REDUCTIONS OF 2-DIMENSIONAL SEMI-STABLE REPRESENTATIONS WITH LARGE $\mathcal{L}$-INVARIANT

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Abstract. We determine reductions of 2-dimensional, irreducible, semi-stable, and non-crystalline representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ with Hodge–Tate weights $0 < k-1$ and with $\mathcal{L}$-invariant whose p-adic norm is sufficiently large, depending on k. Our main result provides the first systematic examples of the reductions for $k \geq p$.

Contents

1. Introduction

Let p be a prime number and $\overline{\mathbb{Q}}_p$ be an algebraic closure of the p-adic numbers $\mathbb{Q}_p$. The goal of this article is to determine the reductions of certain 2-dimensional p-adic representations of $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ that are semi-stable and not crystalline in the sense of Fontaine ([15]). Examples of such representations arise from local p-adic representations associated with eigenforms with $\Gamma_0(p)$-level.

1.1. **Main result.** Write v_p for the p-adic valuation on $\overline{\mathbb{Q}}_p$, normalized so that $v_p(p) = 1$. Choose $\varpi \in \overline{\mathbb{Q}}_p$ such that $\varpi^2 = p$. Then, for each integer $k \geq 2$ and each $\mathcal{L} \in \overline{\mathbb{Q}}_p$, there is a 2-dimensional filtered (φ, N)-module $D_{k,\mathcal{L}} = \overline{\mathbb{Q}}_p e_1 \oplus \overline{\mathbb{Q}}_p e_2$ where, in the basis (e_1, e_2), we have:

$$\varphi = \begin{pmatrix} \varpi^k & 0 \\ 0 & \varpi^{k-2} \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \mathrm{Fil}^i D_{k,\mathcal{L}} = \begin{cases} D_{k,\mathcal{L}} & \text{if } i \leq 0; \\ \overline{\mathbb{Q}}_p \cdot (e_1 + \mathcal{L} e_2) & \text{if } 1 \leq i \leq k-1; \\ \{0\} & \text{if } k \leq i. \end{cases} \tag{1.1}$$

Each $D_{k,\mathcal{L}}$ is weakly-admissible, so a theorem of Colmez and Fontaine implies there is a unique 2-dimensional $\overline{\mathbb{Q}}_p$-linear representation $V_{k,\mathcal{L}}$ of $G_{\mathbb{Q}_p}$ such that $D_{k,\mathcal{L}} = D^*_{\mathrm{st}}(V_{k,\mathcal{L}})$. Up to a twist by

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a crystalline character, the representations $V_{k,\mathcal{L}}$ enumerate all $\overline{\mathbb{Q}}_p$-linear 2-dimensional semi-stable and non-crystalline representations of $G_{\mathbb{Q}_p}$. They are irreducible except if $k = 2$.

We aim to determine the semi-simple mod p reductions $\overline{V}_{k,\mathcal{L}}$ of $V_{k,\mathcal{L}}$. Twenty years ago, Breuil and Mézard determined $\overline{V}_{k,\mathcal{L}}$ for even $k < p$ and any $\mathcal{L}$ ([7, Théorème 4.2.4.7]). Guerberoff and Park recently studied odd $k < p$ ([17, Theorem 5.0.5]). The reader who takes a moment to examine the cited theorems should be left with an impression of the complicated dependence of $\overline{V}_{k,\mathcal{L}}$ on $\mathcal{L}$, and that is just for $k < p$.

Prior results are limited by their ambition to determine $\overline{V}_{k,\mathcal{L}}$ for all $\mathcal{L}$. Here, we focus on determining $\overline{V}_{k,\mathcal{L}}$ for any k while restricting to $\mathcal{L}$ that place $V_{k,\mathcal{L}}$ in a p-adic neighborhood of a crystalline representation (see Section 1.2). Write $\mathbb{Q}_{p^2}$ for the unramified quadratic extension of $\mathbb{Q}_p$, χ for its quadratic character modulo p, and ω_2 for a niveau 2 fundamental character on $G_{\mathbb{Q}_{p^2}}$.

Theorem 1.1 (Theorem 4.1)**.** *Assume $k \geq 4$ and $p \neq 2$. Then, if*

$$v_p(\mathcal{L}) < 2 - \frac{k}{2} - v_p((k-2)!),$$

then $\overline{V}_{k,\mathcal{L}} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}(\omega_2^{k-1}\chi)$.

To be accurate, our method proves Theorem 1.1 when $k \geq 5$ or $p = 3$ and $k = 4$. The theorem holds for $k = 4$ and $p \geq 5$ by the work of Breuil–Mézard, and it is consistent with their work and the work of Guerberoff–Park for $5 \leq k < p$. Our method also directly obtains a result for $k = 3$ and $k = 4$ with a weaker bound. See Remark 4.8 for a more detailed discussion. Our exclusion of $p = 2$ is more fundamental (see Remark 1.4).

Remark 1.2. When $k < p$ and k is even, the bound in Theorem 1.1 is optimal by the results of Breuil–Mézard. The same can be said if $5 \leq k < p$ and k is odd, by the work of Guerberoff–Park. We do not know to what extent the bound is optimal for higher weights (see Section 1.3).

Theorem 1.1 is a natural analog of widely-studied theorems that determine reductions of 2-dimensional, irreducible, crystalline representations of $G_{\mathbb{Q}_p}$. For instance, Buzzard and Gee ([9]) developed a strategy to determine reductions of certain crystalline representations, with unbounded Hodge–Tate weights, using the p-adic local Langlands correspondence. We do not know whether a direct analog for semi-stable, but non-crystalline, representations has been tried or, even, if such an approach would be feasible.

Another approach in the crystalline case is via integral p-adic Hodge theory. Berger, Li, and Zhu and Berger proved local constancy results for reductions of crystalline representations using Wach modules ([5, 4]). Recently, the first two named authors of this article improved the Berger–Li–Zhu result using Kisin modules ([3]). Those are what we will use here, also. One incentive to write the prior article was as training to conduct the current research.

Finally, an indirect approach to calculating $\overline{V}_{k,\mathcal{L}}$ is explained in a recent preprint of Chitrao, Ghate, and Yasuda ([10]), though their investigation heads in a interesting separate direction from ours.

1.2. **Overview of strategy.** We now describe our strategy, first re-contextualizing Theorem 1.1 through the lens of local constancy of reductions as in [5, 4, 3].

The parametrization of semi-stable and non-crystalline representations by $\mathcal{L} \in \overline{\mathbb{Q}}_p$ extends to a $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$-parametrization with a crystalline representation at ∞. Namely, for $\mathcal{L} \neq 0$ we consider $D_{k,\mathcal{L}}$ with basis $(e_1', e_2') = (e_1, \mathcal{L}e_2)$ in which case, rather than (1.1), we have

$$\varphi = \begin{pmatrix} \varpi^k & 0 \\ 0 & \varpi^{k-2} \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 0 \\ \mathcal{L}^{-1} & 0 \end{pmatrix} \qquad \mathrm{Fil}^i D_{k,\mathcal{L}} = \begin{cases} D_{k,\mathcal{L}} & \text{if } i \leq 0; \\ \overline{\mathbb{Q}}_p \cdot (e_1' + e_2') & \text{if } 1 \leq i \leq k-1; \\ \{0\} & \text{if } k \leq i. \end{cases} \tag{1.2}$$

Thus, $D_{k,\mathcal{L}} \to D_{k,\infty}$ as $\mathcal{L}^{-1} \to 0$, where $D_{k,\infty}$ is the filtered (φ, N)-module with the same φ and filtration as (1.2) but with $N = 0$. In fact, $D_{k,\infty} \cong D^*_{\mathrm{crys}}(V_{k,\infty})$ where $V_{k,\infty}$ is a 2-dimensional *crystalline* representation of $G_{\mathbb{Q}_p}$ whose Frobenius trace is $a_p = \varpi^{k-2} + \varpi^k$. Replacing the filtered (φ, N)-modules with Galois representations, we have $V_{k,\mathcal{L}} \to V_{k,\infty}$ as $\mathcal{L}^{-1} \to 0$ (see the description in [12, Section 4.5-4.6] in terms of the space of trianguline representations, for instance). Thus, $\overline{V}_{k,\mathcal{L}} \cong \overline{V}_{k,\infty}$ for $\mathcal{L}^{-1} \to 0$. Furthermore, $v_p(a_p) = \frac{k-2}{2}$ and so $\lfloor \frac{k-1}{p} \rfloor < v_p(a_p)$, except if $p = 2$ or k is small, and so $\overline{V}_{k,\infty} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}(\omega_2^{k-1}\chi)$ by [3, Corollary 5.2.3]. We have reduced the theorem to the question: at which point as $\mathcal{L}^{-1} \to 0$, do we have $\overline{V}_{k,\mathcal{L}} \cong \overline{V}_{k,\infty}$?

We recall the relationship between reductions and Kisin modules, now. To ease notations, assume for the remainder of this subsection that k is even and $\mathcal{L} \in \mathbb{Q}_p$, so $V_{k,\mathcal{L}}$ and $V_{k,\infty}$ are defined over $\mathbb{Q}_p$. Let $\mathfrak{S} = \mathbb{Z}_p[\![u]\!]$, and write $\varphi : \mathfrak{S} \to \mathfrak{S}$ for the Frobenius map $\varphi(u) = u^p$. Then, consider the category $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\leq k-1}$ of φ-modules over $\mathfrak{S}$ with height $\leq k-1$ ([18]). Objects in this category, which are called Kisin modules, are finite free $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a φ-semilinear operator $\varphi : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of the linearization $\varphi^*\mathfrak{M} \to \mathfrak{M}$ is annihilated by $E(u)^{k-1}$, where $E(u) = u + p$. When $\mathfrak{M}$ satisfies the *monodromy condition*, Kisin's theory constructs a canonical semi-stable representation $V_{\mathfrak{M}}$ such that $D^*_{\mathrm{st}}(V_{\mathfrak{M}}) \cong \mathfrak{M}/u\mathfrak{M}[1/p]$, for a certain filtration and monodromy on the right-hand side. Furthermore, $\overline{V}_{\mathfrak{M}}$ is determined by $\mathfrak{M}/p\mathfrak{M}[u^{-1}]$ as a φ-module over $\mathbb{F}_p(\!(u)\!)$. The challenge in calculating $\overline{V}_{\mathfrak{M}}$ this way is determining $\mathfrak{M}$ from $V_{\mathfrak{M}}$ or, equivalently, $D^*_{\mathrm{st}}(V_{\mathfrak{M}})$. That task was carried out for $V_{k,\infty}$ in [3, Theorem 5.2.1].

The heart of this article is a two-step argument to do the same for $V_{k,\mathcal{L}}$ as $\mathcal{L}^{-1} \to 0$. The difficulty presented by non-trivial monodromy on $D_{k,\mathcal{L}}$ requires us to develop a new technique to pass from filtered (φ, N)-modules to Kisin modules. We make use of a category intermediate between filtered (φ, N)-modules and Kisin modules. Namely, write $\mathrm{Mod}_{S_{\mathbb{Q}_p}}^{\varphi,\leq k-1}$ for the category of φ-modules over $S_{\mathbb{Q}_p} = \mathbb{Z}_p[\![u, \frac{E^p}{p}]\!][\frac{1}{p}]$ with height $\leq k-1$. This category is close to certain filtered (φ, N)-modules considered by Breuil ([6]). Adapting Breuil's work, we explicitly construct a canonical object $\mathcal{M}_{k,\mathcal{L}} \in \mathrm{Mod}_{S_{\mathbb{Q}_p}}^{\varphi,\leq k-1}$ such that if $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,\leq k-1}$ and $\mathcal{M}_{k,\mathcal{L}} \cong \mathfrak{M} \otimes_{\mathfrak{S}} S_{\mathbb{Q}_p}$, then $V_{\mathfrak{M}} \cong V_{k,\mathcal{L}}$. Explicit means, for any (non-zero) $\mathcal{L}$, we determine a basis of $\mathcal{M}_{k,\mathcal{L}}$ and an exact formula for φ in that basis. This is where we overcome the difficulty of non-trivial monodromy on $D_{k,\mathcal{L}}$.

The second step is to descend $\mathcal{M}_{k,\mathcal{L}}$ from $S_{\mathbb{Q}_p}$ to $\mathfrak{S}$ when $\mathcal{L}^{-1} \to 0$, thus producing an $\mathfrak{M}$ for $V_{k,\mathcal{L}}$. Here, we view $S_{\mathbb{Q}_p}$ as subring of R_2, where R_2 is the ring of p-adic rigid analytic functions on $|u| \leq p^{-1/2}$ (using that $p \neq 2$). Section 4 of [3] presents a row reduction algorithm for semilinear operators that, under certain conditions, can descend from R_2 to $\mathfrak{S}$. Specifically, the main theorem in *loc. cit.* gives a sufficient condition to descend $\mathcal{M}_{k,\mathcal{L}} \otimes_{S_{\mathbb{Q}_p}} R_2$ to $\mathfrak{S}$. Saving the details for later, we use the explicit calculation of $\mathcal{M}_{k,\mathcal{L}}$ to check those conditions are met when $v_p(\mathcal{L}) < 2-\frac{k}{2}+v_p((k-2)!)$.

Remark 1.3. As already discussed, our approach in the first step is more general than [3] as it applies in the semi-stable, non-crystalline case. In fact, the method is quite general and can be used (with a suitable descent process) to compute reductions for higher dimensional semi-stable representations. For example, the third author has used the strategy here to compute reductions of irreducible 3-dimensional crystalline representations of $G_{\mathbb{Q}_p}$ with Hodge-Tate weights $\{0, r, s\}$ satisfying $2 \leq r \leq p-2$ and $p+2 \leq s \leq r+p-2$. See [22].

