Weak approximation on del Pezzo surfaces of degree 1

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Algebraic Geometry Seminar, Rice, September 2009
The surface

Proposition

Let $X$ be the del Pezzo surface of degree 1 over $k = \mathbb{Q}(\zeta_3)$ given by

$$w^2 = z^3 + 16x^6 + 16y^6$$

in $\mathbb{P}_k(1, 1, 2, 3)$. Then $X$ is $k$-minimal and does not satisfy weak approximation.

Proof. The exceptional curves of $X$ are defined over $K := k(\sqrt[3]{2})$. Let $G := \text{Gal}(K/k) = \langle \sigma \rangle$. Since $G$ is already cyclic, we take $L = K$. 

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The Picard group of $X$

Let $s = \sqrt[3]{2}$. Consider the exceptional curves on $X$ given by

$E_1 = V(z + 2sx^2, w - 4y^3),$

$E_2 = V(z - (-3 + 1)2sx^2, w + 4y^3),$

$E_3 = V(z - 2\zeta_3 sx^2 + 4y^2, w - 4s(\zeta_3 - 2)x^2 y - 4(-2\zeta_3 + 1)y^3),$

$E_4 = V(z + 4\zeta_3 sx^2 - 2s^2(2\zeta_3 - 1)xy - 4(-\zeta_3 + 1)y^2,$

$E_5 = V(z + 4\zeta_3 sx^2 - 2s^2(\zeta_3 - 2)xy - 4\zeta_3 y^2,$

$E_6 = V(z - 2s(-s^2\zeta_3 + s^2 - 2s + 2\zeta_3)x^2 - 2s(2s^2\zeta_3 - 2s^2 + 3s - 4\zeta_3)xy - 2s(-s^2\zeta_3 + s^2 - 2s + 2\zeta_3)y^2,$

$E_7 = V(z - 2s(-s^2 - 2s\zeta_3 + 2s + 2\zeta_3)x^2 - 2s(-2s^2\zeta_3 + 3s + 4\zeta_3 - 4)xy - 2s(-s^2\zeta_3 + s^2 + 2s\zeta_3 - 2)y^2,$

$E_8 = V(z - 2s(s^2\zeta_3 + 2s\zeta_3 + 2\zeta_3)x^2 - 2s(2s^2 + 3s + 4)xy - 2s(-s^2\zeta_3 + s^2 - 2s\zeta_3 + 2s - 2\zeta_3 + 2)y^2.$
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$$E_3 = V(z - 2\zeta_3 sx^2 + 4y^2, w - 4s(\zeta_3 - 2)x^2y - 4(-2\zeta_3 + 1)y^3),$$
$$E_4 = V(z + 4\zeta_3 sx^2 - 2s^2(2\zeta_3 - 1)xy - 4(-\zeta_3 + 1)y^2,$$
$$w - 12x^3 - 8s(-\zeta_3 - 1)x^2y - 12\zeta_3 s^2xy^2 - 4(-2\zeta_3 + 1)y^3),$$
$$E_5 = V(z + 4\zeta_3 sx^2 - 2s^2(\zeta_3 - 2)xy - 4\zeta_3 y^2,$$
$$w + 12x^3 - 8s(2\zeta_3 - 1)x^2y - 12s^2xy^2 - 4(-2\zeta_3 + 1)y^3),$$
$$E_6 = V(z - 2s(-s^2\zeta_3 + s^2 - 2s + 2\zeta_3)x^2 - 2s(2s^2\zeta_3 - 2s^2 + 3s - 4\zeta_3)xy - 2s(-s^2\zeta_3 + s^2 - 2s + 2\zeta_3)y^2,$$
$$w - 4(2s^2\zeta_3 - 4s^2 + 2s\zeta_3 + 2s - 6\zeta_3 + 3)x^3 - 4(-5s^2\zeta_3 + 10s^2 - 6s\zeta_3 - 6s + 16\zeta_3 - 8)x^2y$$
$$- 4(5s^2\zeta_3 - 10s^2 + 6s\zeta_3 + 6s - 16\zeta_3 + 8)xy^2 - 4(-2s^2\zeta_3 + 4s^2 - 2s\zeta_3 - 2s + 6\zeta_3 - 3)y^3),$n
$$E_7 = V(z - 2s(-s^2 - 2s\zeta_3 + 2s + 2\zeta_3)x^2 - 2s(-2s^2\zeta_3 + 3s + 4\zeta_3 - 4)xy - 2s(-s^2\zeta_3 + s^2 + 2s\zeta_3 - 2)y^2,$$
$$w - 4(2s^2\zeta_3 + 2s^2 + 2s\zeta_3 - 4s - 6\zeta_3 + 3)x^3 - 4(10s^2\zeta_3 - 5s^2 - 6s\zeta_3 - 6s - 8\zeta_3 + 16)x^2y$$
$$- 4(5s^2\zeta_3 - 10s^2 - 12s\zeta_3 + 6s + 8\zeta_3 + 8)xy^2 - 4(-2s^2\zeta_3 - 2s^2 - 2s\zeta_3 + 4s + 6\zeta_3 - 3)y^3),$n
$$E_8 = V(z - 2s(s^2\zeta_3 + 2s\zeta_3 + 2\zeta_3)x^2 - 2s(2s^2 + 3s + 4)xy - 2s(-s^2\zeta_3 + s^2 - 2s\zeta_3 + 2s - 2\zeta_3 + 2)y^2,$$
$$w - 4(-4s^2\zeta_3 + 2s^2 - 4s\zeta_3 + 2s - 6\zeta_3 + 3)x^3 - 4(-5s^2\zeta_3 - 5s^2 - 6s\zeta_3 - 6s - 8\zeta_3 - 8)x^2y$$
$$- 4(5s^2\zeta_3 - 10s^2 + 6s\zeta_3 - 12s + 8\zeta_3 - 16)xy^2 - 4(4s^2\zeta_3 - 2s^2 + 4s\zeta_3 - 2s + 6\zeta_3 - 3)y^3).
The Picard group of $X$

