighbors with four different colors. Hence, $\chi(\Gamma)$ will be 5.

Figure 7: The relevant part of our graph. Shown here are all the paths that touch P or R that are colored with **1** and **3**.

subcase (ii) Suppose there exists a walk between P and R consisting of colors **1** and **3** only (Figure 8).

Figure 8: The relevant part of our graph. This time, all paths that touch S or Q and consist entirely of colors **2** and **4** are shown.

We claim no walk between Q and S can consist entirely of colors **2** and **4**. The 'subgraph' $PUWXYRAP$ surrounds Q. All vertices of this subgraph are colored **1** or **3**, except for A, which we know cannot be colored using **2** or **4**. Any path from

Q to S must cross our subgraph[2]. This means there is no path from Q to S which can have only colors **2** and **4**. We are back in case (i), just let Q be our P from before, S be our R, color **2** what used to be color **1** and color **4** what used to be color **3**.

$\square$

References

[1] Taylor & Garnier *Discrete Mathematics for New Technology* ITP, Oxford, 1993.

[2] R. Trudeau *Introduction to Graph Theory* Dover, New York, 1993.

[2]We have inplicitly used here a powerful theorem from algebraic topology known as the *Jordan Curve Theorem*. This theorem says that a closed simple curve divides the plane into two different regions. The statement is intuitive but the theorem is really hard to prove!