Remark 1.4. We exclude $p = 2$ twice. The second time, when we embed $S_{\mathbb{Q}_p}$ into R_2 is likely technical. However, we also exclude $p = 2$ when referencing the calculation of $\overline{V}_{k,\infty}$ in [3], and that seems crucial: our strategy is based not just on knowing $\overline{V}_{k,\infty}$, but also how to construct a Kisin module for $V_{k,\infty}$. Including $p = 2$, here would necessarily require calculating $\overline{V}_{k,\infty}$ when $p = 2$ as well. We note the formula $\overline{V}_{k,\infty} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}(\omega_2^{k-1}\chi)$ should still be true, but we cannot justify it.

1.3. **Global context.** We end this introduction with a discussion of the global situation. Suppose $N \geq 1$ and $f = \sum a_n(f)q^n$ is a cuspidal (normalized) eigenform of level $\Gamma_1(N)$, weight $k \geq 2$, and nebentype character ψ_f. Eichler–Shimura and Deligne famously associated with f a 2-dimensional, irreducible, continuous representation V_f of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We normalize V_f so that for $\ell \nmid Np$ the restriction $V_f|_{D_\ell}$ to D_ℓ, a decomposition group at ℓ, is unramified and the characteristic polynomial of a geometric Frobenius element is $t^2 - a_\ell(f)t + \psi_f(\ell)\ell^{k-1}$. The representation $V_f|_{D_p}$ is semi-stable when $p^2 \nmid N$ and the conductor of ψ_f is prime-to-p; it is crystalline when $p \nmid N$ ([24]).

We assume now that $V_f|_{D_p}$ is semi-stable and non-crystalline, in which case we define the $\mathcal{L}$-invariant of f to be the unique $\mathcal{L}_f \in \overline{\mathbb{Q}}_p$ such that $V_f|_{D_p} \cong V_{k,\mathcal{L}_f}$. The $\mathcal{L}$-invariant defined this way is called the Fontaine–Mazur $\mathcal{L}$-invariant (it agrees with [23, Section 12] up to a sign). It is a local quantity, but it famously arises in global situations. Examining how it arises allows us to provide global examples where Theorem 1.1 applies and to connect $\mathcal{L}$-invariants to global phenomena on p-adic families.

Theorem 1.1 determines $(\overline{V_f}|_{D_p})^{\mathrm{ss}}$ in arbitrary weights $k \geq p$ as long as $v_p(\mathcal{L}_f)$ is sufficiently negative, but it is not immediately obvious that eigenforms exist with $v_p(\mathcal{L}_f)$ so negative. Recent research, however, sheds light on the situation. For instance, Gräf ([16]) and Anni, Böckle, Gräf, and Troya (see [1], which builds on [11]) have developed the theory and practice needed to calculate the multiset of valuations of $\mathcal{L}$-invariants in a fixed weight and level. Pollack has also developed computer code to calculate $\mathcal{L}$-invariants. His method, which dates to the early 2000's, uses the appearance of $\mathcal{L}$-invariants in exceptional zero phenomena for p-adic L-functions. That method is being written up as part of a joint investigation by Pollack and the first author.

Using their works, both Pollack and Gräf kindly calculated some $\mathcal{L}$-invariants for us. In Table 1, we partially list the p-adic valuations found when $p = 3$ and $N = 51 = 3 \cdot 17$. Note, the bound in

TABLE 1. 3-adic valuations of some $\mathcal{L}$-invariants.

k	$v_3(\mathcal{L}_f)$ for newforms $f \in S_k(\Gamma_0(51))$
4	$-2, -1, 0, 0, \ldots$
6	$-3, -2, -1, -1, -1, \ldots$
8	$-3, -3, -\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}, -1, \ldots$

Theorem 1.1 is $v_3(\mathcal{L}_f) < 0$ in weight $k = 4$ and $v_3(\mathcal{L}_f) < -2$ in weight $k = 6$, so Table 1 provides two examples of Theorem 1.1 in weight $k = 4$ and one example of Theorem 1.1 in weight $k = 6$, though none in weight $k = 8$.

Let's look further at $p = 3$ and $k = 6$ and the boundary case $v_3(\mathcal{L}) = -2$ in Theorem 1.1. Pollack's code, in fact, reports not just $v_3(\mathcal{L}_f)$ for each newform f, but also $\overline{V}_f$. This refined data shows that the eigenforms with weight $k = 6$ and $v_3(\mathcal{L}_f)$ equal to -3 and -2 have isomorphic global Galois representations modulo 3. Since Theorem 1.1 applies to $v_3(\mathcal{L}) = -3$, we see that there exists $\mathcal{L}$-invariants with $v_3(\mathcal{L}) = -2$ for which the conclusion of Theorem 1.1 continues to hold. More numerical data is required before theorizing about the sharpness of the bound in Theorem 1.1.

The $\mathcal{L}$-invariants also arise, globally, from p-adic families. Namely, f lives in a p-adic family of eigenforms parametrized by weights $k \in \mathbb{Z}_p$ and $\mathcal{L}_f = -2\,\mathrm{dlog}\, a_p(k) = -2\frac{a_p'(k)}{a_p(f)}$ ([13, Corollarie 0.7]). This appearance reveals an obstruction to the "radius" of the largest "constant slope" family through f. Indeed, for $p \neq 2$, [2, Theorem 4.3] implies $v_p(\mathcal{L}_f^{-1}) \leq m(f)$ where $m(f)$ is the least positive integer such that f lives in a p-adic family of eigenforms f' with $v_p(a_p(f')) = v_p(a_p(f))$ and weight $k' \equiv k \bmod (p-1)p^{m(f)}$.

So, ruling out exceptions to Theorem 1.1, we have $v_p(\mathcal{L}_f) < 2 - \frac{k}{2} - v_p((k-2)!)$ implies

- $(\overline{V}_f|_{D_p})^{\mathrm{ss}} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}(\omega_2^{k-1}\chi)$, and
- $m(f) > \frac{k}{2} - 2 + v_p((k-2)!) \approx \frac{k-2}{2} + \frac{k}{p-1}$.

To connect these, if $k \not\equiv 1 \bmod p+1$, then $\overline{V}_f|_{D_p}$ is irreducible. On the other hand, condition (2) generically implies $m(f) > \frac{k-2}{2} = v_p(a_p(f))$. The fact that $m(f) > v_p(a_p(f))$ occurs in a situation where $\overline{V}_f|_{D_p}$ is irreducible is not a coincidence. It follows a pattern of counter-examples to a conjecture of Gouvêa and Mazur, which is related to the $m(f)$, found by Buzzard and Calegari ([8]). See [2, Section 9] for more discussion.

1.4. **Acknowledgements.** We owe the heuristic reframing in Section 1.2 to comments by Laurent Berger and Christophe Breuil during the conference "Géométrie arithmétique, théorie des représentations et applications" at the Centre International de Rencontres Mathématiques (CIRM) in Luminy, France. Part of this collaboration also took place during the workshop "Moduli spaces and modularity" at Casa Matemática Oaxaca (CMO). We thank both Berger and Breuil for their comments and both CIRM and CMO for their hospitality.

Finally, acknowledgments are due for the discussion in Section 1.3. First, the data reported in Table 1 in the first preprint version of this article was inaccurate. Because of that, we drew faulty conclusions, which have now been removed, on the strength of the bound in Theorem 1.1. We thank Robert Pollack for calculating the original data and then alerting us to the error. We also especially thank Peter Gräf for replicating the newly reported data, using his alternative method.

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2. Theoretical background

In this section, we recall filtered (φ, N)-modules and Breuil and Kisin modules. We explain, in theory, how to calculate a finite height φ-module, over a ring larger than $\mathfrak{S}$, associated with a filtered (φ, N)-module (Theorem 2.7). In Section 3, we carry this out in practice in a special case.

2.1. **Notations.** Let k be a finite field and $W(k)$ be the Witt vectors over k. Set $K_0 = W(k)[1/p]$ and assume K/K_0 is a totally ramified extension of degree e. Let Λ_K be the ring of integers of K, $\pi \in \Lambda_K$ a uniformizer and $E = E(u) \in W(k)[u]$ its Eisenstein polynomial. Choosing $\pi_0 = \pi$ and $\pi_1, \pi_2, \ldots$ a sequence in $\overline{K}$ such that $\pi_{i+1}^p = \pi_i$, we let G_∞ be the absolute Galois group of $\varinjlim K(\pi_i)$. Let $\mathcal{O} \subseteq K_0[\![u]\!]$ be the rigid analytic functions on $|u| < 1$ and $\mathfrak{S} = W(k)[\![u]\!] \subseteq \mathcal{O}$. The action of φ on $K_0[\![u]\!]$, by the Frobenius on K_0 and $\varphi(u) = u^p$, preserves $\mathfrak{S} \subseteq \mathcal{O} \subseteq K_0[\![u]\!]$.

We also choose $F/\mathbb{Q}_p$ a finite extension, which will play the role of linear coefficients. In Section 2.4, we assume F contains a subfield isomorphic the Galois closure of K. We write $\Lambda \subseteq F$ for the ring of integers, $\mathfrak{m}_F \subseteq \Lambda$ the maximal ideal, and $\mathbb{F}$ for the residue field. Define $\mathfrak{S}_\Lambda = \mathfrak{S} \otimes_{\mathbb{Z}_p} \Lambda$ and $\mathcal{O}_F = \mathcal{O} \otimes_{K_0} F$. Extending φ linearly, we have φ-stable subrings of $\mathfrak{S}_\Lambda \subseteq S_F \subseteq (K_0 \otimes_{\mathbb{Q}_p} F)[\![u]\!]$, where $S_F = \mathfrak{S}[\![\frac{E^p}{p}]\!] \otimes_{\mathbb{Q}_p} F$.

2.2. **Kisin modules.** Let $R \subseteq (K_0 \otimes_{\mathbb{Q}_p} F)[\![u]\!]$ be a φ-stable subring containing E. A φ-module over R is a finite free R-module M equipped with an injective φ-semilinear operator $\varphi_M : M \to M$. Let Mod_R^φ be the category of φ-modules over R with morphisms being R-linear maps that commute with φ. For a φ-module M, write $\varphi^* M = R \otimes_{\varphi, R} M$, so $1 \otimes \varphi_M$ defines an R-linear map $\varphi^* M \to M$ called the linearization of φ. For $h \geq 0$, an element $M \in \mathrm{Mod}_R^\varphi$ has (E)-height $\leq h$ if its linearization has cokernel annihilated by E^h. The subcategory of φ-modules over R with height $\leq h$ is denoted $\mathrm{Mod}_R^{\varphi, \leq h}$. A *Kisin module* over $\mathfrak{S}_\Lambda$ with height $\leq h$ is an object in $\mathrm{Mod}_{\mathfrak{S}_\Lambda}^{\varphi, \leq h}$.

Let $\mathrm{MF}_F^{\varphi, N}$ be the category of *positive* filtered (φ, N, K, F)-modules, which we shorten to just filtered (φ, N)-modules over F (see [7, Section 3.1.1]). For $D \in \mathrm{MF}_F^{\varphi, N}$ set $D_K = K \otimes_{K_0} D$; here, positive means $\mathrm{Fil}^0 D_K = D_K$. Let $\mathrm{Rep}_F^{\mathrm{st}, h}$ be the category of F-linear semi-stable representations V of G_K whose Hodge-Tate weights lie in $\{0, \ldots, h\}$. Then, there exists a fully faithful, contravariant, functor

$$D_{\mathrm{st}}^* : \mathrm{Rep}_F^{\mathrm{st}, h} \to \mathrm{MF}_F^{\varphi, N}$$

whose image is the subcategory of weakly-admissible filtered (φ, N)-modules over F (see [15, 14] and [7, Corollaire 3.1.1.3]). For $V \in \mathrm{Rep}_F^{\mathrm{st}, h}$ and $T \subseteq V$ a G_∞-stable and Λ-linear lattice there exists, by [20, Theorem 5.4.1], a canonical Kisin module $\mathfrak{M} = \mathfrak{M}(T)$ over $\mathfrak{S}_\Lambda$ with height $\leq h$. Naturally, we say a Kisin module $\mathfrak{M}$ is associated with V if $\mathfrak{M} = \mathfrak{M}(T)$ for some T. By [3, Corollary 2.3.2], the semi-simple mod p representation $\overline{V}$ can be determined from any associated Kisin module.

One category that intervenes in determining an $\mathfrak{M}$ associated with $V \in \mathrm{Rep}_F^{\mathrm{st},h}$ is the category of (φ, N_∇)-modules over $\mathcal{O}_F$ ([18]). Let $\lambda = \prod_{n\geq 0} \varphi^n(E(u)/E(0)) \in \mathcal{O}_F$. An object $\mathcal{M}_{\mathcal{O}_F} \in \mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N_\nabla}$ is a finite height φ-module over $\mathcal{O}_F$ equipped with a differential operator N_∇ lying over $-u\lambda\frac{d}{du}$ on $\mathcal{O}_F$ and satisfying $N_\nabla\varphi = p\frac{E(u)}{E(0)}\varphi N_\nabla$. By [18, Theorem 1.2.15], we have quasi-inverse equivalences of categories

$$\mathrm{MF}_F^{\varphi,N} \underset{\underline{\mathcal{M}}_{\mathcal{O}_F}}{\overset{\underline{D}_{\mathcal{O}_F}}{\rightleftarrows}} \mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N_\nabla}. \tag{2.1}$$

For $s > 0$, write $\mathcal{O}_s$ for the $\mathcal{O}_F$-algebra of rigid analytic functions converging on $|u| < p^{-s}$.