...as well as the exceptional curve

$$E_9 = V(z - 2\zeta_3 s^2 xy, w - 4x^3 + 4y^3).$$

Then

$$\text{Pic } X_k = \text{Pic } X_K \cong \bigoplus_{i=1}^{8} \mathbb{Z}[E_i] \oplus \mathbb{Z}[H] = \mathbb{Z}^9,$$

where $H = E_1 + E_2 + E_9$. 
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where $H = E_1 + E_2 + E_9$. 
The group $\text{Br}_{\text{cyc}}(X, K)$

Recall we have maps

$$\overline{N}_{K/k}: \text{Pic } X_K \to \text{Pic } X$$

$$[D] \mapsto [D + \sigma D + \sigma^2 D]$$

$$\Delta: \text{Pic } X_K \to \text{Pic } X$$

$$[D] \mapsto [D - \sigma D]$$

and that $\ker \overline{N}_{K/k} / \text{im } \Delta \cong \text{Br}_{\text{cyc}}(X, K) \subseteq \text{Br } k(X)/\text{Br } k$.

We compute

$$\ker \overline{N}_{L/k} / \text{im } \Delta \cong (\mathbb{Z}/3\mathbb{Z})^4;$$

and the classes

$$h_1 = [E_2 + 2E_8 - H], \quad h_2 = [E_5 + 2E_8 - H],$$

$$h_3 = [E_7 + 2E_8 - H], \quad h_4 = [3E_8 - H]$$

of $\text{Pic } X_K$ determine generators for this group.
The group $\text{Br}_{\text{cyc}}(X, K)$

Recall we have maps

$$\overline{N}_{K/k} : \text{Pic} \times K \rightarrow \text{Pic} X$$

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$$[D] \mapsto [D + \sigma D + \sigma^2 D]$$

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of $\text{Pic } X_K$ determine generators for this group.
An Azumaya Algebra

Consider the divisor class $h_1 - h_2 = [E_2 - E_5] \in \text{Pic } X_K$. It gives rise to a cyclic algebra $\mathcal{A} := (K/k, f) \in \text{Br}_{\text{cyc}}(X, K)$, where $f \in k(X)^*$ is any function such that

$$N_{K/k}(E_2 - E_5) = (f),$$

To wit, $f$ is a function with zeroes along

$$E_2 + \sigma E_2 + \sigma^2 E_2$$

and poles along

$$E_5 + \sigma E_5 + \sigma^2 E_5.$$

Using the explicit equations for $E_2$ and $E_5$ we find

$$f := \frac{w + 4y^3}{w + (2\zeta + 2)zy + (-8\zeta + 4)y^3 + 12x^3}$$

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does the job.
The Brauer-Manin obstruction

Recall $X$ is given by $w^2 = z^3 + 16x^6 + 16y^6$. Note that

$$P_1 = [1 : 0 : 0 : 4] \quad \text{and} \quad P_2 = [0 : 1 : 0 : 4].$$

are in $X(k)$.

Let $p$ be the unique prime above 3 in $k$. We compute

$$\text{inv}_p(\mathcal{A}(P_1)) = 0 \quad \text{and} \quad \text{inv}_p(\mathcal{A}(P_2)) = 1/3.$$

Let $P \in X(\mathbb{A}_k)$ be the point that is equal to $P_1$ at all places except $p$, and is $P_2$ at $p$. Then

$$\sum_v \text{inv}_v(\mathcal{A}(P_v)) = 1/3,$$

so $P \in X(\mathbb{A}_k) \setminus X(\mathbb{A}_k)^{\text{Br}}$ and $X$ is a counterexample to weak approximation.
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An Example

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