Proposition 2.1. *Suppose $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_\Lambda}^{\varphi,\leq h}$, $V \in \mathrm{Rep}_F^{\mathrm{st},h}$, and s is such that $1/pe < s < 1/e$ and $\mathfrak{M} \otimes_{\mathfrak{S}_\Lambda} \mathcal{O}_s \cong \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{\mathrm{st}}^*(V)) \otimes_{\mathcal{O}_F} \mathcal{O}_s$ in $\mathrm{Mod}_{\mathcal{O}_s}^{\varphi,\leq h}$. Then, $\mathfrak{M} = \mathfrak{M}(T)$ for some $T \subseteq V$ as above.*

Proof. Since $s < 1/e$, π lies in the disc $|u| < p^{-s}$. Since $\mathfrak{M} \otimes_{\mathfrak{S}_\Lambda} \mathcal{O}_s \cong \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{\mathrm{st}}^*(V)) \otimes_{\mathcal{O}_F} \mathcal{O}_s$, [3, Corollary 2.2.5] implies that $\mathcal{M}_{\mathcal{O}_F} := \mathfrak{M} \otimes_{\mathfrak{S}_\Lambda} \mathcal{O}_F$ is canonically an object in $\mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N_\nabla}$. Then, [20, Theorem 5.4.1] implies that there exists a $V' \in \mathrm{Rep}_F^{\mathrm{st},h}$ such that $\mathfrak{M} = \mathfrak{M}(T)$ for a lattice $T \subseteq V'$ for some T. We claim that $V \cong V'$. Indeed, since $1/pe < s < 1/e$, the definition of $\underline{D}_{\mathcal{O}_F}(\mathcal{M}_{\mathcal{O}_F})$ in [18, Section 1.2.5-7] depends only the finite height φ-module $\mathcal{M}_{\mathcal{O}_F} \otimes_{\mathcal{O}_F} \mathcal{O}_s$ over $\mathcal{O}_s$. Thus, we have

$$D_{\mathrm{st}}^*(V') \cong \underline{D}_{\mathcal{O}_F}(\mathcal{M}_{\mathcal{O}_F}) \cong \underline{D}_{\mathcal{O}_F}(\underline{\mathcal{M}}_{\mathcal{O}_F}(D_{\mathrm{st}}^*(V))) \cong D_{\mathrm{st}}^*(V).$$

Since D_{st}^* is fully faithful, we have $V \cong V'$, completing the proof. $\square$

Remark 2.2. To be accurate, the equivalence (2.1) is constructed in [18] only when $F = \mathbb{Q}_p$. We use multiple references with the same technical limitation. We pause to detail one approach to resolving the issue. Later, we omit details for other functors.

First, we may define the functors $\underline{D}_{\mathcal{O}_F}$ and $\underline{\mathcal{M}}_{\mathcal{O}_F}$ using the same formulas as (2.1), or, equivalently, we can define them by forcing the diagram

$$\begin{tikzcd} \mathrm{MF}_F^{\varphi,N} \arrow[d, "\mathrm{forget}"'] \arrow[r, shift right, "\underline{\mathcal{M}}_{\mathcal{O}_F}"'] & \mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N_\nabla} \arrow[l, shift right, "\underline{D}_{\mathcal{O}_F}"'] \arrow[d, "\mathrm{forget}"] \\ \mathrm{MF}_{\mathbb{Q}_p}^{\varphi,N} \arrow[r, shift right, "\underline{\mathcal{M}}_{\mathcal{O}}"'] & \mathrm{Mod}_{\mathcal{O}}^{\varphi,N_\nabla} \arrow[l, shift right, "\underline{D}_{\mathcal{O}}"'] \end{tikzcd}$$

to commute. Here, the vertical arrows are the natural forgetful functors and the bottom arrows are as in [18], where they are proved to be quasi-inverses. If $\mathcal{M}_{\mathcal{O}_F} \in \mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N_\nabla}$, we thus have a natural isomorphism $\alpha : \underline{\mathcal{M}}_{\mathcal{O}_F}(\underline{D}_{\mathcal{O}_F}(\mathcal{M}_{\mathcal{O}_F})) \cong \mathcal{M}_{\mathcal{O}_F}$ in $\mathrm{Mod}_{\mathcal{O}}^{\varphi,N_\nabla}$. Since multiplication by $x \in F$ defines an endomorphism of $\mathcal{M}_{\mathcal{O}_F}$ in $\mathrm{Mod}_{\mathcal{O}}^{\varphi,N_\nabla}$ and α is natural, we see α is an isomorphism in

$\mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N\nabla}$. Thus, $\underline{\mathcal{M}}_{\mathcal{O}_F}$ is a left quasi-inverse to $\underline{D}_{\mathcal{O}_F}$. Proving $\underline{D}_{\mathcal{O}_F}$ is a right quasi-inverse to $\underline{\mathcal{M}}_{\mathcal{O}_F}$ is analogous.

2.3. **Breuil modules.** Let S_{Br} be the p-adic completion of the divided power envelope of $W(k)[u]$ with respect to the ideal generated by E. Breuil ([6]) classically identified $\mathrm{MF}_{\mathbb{Q}_p}^{\varphi,N}$ with a category of filtered (φ, N)-modules over $S_{\mathrm{Br}}[\frac{1}{p}]$. We recall this, replacing S_{Br} with a simpler ring.

One extends the Frobenius φ to $K_0[\![u]\!]$ via $\varphi(u) = u^p$. We define $N = -u\frac{d}{du}$ on $K_0[\![u]\!]$. Let $\widehat{S}_E$ be the E-completion of $W(k)[u][\frac{1}{p}]$. For a subring $R \subseteq \widehat{S}_E$ and $j \geq 0$, set $\mathrm{Fil}^j R = R \cap E^j \widehat{S}_E$. In particular, we can take $R = S := W(k)[\![u, \frac{E^p}{p}]\!]$. As a subring of $K_0[\![u]\!]$, S is closed under φ and N. We define $S_\Lambda = S \otimes_{\mathbb{Z}_p} \Lambda$ and $S_F = S \otimes_{\mathbb{Z}_p} F$, extending φ, N, and $\mathrm{Fil}^\bullet$ linearly.

Clearly $S \subseteq S_{\mathrm{Br}} \subseteq \widehat{S}_E$ which are compatible with the $(u, \frac{E^p}{p})$-topology on S, the p-adic topology on S_{Br} and the (E)-topology on $\widehat{S}_E$. One advantage S enjoys over S_{Br} is that $\mathrm{Fil}^j S_F = E^j S_F$. To see this, note that any element $f \in \mathrm{Fil}^j S_F$ can be uniquely written in the form $f = \sum_i a_i(u) \frac{E^{pi}}{p^i}$ with $a_i(u) \in K_0[u]$ a polynomial of degree strictly less than ep (e is the degree of E). Then, when $j < pi$, we have $\frac{E^{pi-j}}{p^i} = \frac{1}{p^{i-l}} E^{pi-pl} (\frac{E^p}{p})^l$ with $l = \lfloor \frac{pi-j}{p} \rfloor$. In this situation, $i - l$ depends only on j, so factoring E^j out of the expression for f and examining the leftover summation, one sees at once that $f \in E^j S_F$. Note as well: S_F is an $\mathcal{O}_F$-algebra, and $\varphi(E) = p\mathfrak{c}$ with $\mathfrak{c} \in S^\times$. In particular, $\varphi(\lambda) \in S^\times \subseteq S_F^\times$.

The category $\mathrm{MF}_{S_F}^{\varphi,N}$ of filtered (φ, N)-modules over S_F, or *Breuil modules* over S_F, are objects $(\mathcal{D}, \varphi_{\mathcal{D}}) \in \mathrm{Mod}_{S_F}^{\varphi}$ such that the linearization of $\varphi_{\mathcal{D}}$ is an isomorphism, and $\mathcal{D}$ is equipped with:

- a decreasing filtration $\mathrm{Fil}^\bullet \mathcal{D}$ by S_F-submodules such that $\mathrm{Fil}^0 \mathcal{D} = \mathcal{D}$ and $\mathrm{Fil}^i S_F \cdot \mathrm{Fil}^j \mathcal{D} \subseteq \mathrm{Fil}^{i+j} \mathcal{D}$ for all $i, j \geq 0$;
- an operator $N_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ that acts as a derivation over N, and
  - $N_{\mathcal{D}}\varphi_{\mathcal{D}} = p\varphi_{\mathcal{D}}N_{\mathcal{D}}$, and
  - $N_{\mathcal{D}}(\mathrm{Fil}^i \mathcal{D}) \subseteq \mathrm{Fil}^{i-1} \mathcal{D}$ for all $i \geq 1$.

A morphism in $\mathrm{MF}_{S_F}^{\varphi,N}$ is an S_F-linear map equivariant for φ, N, and $\mathrm{Fil}^\bullet$.

We define a functor $\underline{\mathcal{D}} : \mathrm{MF}_F^{\varphi,N} \to \mathrm{MF}_{S_F}^{\varphi,N}$ as follows:

- $\mathcal{D} := \underline{\mathcal{D}}(D) = S_F \otimes_{K_0 \otimes_{\mathbb{Q}_p} F} D$ as an S_F-module;
- $\varphi_{\mathcal{D}} = \varphi \otimes \varphi_D$;
- $N_{\mathcal{D}} = N \otimes 1 + 1 \otimes N_D$;
- $\mathrm{Fil}^0(\mathcal{D}) = \mathcal{D}$ and
$$\mathrm{Fil}^i(\mathcal{D}) = \{x \in \mathcal{D} \mid N_{\mathcal{D}}(x) \in \mathrm{Fil}^{i-1} \mathcal{D} \text{ and } (\mathrm{ev}_\pi \otimes 1)(x) \in \mathrm{Fil}^i D_K\}$$
  for $i \geq 1$.

Here, $\mathrm{ev}_\pi : S_F \to F \otimes_{\mathbb{Q}_p} K$ is the scalar extension of $\mathrm{ev}_\pi : W(k)[u] \twoheadrightarrow \Lambda_K$, the evaluation at π map.

Theorem 2.3 (Breuil)**.** *The functor $\underline{\mathcal{D}} : \mathrm{MF}_F^{\varphi,N} \to \mathrm{MF}_{S_F}^{\varphi,N}$ is an equivalence of categories.*

Breuil proves in [6, Section 6] that $\underline{\mathcal{D}}$ is an equivalence of categories when $F = \mathbb{Q}_p$ and S is replaced by S_{Br}. That one can replace S_{Br} by S is known to some, but there does not appear to be a reference. The only step in the proof of Breuil that requires honestly new justification is the following analog of [6, Proposition 6.2.1.1]. (This version is even easier to prove.)

Lemma 2.4. *Let $\mathcal{D} \in \mathrm{MF}_{S_F}^{\varphi,N}$ and $D = \mathcal{D}/u\mathcal{D}$. Then, there exists a unique $F \otimes_{\mathbb{Q}_p} K_0$-linear φ-equivariant section $s : D \to \mathcal{D}$ of the reduction map.*

Proof. First, suppose $F = \mathbb{Q}_p$ and let $(\widehat{e}_1, \ldots, \widehat{e}_d)$ be an $S[\frac{1}{p}]$-basis of $\mathcal{D}$. Write $\varphi_{\mathcal{D}}(\widehat{e}_1, \ldots, \widehat{e}_d) = (\widehat{e}_1, \ldots, \widehat{e}_d)X$ and set $X_0 = X \bmod u$. Then, $X \in p^k \operatorname{Mat}_d(S)$, $X_0^{-1} \in p^\ell \operatorname{Mat}_d(W(k))$, and $XX_0^{-1} \in I + up^m \operatorname{Mat}_d(S)$ for some $k, \ell, m \in \mathbb{Z}$. As in the proof of [6, Proposition 6.2.1.1], we need to show

$$Y_n := X\varphi(X)\cdots\varphi^n(X)\varphi^n(X_0^{-1})\cdots\varphi(X_0^{-1})X_0^{-1}$$

converges in $\operatorname{Mat}_d(S[\frac{1}{p}])$ as $n \to \infty$. But, in the notation above,

$$Y_n - Y_{n-1} \in \varphi^n(u)p^{n(k+\ell)+m} \operatorname{Mat}_d(S).$$

Since $\varphi^n(u)p^{nr} \to 0$ in $S[\frac{1}{p}]$ for any fixed r, we see that $Y_n - Y_{n-1} \to 0$ in $\operatorname{Mat}_d(S[\frac{1}{p}])$, as needed.

If $F \neq \mathbb{Q}_p$, the proof already given implies there exists a unique K_0-linear φ-equivariant section $s : D \to \mathcal{D}$. If $x \in F^\times$ then $x^{-1}sx$ also K_0-linear and φ-equivariant and thus s is F-linear. □

Proof of Theorem 2.3. Define $\underline{D}_{S_F} : \mathrm{MF}_{S_F}^{\varphi,N} \to \mathrm{MF}_F^{\varphi,N}$ as follows. Set $D = \underline{D}_{S_F}(\mathcal{D}) = \mathcal{D}/u\mathcal{D}$ with its induced action of φ and N. For s in Lemma 2.4, $(\mathrm{ev}_\pi \otimes 1) \circ s : D \to \mathcal{D}/E\mathcal{D}$ induces a canonical isomorphism $D_K \cong \mathcal{D}/E\mathcal{D}$. The filtration $\mathrm{Fil}^i(D_K)$ is the pullback of the filtration on $\mathcal{D}/E\mathcal{D}$ defined as the image $\mathrm{Fil}^i(\mathcal{D}) \to \mathcal{D}/E\mathcal{D}$. The arguments in [6], with Lemma 2.4 replacing Proposition 6.2.1.1 of *loc. cit.*, show $\underline{D}_{S[\frac{1}{p}]}$ and $\underline{\mathcal{D}}$ are quasi-inverses when $F = \mathbb{Q}_p$. In general, see Remark 2.2. □

2.4. **Comparison.** We now assume that F contains a subfield isomorphic to the Galois closure of K (see Lemma 2.5). In practice, as in Sections 3 and 4, we take $K = \mathbb{Q}_p$ so this is no hindrance.

In the prior sections, we have described equivalences

$$\mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N\nabla} \xrightarrow{\cong} \mathrm{MF}_F^{\varphi,N} \xrightarrow{\cong} \mathrm{MF}_{S_F}^{\varphi,N}. \tag{2.2}$$

An analog of [21, Corollary 3.2.3] allows for a description of the composition that, unfortunately, is not practical for calculations. Below, though, we explain how to determine $\underline{\mathcal{M}}_{\mathcal{O}_F}(D) \otimes_{\mathcal{O}_F} S_F$ as a φ-module over S_F from D, up to determining $\mathcal{D} = \underline{\mathcal{D}}(D)$. A key technical point, which follows from the next lemma, is that filtrations on Breuil modules over S_F are always free, in contrast to the filtrations on objects in $\mathrm{MF}_F^{\varphi,N}$ (cf. [7, Exemple 3.1.1.4]).

Lemma 2.5. *Suppose that $\mathcal{N}$ is a finite free S_F-module and $\mathcal{H} \subseteq \mathcal{N}$ is an S_F-submodule such that $E^j\mathcal{N} \subseteq \mathcal{H}$ for some $j \geq 0$. Then, $\mathcal{H}$ is finite free over S_F.*

Proof. We may assume $j = 1$. Indeed, consider the nested sequence $\mathcal{H}_i = \mathcal{H} + E^i\mathcal{N}$ of S_F-modules, which satisfy $E\mathcal{H}_i \subseteq \mathcal{H}_{i+1} \subseteq \mathcal{H}_i$. By the $j = 1$ case we deduce $\mathcal{H}_1 \subseteq \mathcal{N}$ is free, and then $\mathcal{H}_2$, and so on until $\mathcal{H}_j = \mathcal{H}$ is free. We may also assume $\mathcal{N} \cong S_F$. Indeed, if $0 \to \mathcal{N}'' \to \mathcal{N} \xrightarrow{f} \mathcal{N}' \to 0$ is an exact sequence of finite free S_F-modules, then $\mathcal{H}' = f(\mathcal{H})$ and $\mathcal{H}'' = \ker(f) \cap \mathcal{H}$ satisfy $E\mathcal{N}'' \subseteq \mathcal{H}''$ and $E\mathcal{N}' \subseteq \mathcal{H}'$. So, if both $\mathcal{H}''$ and $\mathcal{H}'$ are free, then $\mathcal{H} \cong \mathcal{H}'' \oplus \mathcal{H}'$ is free as well.

We have reduced to proving: if $I \subseteq S_F$ is an ideal containing E, then I is free. Since F contains a subfield isomorphic to the Galois closure of K, we may decompose $S_F = \prod_{\sigma \in \mathrm{Hom}(K_0,F)} S_{F,\sigma}$ where $S_{F,\sigma} = \Lambda[\![u, \frac{\sigma(E)^p}{p}]\!][\frac{1}{p}]$ is a domain. The ideal I decomposes as a product of ideals I_σ such that $\sigma(E)S_{F,\sigma} \subseteq I_\sigma$. Since $\sigma(E)$ is non-zero, it suffices to show each I_σ is principal. Write $\mathrm{Hom}_\sigma(K,F)$ for the embeddings $\tau : K \to F$ lifting σ. Then, we have a canonical isomorphism

$$S_{F,\sigma}/\sigma(E)S_{F,\sigma} \cong K \otimes_{K_0,\sigma} F \cong F^{\mathrm{Hom}_\sigma(K,F)}.$$

So, $I_\sigma/\sigma(E)S_{F,\sigma} \cong F^T$ for some subset $T \subseteq \mathrm{Hom}_\sigma(K,F)$. But, $J_T = \prod_{\tau \in T}(u - \tau(\pi)) \cdot S_F$ also contains $\sigma(E)S_{F,\sigma}$ and $J_T/\sigma(E)S_{F,\sigma} \cong F^T$. Thus $I_\sigma = J_T$ is principal, completing the proof. $\square$

We now consider an *ad hoc* category of "Breuil modules without monodromy". Let $\mathrm{MF}^{\varphi,h}_{S_F}$ denote the category whose objects are $(\mathcal{D}, \varphi_{\mathcal{D}}) \in \mathrm{Mod}^{\varphi}_{S_F}$ such that the linearization of $\varphi_{\mathcal{D}}$ is an isomorphism, and $\mathcal{D}$ is equipped with a finite *free* S_F-submodule $\mathrm{Fil}^h \mathcal{D} \subseteq \mathcal{D}$ such that $\mathrm{Fil}^h S_F \cdot \mathcal{D} \subseteq \mathrm{Fil}^h \mathcal{D}$. By Lemma 2.5 there is a natural forgetful functor $\mathrm{MF}^{\varphi,N}_{S_F} \to \mathrm{MF}^{\varphi,h}_{S_F}$.

Now define $\underline{\mathcal{D}}' : \mathrm{Mod}^{\varphi,\leq h}_{S_F} \to \mathrm{MF}^{\varphi,h}_{S_F}$ by declaring $\underline{\mathcal{D}}'(\mathcal{M}) = S_F \otimes_{\varphi,S_F} \mathcal{M}$ as an S_F-module, and

- $\varphi_{\underline{\mathcal{D}}'(\mathcal{M})} = \varphi \otimes \varphi_{\mathcal{M}}$, and
- $\mathrm{Fil}^h \underline{\mathcal{D}}'(\mathcal{M}) = \{x \in \underline{\mathcal{D}}'(\mathcal{M}) \mid (1 \otimes \varphi_{\mathcal{M}})(x) \in \mathrm{Fil}^h S_F \cdot \mathcal{M}\}$.

Since $E^h \underline{\mathcal{D}}'(\mathcal{M}) \subseteq \mathrm{Fil}^h \underline{\mathcal{D}}'(\mathcal{M})$, Lemma 2.5 implies $\mathrm{Fil}^h \underline{\mathcal{D}}'(\mathcal{M})$ is finite free over S_F.

Proposition 2.6. *The functor $\underline{\mathcal{D}}'$ is an equivalence.*

Proof. We first show $\underline{\mathcal{D}}'$ is fully faithful. Suppose $\mathcal{M}$ and $\mathcal{M}'$ are in $\mathrm{Mod}^{\varphi,\leq h}_{S_F}$. Write $\mathcal{D} := \underline{\mathcal{D}}'(\mathcal{M})$ and $\mathcal{D}' := \underline{\mathcal{D}}'(\mathcal{M}')$. Choose a basis $(e_1, \ldots, e_d)$ of $\mathcal{M}$ and write $\varphi_{\mathcal{M}}(e_1, \ldots, e_d) = (e_1, \ldots, e_d)A$ with $A \in \mathrm{Mat}_d(S_F)$. Since $\mathcal{M}$ has height $\leq h$, there exists a matrix $B \in \mathrm{Mat}_d(S_F)$ such that $AB = BA = E^h I_d$. By assumption, $\mathrm{Fil}^h \mathcal{D}$ has basis $(\alpha_1, \ldots, \alpha_d) = (\widetilde{e}_1, \ldots, \widetilde{e}_d)B$ where $\widetilde{e}_i = 1 \otimes e_i \in \mathcal{D}$ compose a basis of $\mathcal{D}$. Similarly, we get A', B' and $\widetilde{e}'_i$ from a basis $(e'_1, \ldots, e'_{d'})$ of $\mathcal{M}'$.

Now suppose $f : \mathcal{D} \to \mathcal{D}'$ is a morphism in $\mathrm{MF}^{\varphi,h}_{S_F}$. We write $f(\widetilde{e}_1, \ldots, \widetilde{e}_d) = (\widetilde{e}'_1, \ldots, \widetilde{e}'_{d'})X$ for $X \in \mathrm{Mat}_d(S_F)$. Since f is φ-equivariant, we have $X\varphi(A) = \varphi(A')\varphi(X)$, and, since $f(\mathrm{Fil}^h \mathcal{D}) \subseteq \mathrm{Fil}^h \mathcal{D}'$, we have $XB = B'Y$ for some $Y \in \mathrm{Mat}_d(S_F)$. Using $AB = BA = E^h I_d$ and $A'B' = B'A' = E^h I_{d'}$, we see $\varphi(Y)\varphi(E^h) = X\varphi(E^h)$, and so $X = \varphi(Y)$ because $\varphi(E) \in S_F^\times$. It follows that $YA = A'\varphi(Y)$. Define $\mathfrak{f} : \mathcal{M} \to \mathcal{M}'$ by $\mathfrak{f}(e_1, \ldots, e_d) = (e'_1, \ldots e'_{d'})Y$. Then, $\mathfrak{f}$ is φ-equivariant and $f = \underline{\mathcal{D}}'(\mathfrak{f})$ since $X = \varphi(Y)$. This shows $\underline{\mathcal{D}}'$ is full, and since Y determines X, we also see $\underline{\mathcal{D}}'$ is faithful.

Now we prove $\underline{\mathcal{D}}'$ is essentially surjective. Given a $\mathcal{D} \in \mathrm{MF}^{\varphi,h}_{S_F}$, choose bases $(e_1, \ldots, e_d)$ of $\mathcal{D}$ and $(\alpha_1, \ldots, \alpha_d)$ of $\mathrm{Fil}^h \mathcal{D}$. Write $(\alpha_1, \ldots, \alpha_d) = (e_1, \ldots, e_d)B$ and $\varphi_{\mathcal{D}}(e_1, \ldots, e_d) = (e_1, \ldots, e_d)X$ with $\det(X) \in S_F^\times$. Since $E^h\mathcal{D} \subseteq \mathrm{Fil}^h \mathcal{D}$, there exists $A \in \mathrm{Mat}_d(S_F)$ such that $AB = BA = E^h I_d$. Since $\varphi(E) = p\mathfrak{c} \in S_F^\times$, we see $X\varphi(B) \in \mathrm{GL}_d(S_F)$, whereas $\varphi_{\mathcal{D}}(\alpha_1, \ldots, \alpha_d) = (e_1, \ldots, e_d)X\varphi(B)$. Thus $(f_1, \ldots, f_d) = (e_1, \ldots, e_d)X\varphi(B)p^{-h}\mathfrak{c}^{-h}$ is a basis of $\mathcal{D}$ and $\varphi_{\mathcal{D}}(\alpha_1, \ldots, \alpha_d) = (f_1, \ldots, f_d)p^h\mathfrak{c}^h$. Finally, $(\alpha_1, \ldots, \alpha_d) = (f_1, \ldots, f_d)B'$ where $B' = YB$ and $Y = (X\varphi(B)p^{-h}\mathfrak{c}^{-h})^{-1}$, so there exists an A' such that $A'B' = B'A' = E^h I_d$. Now define $\mathcal{M} = \bigoplus_{i=1}^{d} S_F \mathfrak{f}_i$ and set $\varphi_{\mathcal{M}}(\mathfrak{f}_1, \ldots, \mathfrak{f}_d) = (\mathfrak{f}_1, \ldots, \mathfrak{f}_d)A'$. Then, $\mathcal{M} \in \mathrm{Mod}_{S_F}^{\varphi, \leq h}$ and $\underline{\mathcal{D}}'(\mathcal{M}) = \mathcal{D}$ (set $f_i = 1 \otimes \mathfrak{f}_i$). $\square$

We now reach the main theorem of this section, which provides a mechanism to calculate a finite height φ-module over S_F explicitly from $D \in \mathrm{MF}_F^{\varphi,N}$. We write $\varphi(E) = p\mathfrak{c}$ with $\mathfrak{c} \in S^{\times}$ as above.

Theorem 2.7. *Suppose $D \in \mathrm{MF}_F^{\varphi,N}$. Write $\mathcal{D}' \in \mathrm{MF}_{S_F}^{\varphi,h}$ for the image of $\underline{\mathcal{D}}(D)$ under the natural forgetful functor and $\mathcal{M} = \underline{\mathcal{M}}_{\mathcal{O}_F}(D) \otimes_{\mathcal{O}_F} S_F$. Then, there is a natural isomorphism $\underline{\mathcal{D}}'(\mathcal{M}) \cong \mathcal{D}'$.*

In particular, $\mathcal{M}$ is recovered from D via the following steps:

1. *Select S_F-bases $(e_1, \ldots, e_d)$ of $\mathcal{D} = \underline{\mathcal{D}}(D)$ and $(\alpha_1, \ldots, \alpha_d)$ of $\mathrm{Fil}^h \mathcal{D}$.*
2. *Let $\varphi_{\mathcal{D}}(e_1, \ldots, e_d) = (e_1, \ldots, e_d)X$ and $(\alpha_1, \ldots, \alpha_d) = (e_1, \ldots, e_d)B$ with $X, B \in \mathrm{Mat}_d(S_F)$.*
3. *Then, $\mathcal{M}$ has an S_F-basis $(\mathfrak{f}_1, \ldots, \mathfrak{f}_d)$ in which $\varphi_{\mathcal{M}}(\mathfrak{f}_1, \ldots, \mathfrak{f}_d) = (\mathfrak{f}_1, \ldots, \mathfrak{f}_d)A$, where*

$$A = E^h B^{-1} X \varphi(B) p^{-h} \mathfrak{c}^{-h}.$$

Proof. To start, once the isomorphism $\underline{\mathcal{D}}'(\mathcal{M}) \cong \mathcal{D}'$ is justified, the "in particular" follows by tracing through the second half of the proof of Proposition 2.6.

For $\mathcal{M}_{\mathcal{O}_F} \in \mathrm{Mod}_{\mathcal{O}_F}^{\varphi, N_\nabla}$ we define $\mathcal{D} = \underline{\mathcal{D}}_{\mathcal{O}_F}(\mathcal{M}_{\mathcal{O}_F}) = S_F \otimes_{\varphi, \mathcal{O}_F} \mathcal{M}_{\mathcal{O}_F}$, which is a finite free S_F-module, and equip it with the following structure of a Breuil module over S_F:

- $\varphi_{\mathcal{D}} = \varphi \otimes \varphi_{\mathcal{M}}$;
- $N_{\mathcal{D}} = N \otimes 1 + \frac{p}{\varphi(\lambda)} \otimes N_\nabla$;
- $\mathrm{Fil}^i(\mathcal{D}) = \{x \in \mathcal{D} \mid (1 \otimes \varphi_{\mathcal{M}})(x) \in \mathrm{Fil}^i S_F \otimes_{\mathcal{O}_F} \mathcal{M}_{\mathcal{O}_F}\}$.

Following the proof of [20, Proposition 3.2.1], replacing S by S_{Br} and adding linear F-coefficients, we see $\underline{\mathcal{D}}_{\mathcal{O}_F} : \mathrm{Mod}_{\mathcal{O}_F}^{\varphi, N_\nabla} \to \mathrm{MF}_{S_F}^{\varphi,N}$ defines a functor. Moreover, if $\mathcal{M}_{\mathcal{O}_F}$ has height $\leq h$, then

$$\underline{\mathcal{D}}_{\mathcal{O}_F}(\mathcal{M}_{\mathcal{O}_F}) \cong \underline{\mathcal{D}}'(\mathcal{M}_{\mathcal{O}_F} \otimes_{\mathcal{O}_F} S_F)$$

in the category $\mathrm{MF}_{S_F}^{\varphi,h}$. Thus, it remains to show that $\underline{\mathcal{D}}_{\mathcal{O}_F}$ makes the diagram of functors

(2.3)
$$\begin{tikzcd} \mathrm{MF}_F^{\varphi,N} \arrow[r, "\underline{\mathcal{D}}"] & \mathrm{MF}_{S_F}^{\varphi,N} \\ \mathrm{Mod}_{\mathcal{O}_F}^{\varphi, N_\nabla} \arrow[u, "\underline{D}_{\mathcal{O}_F}"] \arrow[ur, "\underline{\mathcal{D}}_{\mathcal{O}_F}"'] & \end{tikzcd}$$

commute as well. (In particular, $\underline{D}_{\mathcal{O}_F}$ is an equivalence.) It is enough to check this when $F = \mathbb{Q}_p$ (by Remark 2.2). In that case, if S is replaced by S_{Br}, this is the statement of [20, Corollary 3.2.3]. The proof in *loc. cit.* goes through here with only one adjustment. Namely, the isomorphism $S_{\mathrm{Br}}[\frac{1}{p}] \otimes_{K_0} \underline{D}_{\mathcal{O}}(\mathcal{M}_{\mathcal{O}}) \cong S_{\mathrm{Br}}[\frac{1}{p}] \otimes_{\varphi, \mathcal{O}} \mathcal{M}_{\mathcal{O}}$ implicit in the first two displayed equations of *loc. cit.* needs to have S_{Br} replaced by S. To make this adjustment, consider the map $\xi : \mathcal{O} \otimes_{K_0} \underline{D}(\mathcal{M}_{\mathcal{O}}) \to \mathcal{M}_{\mathcal{O}}$ constructed in [18, Lemma 1.2.6]. Thus ξ is a φ-equivariant injection with cokernel annihilated by

λ^h for some $h \geq 0$. From the diagram in the middle of the proof of *loc. cit.* we have ξ factors

$$\begin{tikzcd} \mathcal{O} \otimes_{K_0} \underline{D}(\mathcal{M}_\mathcal{O}) \arrow[r, "\xi"] \arrow[d] & \mathcal{M}_\mathcal{O} \\ \mathcal{O} \otimes_{\varphi, \mathcal{O}} \mathcal{M}_\mathcal{O}. \arrow[ur, "1\otimes\varphi"'] & \end{tikzcd} \tag{2.4}$$

We deduce the vertical arrow in (2.4) has cokernel annihilated by $\varphi(\lambda)^h$. Since $\varphi(\lambda) \in S^\times$, we have

$$S[1/p] \otimes_{K_0} \underline{D}(\mathcal{M}_\mathcal{O}) \overset{1\otimes\xi}{\cong} S[1/p] \otimes_{\varphi,\mathcal{O}} \mathcal{M}_\mathcal{O}.$$

This completes the proof. □

Remark 2.8. The above proof makes it clear to see that for $D \in \mathrm{MF}_F^{\varphi,N}$ and $\mathcal{D} = \underline{\mathcal{D}}(D) \in \mathrm{MF}_{S_F}^{\varphi,N}$, the map ev_π induces an isomorphism $\mathrm{Fil}^{i+1}\mathcal{D}/E\,\mathrm{Fil}^i\mathcal{D} \cong \mathrm{Fil}^{i+1} D_K$. Indeed, since $\mathrm{ev}_\pi(\mathrm{Fil}^{i+1}\mathcal{D}) = \mathrm{Fil}^{i+1} D_K$, it suffices to show that $E\mathcal{D} \cap \mathrm{Fil}^{i+1}\mathcal{D} = E\,\mathrm{Fil}^i\mathcal{D}$. Pick $y = Ex \in \mathrm{Fil}^{i+1}\mathcal{D}$ with $x \in \mathcal{D}$. The proof of the theorem, especially the fact that (2.3) commutes, shows that

$$\mathrm{Fil}^{i+1}(\mathcal{D}) = \{x \in \mathcal{D} \mid (1 \otimes \varphi_\mathcal{M})(x) \in \mathrm{Fil}^{i+1} S_F \otimes_{\mathcal{O}_F} \mathcal{M}_{\mathcal{O}_F}\}.$$

Thus, we see that $(1 \otimes \varphi_\mathcal{M})(Ex) = E(1 \otimes \varphi_\mathcal{M})(x) \in \mathrm{Fil}^{i+1} S_F \otimes_{\mathcal{O}_F} \mathcal{M}_{\mathcal{O}_F}$. Since $\mathrm{Fil}^n S_F = E^n S_F$, it is clear that $(1 \otimes \varphi_\mathcal{M})(x) \in \mathrm{Fil}^i S_F \otimes_{\mathcal{O}_F} \mathcal{M}_{\mathcal{O}_F}$ and hence $x \in \mathrm{Fil}^i \mathcal{D}$ as required. (Compare with the end of the proof of [21, Proposition 3.2.1].)

Example 2.9. Suppose $K = \mathbb{Q}_p$ and V is crystalline. By [19], $D = D^*_{\mathrm{st}}(V)$ admits a strongly divisible lattice $(M, \mathrm{Fil}^i M, \varphi_i)$. More precisely, there exists an F-basis $(e_1, \dots, e_d)$ of D and integers $0 = n_0 \leq n_1 \leq \cdots \leq n_h \leq d$ such that $\mathrm{Fil}^i D := \bigoplus_{j \geq n_i} Fe_j$, and $\varphi(e_1, \dots, e_d) = (e_1, \dots, e_d)XP$ where $X \in \mathrm{GL}_d(\Lambda)$ and P is a diagonal matrix whose ii-th entry is p^{s_i} where $s_i = \max\{j \mid n_j \leq i\} = \max\{j \mid e_i \in \mathrm{Fil}^j D\}$. Since $N = 0$ on D, we easily compute that $\mathrm{Fil}^h \mathcal{D}$ admits a basis $(e_1, \dots, e_d)B$ where B is the diagonal matrix with (i,i)-th entry is E^{h-s_i} (cf. Section 3.1 below). By the steps outlined in Theorem 2.7 , using the basis $1 \otimes e_i \in \mathcal{D}$ we see the matrix of φ on $\mathcal{M}$ is given by $A = E^h B^{-1} XP\varphi(B)p^{-h}\mathfrak{c}^{-h}$, where $A = \Lambda XC$, and Λ is a diagonal matrix with (i,i)-th entry is E^{s_i} and C is a diagonal matrix with (i,i)-th entry is $\mathfrak{c}^{-s_i}$.

3. An explicit determination of a Breuil module

In this section, we assume $K = \mathbb{Q}_p$. We choose $\pi = -p$, so $E(u) = u + p$. We keep $F/\mathbb{Q}_p$ as a linear coefficient field and recall Λ is its ring of integers. In Section 3.2, we explain the definition of the filtered (φ, N)-module $D_{h+1,\mathcal{L}} \in \mathrm{MF}_F^{\varphi,N}$, for $h \geq 1$ and $\mathcal{L} \in F$, discussed in the introduction. Let $\mathcal{M}_{h+1,\mathcal{L}} = \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{h+1,\mathcal{L}}) \otimes_{\mathcal{O}_F} S_F \in \mathrm{Mod}_{S_F}^{\varphi,\leq h}$. The ultimate goal (Theorem 3.7) is to describe the matrix of φ in a certain trivialization $\mathcal{M}_{h+1,\mathcal{L}} \cong S_F^{\oplus 2}$, at least if $\mathcal{L} \neq 0$. We begin by describing the Breuil module $\mathcal{D}_{h+1,\mathcal{L}} = \underline{\mathcal{D}}(D_{h+1,\mathcal{L}})$.

3.1. **The filtration on some rank 2 Breuil modules.** In order to minimize notation, in this subsection, we let $D \in \mathrm{MF}_F^{\varphi,N}$ be any 2-dimensional filtered (φ, N)-module with Hodge–Tate weights $0 < h$. We also choose any basis (f_1, f_2) for D such that $\mathrm{Fil}^h D = F f_2$. We write $N_D(f_1, f_2) = (f_1, f_2)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Mat}_2(F)$. (Compare with Lemma 3.6.)

Set $\mathcal{D} = \underline{\mathcal{D}}(D) = S_F \otimes_F D$. For $f \in D$ we write $\widehat{f} = 1 \otimes f \in \mathcal{D}$. In particular, $\mathcal{D}$ is a free S_F-module with basis $(\widehat{f}_1, \widehat{f}_2)$. Recall that $\mathrm{Fil}^i \mathcal{D}$ is defined by $\mathrm{Fil}^0 \mathcal{D} = \mathcal{D}$ and, for $i \geq 1$,

$$\mathrm{Fil}^i \mathcal{D} = \{x \in \mathcal{D} \mid N_{\mathcal{D}}(x) \in \mathrm{Fil}^{i-1} \mathcal{D} \text{ and } \mathrm{ev}_\pi(x) \in \mathrm{Fil}^i D\}.$$

When $i = 1$, the condition $N_{\mathcal{D}}(x) \in \mathrm{Fil}^0 \mathcal{D} = \mathcal{D}$ is a tautology. So, $\mathrm{Fil}^1 \mathcal{D} = S_F \widehat{f}_2 + S_F E \widehat{f}_1$.

Proposition 3.1. *There exists $x_1, \ldots, x_{h-1} \in F$ such that, if $0 \leq i \leq h$, then*

$$\mathrm{Fil}^i \mathcal{D} = S_F \cdot \big(\widehat{f}_2 + (\sum_{j=1}^{i-1} x_j E^j)\widehat{f}_1\big) + S_F \cdot E^i \widehat{f}_1.$$

Proof. Assume by induction on $0 \leq i < h$, that there exists $x_1, \ldots, x_{i-1} \in F$ such that for each $0 \leq j \leq i$ we have $\mathrm{Fil}^j \mathcal{D} = S_F \cdot \widehat{f}_2^{(j)} + S_F \cdot \widehat{f}_1$, where $\widehat{f}_2^{(j)} = \widehat{f}_2 + (\sum_{m=1}^{j-1} x_m E^m)\widehat{f}_1$. Setting $\widehat{f}_2^{(0)} = \widehat{f}_2^{(1)} = \widehat{f}_2$ handles the case $i = 0$ and $i = 1$. So, suppose $1 \leq i < h$.

For the $(i+1)$-st case, we first define $x_i \in F$. By induction, $N_{\mathcal{D}}(\widehat{f}_2^{(i)}) \in \mathrm{Fil}^{i-1} \mathcal{D} = S_F \widehat{f}_2^{(i-1)} + S_F E^{i-1} \widehat{f}_1$. Since $\widehat{f}_2^{(i-1)} = \widehat{f}_2^{(i)} - x_{i-1} E^{i-1} \widehat{f}_1$, we can write

$$N_{\mathcal{D}}(\widehat{f}_2^{(i)}) = d_i \widehat{f}_2^{(i)} + b_i E^{i-1} \widehat{f}_1$$

for some $d_i, b_i \in S_F$ (cf. Lemma 3.2 below). Set $x_i = b_i(\pi)/i\pi$, and then set $\widehat{f}_2^{(i+1)} = \widehat{f}_2^{(i)} + x_i E^i \widehat{f}_1$. Since $2 \leq i+1 \leq h$, we have $\mathrm{Fil}^{i+1} D = F f_2$. Thus, $\mathrm{ev}_\pi(\widehat{f}_2^{(i+1)}) = \widehat{f}_2 \in \mathrm{Fil}^{i+1} D$. Further,

$$\begin{aligned} N_{\mathcal{D}}(\widehat{f}_2^{(i+1)}) &= N_{\mathcal{D}}(\widehat{f}_2^{(i)}) - x_i i u E^{i-1} \widehat{f}_1 + x_i E^i N_{\mathcal{D}}(\widehat{f}_1) \\ &= d_i \widehat{f}_2^{(i)} + (b_i - x_i i u) E^{i-1} \widehat{f}_1 + x_i E^i N_{\mathcal{D}}(\widehat{f}_1). \end{aligned} \tag{3.1}$$

Note, the last summand in (3.1) lies in $\mathrm{Fil}^i S_F \cdot \mathcal{D} \subseteq \mathrm{Fil}^i \mathcal{D}$, while the first lies in $\mathrm{Fil}^i \mathcal{D}$. By definition we have $\mathrm{ev}_\pi(b_i - x_i i u) = 0$ and so the the middle summand also lies in $\mathrm{Fil}^i S_F \cdot \mathcal{D} \subseteq \mathrm{Fil}^i \mathcal{D}$. Thus $\widehat{f}_2^{(i+1)} \in \mathrm{Fil}^{i+1} \mathcal{D}$.

For a moment, define $F^{i+1}\mathcal{D} = S_F \widehat{f}_2^{(i+1)} + S_F E^{i+1} \widehat{f}_1 \subseteq \mathrm{Fil}^{i+1} \mathcal{D}$. We want to show equality. Since $E \widehat{f}_2^{(i)} = E \widehat{f}_2^{(i+1)} - x_i E^{i+1} \widehat{f}_1$, we in fact have

$$E \,\mathrm{Fil}^i \mathcal{D} \subseteq F^{i+1}\mathcal{D} \subseteq \mathrm{Fil}^{i+1} \mathcal{D}.$$

Since ev_π gives an isomorphism $\mathrm{Fil}^{i+1} \mathcal{D}/E \,\mathrm{Fil}^i \mathcal{D} \cong F f_2$ by Remark 2.8, and $\mathrm{ev}_\pi(F^{i+1}\mathcal{D}) \neq 0$, we conclude the natural map $F^{i+1}\mathcal{D}/E \,\mathrm{Fil}^i \mathcal{D} \to \mathrm{Fil}^{i+1} \mathcal{D}/E \,\mathrm{Fil}^i \mathcal{D}$ is an isomorphism. Thus, $F^{i+1}\mathcal{D} = \mathrm{Fil}^{i+1} \mathcal{D}$. ∎

The proof of Proposition 3.1 allows for explicit control of the scalars x_j in terms of the monodromy matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. For the next two results, we explain this by re-examining the proof.

Lemma 3.2. *For $1 \leq i \leq h-1$, let $d_i, b_i \in S_F$ be such that $N_{\mathcal{D}}(\widehat{f}_2^{(i)}) = d_i \widehat{f}_2^{(i)} + b_i E^{i-1}\widehat{f}_1$. Then, $d_1 = d$, $b_1 = b$, $x_1 = \frac{b}{\pi}$ and for $1 \leq i < h-1$*

$$\begin{aligned} d_{i+1} &= d_i + cx_iE^i \\ b_{i+1} &= x_i(a - cz_i - d_i) + (b_i - x_i iu)/E \\ x_{i+1} &= \frac{b_{i+1}(\pi)}{(i+1)\pi} \end{aligned}$$

where $z_i = \sum_{j=1}^{i} x_j E^j$.

Proof. The values of d_1, b_1, and x_1 follow immediately from $\widehat{f}_2^{(1)} = \widehat{f}_2$ and $N_{\mathcal{D}}(\widehat{f}_2) = b\widehat{f}_1 + d\widehat{f}_2$. Next, by (3.1) and because $N_{\mathcal{D}}(\widehat{f}_1) = a\widehat{f}_1 + c\widehat{f}_2$, we have

$$N_{\mathcal{D}}(\widehat{f}_2^{(i+1)}) = d_i\widehat{f}_2^{(i)} + (b_i - x_i iu)E^{i-1}\widehat{f}_1 + x_iE^i(a\widehat{f}_1 + c\widehat{f}_2). \tag{3.2}$$

We can write $\widehat{f}_2^{(i)} = \widehat{f}_2^{(i+1)} - x_iE^i\widehat{f}_1$ and, separately, $\widehat{f}_2 = \widehat{f}_2^{(i+1)} - z_i\widehat{f}_1$. Thus (3.2) becomes

$$N_{\mathcal{D}}(\widehat{f}_2^{(i+1)}) = (d_i + cx_iE^i)\widehat{f}_2^{(i+1)} + \big(-d_ix_iE^i + (b_i - x_i iu)E^{i-1} + x_iE^i(a - cz_i)\big)\widehat{f}_1.$$

Factoring E^i out of the $\widehat{f}_1$-coefficient, the result is clear. □

Example 3.3. Below, in Lemma 4.4, we will need an explicit calculation of the x_i and z. This can be done using the recursive formulas above. The calculations we need, both of which are straightforward, are:

$$\begin{aligned} x_2 &= \frac{b}{2\pi^2}(a - d - 1) \\ z_2(0) &= \frac{b}{2}(a - d - 3). \end{aligned}$$

(See Example 3.9, also.)

Lemma 3.4. *Assume that $a - d \in \Lambda$ and $bc \in \Lambda$. Then, for $1 \leq i \leq h-1$, we have*

$$v_p(x_i) + v_p(i!) + i \geq v_p(b).$$

Remark 3.5. The lemma is consistent with $b = 0$ since $x_i = 0$, for all i, in that case.

Proof of Lemma 3.4. Given $v \in \mathbb{R}$ we write

$$A_v = \{\sum_{j\geq 0} y_jE^j \in F[u] \mid v_p(y_j) + v_p(j!) + j \geq v\}.$$

Note that A_v is a subgroup of $F[u]$. Since $v_p((j+k)!) \geq v_p(j!) + v_p(k!)$ for all non-negative integers j, k (because binomial coefficients are integers), we have $A_vA_w \subseteq A_{v+w}$, as well. In particular, A_0 is a ring containing Λ as a subring and each A_v is an A_0-module.

The lemma is equivalent to $x_iE^i \in A_{v_p(b)}$ for all $1 \leq i \leq h-1$, but to show $x_iE^i \in A_{v_p(b)}$ it suffices to show $b_iE^{i-1} \in A_{v_p(b)}$. Indeed, $b_iE^{i-1} \in b_i(\pi)E^{i-1} + E^iF[u]$, and so if $b_iE^{i-1} \in A_v$ (for any v) then $v_p(b_i(\pi)) + v_p((i-1)!) + i - 1 \geq v$. Since $b_i(\pi) = x_i i\pi$, by definition, we would clearly have $v_p(x_i) + v_p(i!) + i \geq v$ as well.

We have reduced to showing $b_iE^{i-1} \in A_{v_p(b)}$ for $1 \leq i \leq h-1$. For $i=1$, by Lemma 3.2, we have $b_1 = b$ and so the claim is clear. Now assume that $b_jE^{j-1} \in A_{v_p(b)}$ for all $j \leq i$. By the previous paragraph we have $x_jE^j \in A_{v_p(b)}$ for all $j \leq i$, and so $z_j \in A_{v_p(b)}$ for all $j \leq i$ (including z_0, which we define to be 0). By Lemma 3.2, we have

$$\begin{aligned} b_{i+1}E^i &= (a - cz_i - d_i)x_iE^i + (b_i - x_iiu)E^{i-1} \\ &= (a - d - c(z_i + z_{i-1}))x_iE^i + b_iE^{i-1} - x_ii\pi E^{i-1} - x_iiE^i. \end{aligned} \tag{3.3}$$

It is clear by induction that the final three summands are in $A_{v_p(b)}$. For the first summand, we know $z_i + z_{i-1} \in A_{v_p(b)}$. Since $v_p(c) + v_p(b) \geq 0$ and $a - d \in \Lambda$, we see $a - d - c(z_i + z_{i-1}) \in A_0$. Since $x_iE^i \in A_{v_p(b)}$, by induction, the first summand also lies in $A_{v_p(b)}$. Thus, $b_{i+1}E^i \in A_{v_p(b)}$. □

3.2. **Explicit filtered (φ, N)-modules.** Now assume F contains an element ϖ such that $\varpi^2 = p$. For $\mathcal{L} \in F$ and $h \geq 1$, we define $D_{h+1,\mathcal{L}} = Fe_1 \oplus Fe_2 \in \mathrm{MF}_F^{\varphi,N}$ where, in the basis (e_1, e_2),

$$\varphi = \begin{pmatrix} \varpi^{h+1} & 0 \\ 0 & \varpi^{h-1} \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \mathrm{Fil}^i D_{h+1,\mathcal{L}} = \begin{cases} D_{h+1,\mathcal{L}} & \text{if } i \leq 0; \\ F \cdot (e_1 + \mathcal{L}e_2) & \text{if } 1 \leq i \leq h; \\ \{0\} & \text{if } h < i. \end{cases}$$

See [7, Exemple 3.1.2.2(iv)]. It is useful make a change of a basis. Set $a_p = \varpi^{h-1} + \varpi^{h+1}$.

Lemma 3.6. *If $\mathcal{L} \neq 0$, then $(f_1, f_2) = (-\varphi(e_1 + \mathcal{L}e_2), e_1 + \mathcal{L}e_2)$ is a basis of $D_{h+1,\mathcal{L}}$ in which*

$$\varphi = \begin{pmatrix} a_p & -1 \\ p^h & 0 \end{pmatrix} \qquad N = \frac{p}{\mathcal{L}(1-p)} \begin{pmatrix} 1 & \varpi^{-h-1} \\ \varpi^{h+1} & -1 \end{pmatrix} \qquad \mathrm{Fil}^i D_{h+1,\mathcal{L}} = \begin{cases} D_{h+1,\mathcal{L}} & \textit{if } i \leq 0; \\ Ff_2 & \textit{if } 1 \leq i \leq h; \\ \{0\} & \textit{if } h < i. \end{cases}$$

Proof. If $\mathcal{L} \neq 0$, then $e_1 + \mathcal{L}e_2$ is *not* an eigenvector of φ, so (f_1, f_2) is a basis. We leave calculating the matrices for the reader. □

Now let $\mathcal{D}_{h+1,\mathcal{L}} = \underline{\mathcal{D}}(D_{h+1,\mathcal{L}})$ and $\mathcal{M}_{h+1,\mathcal{L}} = \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{h+1,\mathcal{L}}) \otimes_{\mathcal{O}_F} S_F \in \mathrm{Mod}_{S_F}^{\varphi,\leq h}$. Recall that $\mathfrak{c} = \varphi(E)/p \in S_F^\times$. Let $\lambda_- = \prod_{n\geq 0} \varphi^{2n+1}(E)/p$ and $\lambda_{++} = \varphi(\lambda_-)$.

Theorem 3.7. *If $\mathcal{L} \neq 0$, there exists a basis of $\mathcal{M}_{h+1,\mathcal{L}}$ in which the matrix of φ is given by*

$$A = \begin{pmatrix} (a_p - p^h z)\left(\frac{\lambda_-}{\lambda_{++}}\right)^h & -1 + \varphi(z)(a_p - p^h z) \\ E^h & E^h\varphi(z)\left(\frac{\lambda_{++}}{\lambda_-}\right)^h \end{pmatrix},$$

where $z = \sum_{j=1}^{h-1} x_jE^j \in F[E]$. Moreover, if $v_p(\mathcal{L}^{-1}) \geq -1$, then

$$v_p(x_j) \geq v_p(\mathcal{L}^{-1}) - \frac{h-1}{2} - v_p(j!) - j \tag{3.4}$$

for each $1 \leq j \leq h-1$.

Proof. Let (f_1, f_2) be the basis as in Lemma 3.6. Set $\widehat{f}_1 = 1 \otimes f_1$ and $\widehat{f}_2 = 1 \otimes f_2$, elements of $\mathcal{D}_{h+1,\mathcal{L}}$, as before. Then, the matrix of φ in the basis $(\widehat{f}_1, \widehat{f}_2)$ of $\mathcal{D}_{h+1,\mathcal{L}}$ is $X = \left(\begin{smallmatrix} a_p & -1 \\ p^h & 0 \end{smallmatrix}\right)$. Moreover, Proposition 3.1 implies that $\mathrm{Fil}^h \mathcal{D}_{h+1,\mathcal{L}} = S_F\alpha_1 \oplus S_F\alpha_2$, where

$$(\alpha_1, \alpha_2) = (\widehat{f}_1, \widehat{f}_2)\begin{pmatrix} E^h & z \\ 0 & 1 \end{pmatrix} =: (\widehat{f}_1, \widehat{f}_2)B$$

for $z = \sum_{j=1}^{h-1} x_j E^j$ and some $x_j \in F$. Theorem 2.7 implies that $\mathcal{M}_{h+1,\mathcal{L}}$ has a basis in which the matrix of φ is given by

$$A' = E^h B^{-1} X \varphi(B) p^{-h}\mathfrak{c}^{-h} = \begin{pmatrix} a - p^h z & p^{-h}\mathfrak{c}^{-h}\left(-1 + \varphi(z)(a_p - p^h z)\right) \\ E^h p^h & p^{-h}\mathfrak{c}^{-h} E^h p^h \varphi(z) \end{pmatrix}. \tag{3.5}$$

Since λ_- and λ_{++} are units in S_F, we can replace A' by $CA'\varphi(C^{-1})$ for $C = \begin{pmatrix} p^h\lambda_-^h & 0 \\ 0 & \lambda_{++}^h \end{pmatrix}$. A short calculation shows $A = CA'\varphi(C^{-1})$, completing the general proof.

Finally, if $v_p(\mathcal{L}^{-1}) \geq -1$, then the matrix of N in Lemma 3.6 satisfies the hypotheses of Lemma 3.4. So, the estimates (3.4) follow from the b-entry of the monodromy matrix being

$$b = \frac{-p}{\varpi^{h+1}\mathcal{L}(1-p)} = \frac{-1}{\varpi^{h-1}\mathcal{L}(1-p)}.$$

This completes the proof. □

Remark 3.8. An analogous calculation in the crystalline case, where $z = 0$ (see Remark 3.5), was made in [3, Section 3]. The technique here, passing through the category $\mathrm{MF}_{S_F}^{\varphi,N}$, is different than *loc. cit.* The descriptions are the same, though. Compare with Example 2.9.

Example 3.9. We need one ad hoc calculation in Lemma 4.4 below. Let $h = 3$. By Example 3.3, the element z in Theorem 3.7 satisfies $z(0) = \frac{b}{2}(a - d - 3)$ where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is the monodromy matrix in Lemma 3.6. For $p = h = 3$, plugging in the explicit matrix, we see $z(0) = \frac{1}{4\mathcal{L}}\left(\frac{1}{\mathcal{L}} + 1\right)$.

4. Descent and reductions

The goal in this section is to prove the main theorem of this article. Given $h \geq 1$ and $\mathcal{L} \in F$ we write $V_{h+1,\mathcal{L}}$ for the unique two-dimensional representation of $G_{\mathbb{Q}_p}$ such that $D_{\mathrm{st}}^*(V_{h+1,\mathcal{L}}) \cong D_{h+1,\mathcal{L}}$ where $D_{h+1,\mathcal{L}}$ is as in Section 3.2. Write $\overline{V}$ for the semi-simple reduction modulo $\mathfrak{m}_F$ of V. Let $\mathbb{Q}_{p^2}$ be the unramified quadratic extension of $\mathbb{Q}_p$, χ the unramified quadratic character of $G_{\mathbb{Q}_{p^2}}$, and ω_2 a niveau 2 fundamental character of $\mathbb{Q}_{p^2}$. Note that $\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}(\omega_2^h\chi)$ has determinant ω^h, where ω is the cyclotomic character, and its restriction to inertia is $\omega_2^h \oplus \omega_2^{ph}$.

Theorem 4.1. *Assume $h \geq 3$ and $p \neq 2$. Then, if $\mathcal{L}$ satisfies*

$$v_p(\mathcal{L}^{-1}) > \frac{h-1}{2} - 1 + v_p((h-1)!),$$

then $\overline{V}_{h+1,\mathcal{L}} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}(\omega_2^h\chi)$.

Remark 4.2. Our contribution toward Theorem 4.1 is limited to $h \geq 4$ and $p = h = 3$. The case of $h = 3$ and $p \geq 5$ follows from the work of Breuil and Mézard. If we were to use the weaker bound $v_p(\mathcal{L}^{-1}) > \frac{h-1}{2} + v_p((h-1)!)$, then our calculation would also cover the cases of $h = 2$ and $h = 3$. See Remark 4.8 for further explanations.

We plan to take the matrix of φ acting on $\mathcal{M}_{h+1,\mathcal{L}} = \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{h+1,\mathcal{L}}) \otimes_{\mathcal{O}_F} S_F$ as in Theorem 3.7 and replace it with a φ-conjugate defined over $\mathfrak{S}_\Lambda$ when $v_p(\mathcal{L}^{-1})$ satisfies the bound in theorem. This defines a Kisin module $\mathfrak{M}$ for $V_{h+1,\mathcal{L}}$ that allows us to calculate the reduction $\overline{V}_{h+1,\mathcal{L}}$. Despite our theorem being limited to $h \geq 3$, we will present many calculations only assuming $h \geq 2$, in order to later justify Remark 4.2. So, we assume without further comment that:

$$p \neq 2 \text{ and } h \geq 2; \tag{4.1}$$
$$v_p(\mathcal{L}^{-1}) > \frac{h-1}{2} - 1 + v_p((h-1)!).$$

We will clarify result-by-result where we need to limit to $h \geq 3$ or $h \geq 4$. Also, fix $z = \sum_{j=1}^{h-1} x_j E^j$ as in Theorem 3.7. Note that by (4.1), we have $v_p(\mathcal{L}^{-1}) \geq -1$ so the estimates (3.4) in Theorem 3.7 hold.

4.1. **Preparing for descent.** Consider the ring

$$R_2 = \{f = \sum a_i u^i \in F[\![u]\!] \mid i + 2v_p(a_i) \to \infty \text{ as } i \to \infty\}.$$

Thus R_2 is the F-Banach algebra of series converging on $|u| \leq p^{-1/2}$. We equip R_2 with the valuation $v_{R_2}(\sum a_i u^i) = \inf_i\{i + 2v_p(a_i)\}$. The canonical map $\mathcal{O}_F \hookrightarrow R_2$ factors through S_F since $v_{R_2}(E^p/p) = p - 2 > 0$. Finally, given $v \in \mathbb{R}$, we define additive subgroups $H_v^\circ \subseteq H_v \subseteq R_2$ by

$$H_v = \{f \in R_2 \mid v_{R_2}(f) \geq v\}; \qquad H_v^\circ = \{f \in R_2 \mid v_{R_2}(f) > v\}.$$

For any v, H_v and H_v° are stable under φ. In fact, for any $j \geq 0$ we have that

$$\varphi(H_v \cap u^j R_2) \subseteq H_{v+j(p-1)} \cap u^{pj} R_2, \tag{4.2}$$

and the same for H_v° replacing H_v. See [3, Lemma 4.1.1], for instance.

Our first lemma, concerning some entries of the matrix in Theorem 3.7, is straightforward so we omit the proof (compare with [3, Lemma 5.1.1]).

Lemma 4.3. *Let $\lambda_- = \prod_{n\geq 0} \varphi^{2n+1}(E)/p$ and $\lambda_{++} = \varphi(\lambda_-)$ be as in Theorem 3.7. Then,*

- *(a) $\lambda_- \in 1 + H_{p-2}$ and $\lambda_{++} \in 1 + H_{p^2-2}$;*
- *(b) $\lambda_-, \lambda_{++} \in R_2^\times$;*
- *(c) $v_{R_2}(\lambda_-^{\pm 1}) = 0 = v_{R_2}(\lambda_{++}^{\pm 1})$.*

We also prepare estimates for z. Note that by (4.1) the estimate (3.4) becomes

$$v_p(x_j) > v_p((h-1)!) - v_p(j!) - j - 1 \geq -j - 1. \tag{4.3}$$

Recall, we write $a_p = \varpi^{h-1} + \varpi^{h+1}$. Thus, $v_p(a_p) = \frac{h-1}{2}$.

Lemma 4.4. *For $z = \sum_{j=1}^{h-1} x_j E^j$ as above, and $\nu = -1 + \varphi(z)(a_p - p^h z)$, we have*

(a) $p^h z \in H^\circ_{h-1}$;
(b) $\varphi(z) \in H^\circ_{-2}$;
(c) $\nu \in -1 + H^\circ_{h-3}$;
(d) *If* $h \geq 3$, *then* $\nu \in R_2^\times$.

Furthermore, if $p = 3$ *and* $h = 3$, *then* $\varphi(z) \in H^\circ_{-1}$ *and* $\nu \in -1 + H^\circ_{h-2} = -1 + H^\circ_1$.

Proof. First, $v_{R_2}(E^j) = j$. By the ultrametric inequality and (4.3), we see

$$v_{R_2}(z) > \inf\{2(-j-1) + j \mid 1 \leq j \leq h-1\} = -1 - h.$$

Part (a) follows because $v_{R_2}(p^h) = 2h$. For (b), note $v_{R_2}(\varphi(E)^j) = 2j$. Thus, using (4.3),

$$v_{R_2}(\varphi(z)) > \inf\{2(-j-1) + 2j \mid 1 \leq j \leq h-1\} = -2.$$

Continuing, $\varphi(z)p^h z \in H^\circ_{h-3}$ by parts (a) and (b) and, since $v_{R_2}(a_p) = h-1$, we have $\varphi(z)a_p \in H^\circ_{h-3}$. This proves (c). Finally, part (d) follows from the geometric series and part (c).

Finally, suppose $p = h = 3$. By the argument for (c) above, it suffices to show $\varphi(z) \in H^\circ_{-1}$. We note $v_{R_2}(\varphi(E)^j - E(0)^j) \geq p + 2j - 2$ for any j. Thus, by (4.3)

$$v_{R_2}(\varphi(z) - \varphi(z)(0)) > p + 2j - 2 - 2(j+1) = p - 4 = -1. \tag{4.4}$$

But, by Example 3.9 we have $\varphi(z)(0) = z(0) = \frac{1}{4\mathcal{L}}\left(\frac{1}{\mathcal{L}} + 1\right)$. Since $v_p(\mathcal{L}^{-1}) > 0$, (4.4) then implies $v_{R_2}(\varphi(z)) > -1$ as we wanted. $\square$

We now write $\mathcal{M}_2 = \mathcal{M}_{h+1,\mathcal{L}} \otimes_{S_F} R_2 \cong \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{h+1,\mathcal{L}}) \otimes_{\mathcal{O}_F} R_2$. Thus, $\mathcal{M}_2 \in \mathrm{Mod}^{\varphi,\leq h}_{R_2}$. We also introduce some notation. Given $A \in \mathrm{Mat}_d(R_2)$ and $C \in \mathrm{GL}_d(R_2)$ we write $C *_\varphi A = C \cdot A \cdot \varphi(C)^{-1}$. Thus, if (e_1, e_2) is a basis of $\mathcal{M}_2$ and A is the matrix of $\varphi_{\mathcal{M}_2}$ in that basis, then $C *_\varphi A$ is the matrix of $\varphi_{\mathcal{M}_2}$ in the basis (e_1', e_2') is given by $(e_1', e_2') = (e_1, e_2)C^{-1}$.

Proposition 4.5. *Assume* $h \geq 4$ *or* $p = h = 3$. *Then, there exists a basis of* $\mathcal{M}_2$ *in which the matrix of* $\varphi_{\mathcal{M}_2}$ *is* $\left(\begin{smallmatrix} G & -1 \\ E^h & 0 \end{smallmatrix}\right)$, *where* $G \in (a_p - p^h z)\left(\frac{\lambda_-}{\lambda_{++}}\right)^h + H^\circ_h$.

Proof. By Theorem 3.7, there is a basis (e_1, e_2) of $\mathcal{M}_2$ such that $\varphi_{\mathcal{M}_2}(e_1, e_2) = (e_1, e_2)A$, where

$$A = \begin{pmatrix} (a_p - p^h z)\left(\frac{\lambda_-}{\lambda_{++}}\right)^h & -1 + \varphi(z)(a_p - p^h z) \\ E^h & E^h \varphi(z)\left(\frac{\lambda_{++}}{\lambda_-}\right)^h \end{pmatrix} = \begin{pmatrix} \mu & \nu \\ E^h & \eta \end{pmatrix},$$

where ν is as in Lemma 4.4 and μ and η are defined by the equality. Assume for now just that $h \geq 3$. Then, by Lemma 4.4(d), $\nu \in R_2^\times$. Making a change of basis on $\mathcal{M}_2$, we replace A by (note that $\mu\eta = (1+\nu)E^h$)

$$A' = \begin{pmatrix} 1 & 0 \\ -\eta/\nu & 1 \end{pmatrix} *_\varphi A = \begin{pmatrix} \mu + \frac{\nu\varphi(\eta)}{\varphi(\nu)} & \nu \\ -E^h \nu^{-1} & 0 \end{pmatrix}.$$

Since $v_{R_2}(\nu + 1) > 0$ by Lemma 4.4(c), we have $\nu(0) \in \Lambda^\times$. Thus $\nu_0 = \nu/\nu(0) \in 1 + (H^\circ_{h-3} \cap uR_2)$. By (4.2), we have $\varphi^k(\nu_0) \in 1 + H_{h-3+m_k}$ where $m_k \to \infty$ as $k \to \infty$. Thus, the infinite product

$\nu_+ = \prod_{n\geq 0}\varphi^{2n}(\nu_0)$ converges in R_2. Set $\nu_- = \varphi(\nu_+)$, so $\nu_\pm \in 1 + H^\circ_{h-3} \subseteq R_2^\times$. We now change basis on $\mathcal{M}_2$ again to get a matrix A'' for $\varphi_{\mathcal{M}_2}$ given by

$$A'' = \begin{pmatrix} \frac{-1}{\nu(0)}\frac{\nu_-}{\nu_+} & 0 \\ 0 & \frac{\nu_+}{\nu_-} \end{pmatrix} *_\varphi A' = \begin{pmatrix} G & -1 \\ E^h & 0 \end{pmatrix},$$

where

$$G = \left(\mu + \frac{\nu\varphi(\eta)}{\varphi(\nu)}\right)\frac{\nu_-^2}{\nu_+\nu_{++}} \tag{4.5}$$

and $\nu_{++} = \varphi(\nu_-)$.

To complete the argument, we justify $G \in \mu + H^\circ_h$. We already know $\nu_-^2/\nu_+\nu_{++} \in 1 + H^\circ_{h-3}$. The same is true for $\nu/\varphi(\nu)$. So,

$$v_{R_2}\left(\frac{\nu\varphi(\eta)}{\varphi(\nu)}\right) \geq v_{R_2}(\varphi(\eta)) \geq v_{R_2}(\varphi(E)^h\varphi^2(z)), \tag{4.6}$$

where we used Lemma 4.3 to remove λ_- and λ_{++} from the estimate. We note $v_{R_2}(\varphi(E)^h) = 2h$ and $v_{R_2}(\varphi^2(z)) \geq v_{R_2}(\varphi(z)) > -2$, by (4.2) and Lemma 4.4(b). Thus from (4.6) we deduce that $v_{R_2}(\nu\varphi(\eta)/\varphi(\nu)) > 2h - 2 = 2(h-1)$. We also note that $a_p - p^h z \in H_{h-1}$. Thus, $\mu \in H_{h-1}$ and so, returning to the definition (4.5) of μ and G, we see

$$G \in \left(\mu + H^\circ_{2(h-1)}\right)\cdot(1 + H^\circ_{h-3}) \subseteq \mu + H^\circ_{2h-4} + H^\circ_{2(h-1)} = \mu + H^\circ_{2h-4}.$$

Now, if $h \geq 4$, then $2h-4 \geq h$ and so $G \in \mu + H^\circ_h$. This completes the proof except if $p = h = 3$. In that case, Lemma 4.4 shows $\nu \in -1 + H^\circ_1$, rather than $-1 + H^\circ_0$, from which we deduce

$$G \in \left(\mu + H^\circ_4\right)\cdot(1 + H^\circ_1) \subseteq \mu + H^\circ_3 = \mu + H^\circ_h$$

anyways. This completes the proof. ∎

4.2. **Descent.** To descend to $\mathfrak{S}_\Lambda$, we use the algorithm from [3, Section 4]. Write $T_{\leq d} : R_2 \to F[u]$ for the "truncation" operation $T_{\leq d}(\sum a_i u^i) = \sum_{i\leq d} a_i u^i$ and $T_{>d}(f) = f - T_{\leq d}(f)$. In the next two proofs, we will use the following principle: if $f \in R_2$ and $v_{R_2}(T_{\leq d}(f)) > d$ (for instance, if $v_{R_2}(f) > d$) then $T_{\leq d}(f) \in \mathfrak{m}_F[u]$.

Proposition 4.6. *Suppose that $G \in R_2$ such that*

(a) $G \in H_{h-1}$;
(b) $T_{>h}(G) \in H^\circ_{h-1}$;
(c) $T_{\leq h}(G) \in \mathfrak{m}_F[u]$.

Then, given $A = \left(\begin{smallmatrix} G & -1 \\ E^h & 0 \end{smallmatrix}\right)$, there exists $C \in \mathrm{GL}_2(R_2)$ and $P \in \mathfrak{m}_F[u]$ such that $C *_\varphi A = \left(\begin{smallmatrix} P & -1 \\ E^h & 0 \end{smallmatrix}\right)$.

Proof. Since $E^h \in u^h + H_{h+1}$, the assumption (a) implies that

$$A \in \begin{pmatrix} 0 & -1 \\ u^h & 0 \end{pmatrix} + \begin{pmatrix} H_{h-1} & 0 \\ H_{h+1} & 0 \end{pmatrix}.$$

In the notation of [3, Section 4.3], set $a = 0$, $b = h$, $a' = \frac{h}{2} - \frac{p-1}{2}$ and $b' = \frac{h}{2} + \frac{p-1}{2}$, and $(c_0, c_h) = (-1, 1)$. Since $h - 1 - a' = \frac{h}{2} - 1 + \frac{p-1}{2} \geq 1$, we see A is γ-allowable with $\gamma = 1$ in the sense of [3, Definition 4.3.1]. The error of A, in the same definition, is $\varepsilon = v_{R_2}(T_{>h}(G)) - a'$. By [3, Theorem 4.3.7], with $R = R_2$ in *loc. cit.*, there exists $C \in \mathrm{GL}_2(R_2)$ such that $A' = C *_{\varphi} A$ satisfies:

(i) Evaluating at $u = 0$, we have $A'|_{u=0} = A|_{u=0}$.
(ii) The matrix A' is of the form $A' = \begin{pmatrix} P & -1 \\ f & 0 \end{pmatrix}$ with P and f polynomials of degree at most h.
(iii) We have an estimate $v_{R_2}(P - T_{\leq h}(G)) \geq \varepsilon + a' + 1$.

(For the reader checking references, note that the role of A versus C is reversed in [3].)

We claim $P \in \mathfrak{m}_F[u]$ and $f = E^h$, which would finish the proof of the proposition. To see $P \in \mathfrak{m}_F[u]$, we start by combining the estimate (iii) and the assumption (b) in order to see that

$$v_{R_2}(P - T_{\leq h}(G)) \geq \varepsilon + a' + 1 = v_{R_2}(T_{>h}(G)) + 1 > h.$$

On the other hand, $P - T_{\leq h}(G)$ has degree at most h by (ii) and so $P - T_{\leq h}(G) \in \mathfrak{m}_F[u]$, which implies $P \in \mathfrak{m}_F[u]$ by assumption (c).

To see $f = E^h$, we evidently have $f = \det(A') = rE^h$ for some $r \in R_2^{\times}$. In particular, f has a root of multiplicity h at $u = -p$. But, f is a polynomial of degree at most h by point (ii), and by point (i) we have $f(0) = E(0)^h$. It now follows quickly that $f = E^h$, since $F[\![u]\!]$ is a unique factorization domain. $\square$

We now verify the G from Proposition 4.5 satisfies the hypothesis of Proposition 4.6.

Lemma 4.7. *Let* $G \in (a_p - p^h z)\left(\frac{\lambda_-}{\lambda_{++}}\right)^h + H_h^{\circ}$. *Then,*

(a) $G \in H_{h-1}$,
(b) $T_{>h}(G) \in H_{h-1}^{\circ}$, *and*
(c) $T_{\leq h}(G) \in \mathfrak{m}_F[u]$.

Proof. First, the conclusions depend only on $G \bmod H_h^{\circ}$, so we suppose $G = (a_p - p^h z)\left(\frac{\lambda_-}{\lambda_{++}}\right)^h$. Part (a) follows from Lemmas 4.3 and 4.4. For part (b), we first have, by Lemma 4.3(a), that $a_p\left(\frac{\lambda_-}{\lambda_{++}}\right)^h \in a_p + a_p H_{p-2}$. So, $T_{>0}\big(a_p\left(\frac{\lambda_-}{\lambda_{++}}\right)^h\big) \in H_{h+p-3} \subseteq H_h$. On the other hand, by Lemma 4.4(a) we have $p^h z \in H_{h-1}^{\circ}$. Thus we've shown in fact $T_{>0}(G) \in H_{h-1}^{\circ}$.

Finally, we consider part (c). Since $E = u + p$, any $f \in S_{\Lambda}$ can be written $f = \sum_{n=0}^{\infty} \alpha_n \frac{u^n}{p^{\lfloor \frac{n}{p} \rfloor}}$ with $\alpha_n \in \Lambda$. Let $f = \frac{\lambda_-}{\lambda_{++}} \in S_{\Lambda}$, in particular. Since $v_p(a_p) = \frac{h-1}{2} > \lfloor \frac{h}{p} \rfloor$ unless $p = h = 3$ (or $p = 2$, which we have excluded in (4.1)), we see immediately that $T_{\leq h}(a_p f^h) \in \mathfrak{m}_F[u]$ except when $h = p = 3$. When $h = p$, however,

$$T_{\leq p}(f^p) = T_{\leq p}\left(\left(\sum_{n=0}^{p-1} \alpha_n u^n + \alpha_p \frac{u^p}{p}\right)^p\right) \in p \cdot \alpha_0^{p-1} \alpha_p \frac{u^p}{p} + \Lambda[u] \subseteq \Lambda[u].$$

Since $v_p(a_p) > 0$, we see $T_{\leq h}(a_p f^h) \in \mathfrak{m}_F[u]$ in every case.

By the prior paragraph, to show (c) it remains to show that $T_{\leq h}(p^h z f^h) \in \mathfrak{m}_F[u]$ as well. By definition, we can write $f^h = \sum_{i=0}^{\infty} \beta_i \frac{E^i}{p^{\lfloor \frac{i}{p} \rfloor}}$ with $\beta_i \in \Lambda$ and recall $z = \sum_{j=1}^{h-1} x_j E^j$. Thus,

$$p^h z f^h = \sum_{n=1}^{\infty} \left(\sum_{i+j=n} p^h x_j \beta_i p^{-\lfloor \frac{i}{p} \rfloor} \right) E^n. \tag{4.7}$$

Using the binomial expansion of $E^n = (u+p)^n$ we see that the u^m-term of (4.7) is exactly equal to

$$\sum_{n=m}^{\infty} \left(\sum_{i+j=n} p^h x_j \beta_i p^{-\lfloor \frac{i}{p} \rfloor} \right) \binom{n}{m} p^{n-m}.$$

We must show this has positive p-adic valuation for $m \leq h$. Since $\beta_i \in \Lambda$ and binomial coefficients are integers, it is enough to show that for all $m \leq h$ and $j < h$, if $n \geq m, j$ then

$$v_p(x_j) + h + n - m - \lfloor \frac{n-j}{p} \rfloor > 0. \tag{4.8}$$

By (4.3) we have $v_p(x_j) > -j - 1$ and so

$$v_p(x_j) + h + n - m - \lfloor \frac{n-j}{p} \rfloor > h - m - 1 + n - j - \lfloor \frac{n-j}{p} \rfloor. \tag{4.9}$$

But, the right-hand side of (4.9) is non-negative. Indeed, when $h > m$, this is clear because $n \geq j$. When $h = m$, on the other hand, we have $n \geq m = h > j$. So, the right-hand side of (4.9) in that case has the form $x - \lfloor x/p \rfloor - 1$ with $x \geq 1$, which is also non-negative. $\square$

4.3. **Proof of Theorem 4.1.** Finally, we give the proof of the main theorem:

Assume that $h \geq 3$ and $p \neq 2$. Then, if $\mathcal{L}$ satisfies

$$v_p(\mathcal{L}^{-1}) > \frac{h-1}{2} - 1 + v_p((h-1)!),$$

then $\overline{V}_{h+1,\mathcal{L}} \cong \mathrm{Ind}_{G_{\mathbb{Q}_p^2}}^{G_{\mathbb{Q}_p}}(\omega_2^h \chi)$.

Proof of Theorem 4.1. First, if $h = 3$ and $p \geq 5$, then the assumption is that $v_p(\mathcal{L}) < 0$. The verification that $\overline{V}_{4,\mathcal{L}} \cong \mathrm{Ind}_{G_{\mathbb{Q}_p^2}}^{G_{\mathbb{Q}_p}}(\omega_2^3 \chi)$ is the first bullet point of [7, Theorem 4.2.4.7(iii)], where the reader should take $k = 4 < p$ and $\ell = v_p(\mathcal{L}) < 0$.

Now we assume that either $h \geq 4$ or $p = h = 3$. Then, applying Proposition 4.5, Lemma 4.7, and Proposition 4.6, we deduce that there exists a basis of $\mathcal{M}_2$ in which the matrix of $\varphi_{\mathcal{M}_2}$ is given by $A = \left(\begin{smallmatrix} P & -1 \\ E^h & 0 \end{smallmatrix}\right)$ and $P \in \mathfrak{m}_F[u]$. Define $\mathfrak{M} = \mathfrak{S}_\Lambda^{\oplus 2}$ with the matrix of φ being given by A. Clearly $\mathfrak{M}$ is a Kisin module over $\mathfrak{S}_\Lambda$ of height $\leq h$, and

$$\mathfrak{M} \otimes_{\mathfrak{S}_\Lambda} R_2 \cong \mathcal{M}_2 = \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{h+1,\mathcal{L}}) \otimes_{\mathcal{O}_F} R_2$$

as φ-modules over R_2. Thus, by Proposition 2.1 we deduce $\mathfrak{M} = \mathfrak{M}(T)$ for some lattice $T \subseteq V_{h+1,\mathcal{L}}$. Furthermore, $\mathfrak{M} \otimes_{\mathfrak{S}_\Lambda} \mathbb{F}[u^{-1}]$ is a φ-module over $\mathbb{F}(\!(u)\!)$ with Frobenius given by $\left(\begin{smallmatrix} 0 & -1 \\ u^h & 0 \end{smallmatrix}\right)$. This shows, in particular, that $\overline{V}_{h+1,\mathcal{L}}$ is the same for any $\mathcal{L}$ satisfying (4.1) (see [3, Corollary 2.3.2]).

Let $V_{h+1,\infty}$ be as in the introduction. By [3, Corollary 5.2.2], for $V_{h+1,\infty}$ there exists a Kisin module $\mathfrak{M}'$ such that $M' := \mathfrak{M}' \otimes_{\mathfrak{S}_\Lambda} \mathbb{F}[u^{-1}]$ has Frobenius also given by $\left(\begin{smallmatrix} 0 & -1 \\ u^h & 0 \end{smallmatrix}\right)$ and M' determines $\overline{V}_{h+1,\infty} \cong \mathrm{Ind}_{G_{\mathbb{Q}_p^2}}^{G_{\mathbb{Q}_p}}(\omega_2^h\chi)$. Therefore, $\overline{V}_{h+1,\mathcal{L}} \cong \overline{V}_{h+1,\infty} \cong \mathrm{Ind}_{G_{\mathbb{Q}_p^2}}^{G_{\mathbb{Q}_p}}(\omega_2^h\chi)$. □

Remark 4.8. We return to Remark 4.2. Suppose we replace (4.1) with

$$v_p(\mathcal{L}^{-1}) > \frac{h-1}{2} + v_p((h-1)!). \tag{4.10}$$

This has the impact of scaling z by a p-adic unit multiple of p, thus increasing $v_{R_2}(z)$ by 2 throughout our estimates in Section 4.1. The reader may check that Proposition 4.5 holds with these new estimates, and so the proof goes through for all $h \geq 2$ and $p \geq 3$ under the assumption (4.10). Of course, this bound is not the sharpest possible when $h = 2$ or $h = 3$. For instance, we've already noted that for $h = 3$ and $p \geq 5$, Breuil and Mézard confirmed Theorem 4.1 with the stronger bound (4.1).

The situation is more complicated when $h = 2$. In that case, for $p \geq 5$, Guerberoff and Park showed that $\overline{V}_{3,\mathcal{L}} \cong \mathrm{Ind}_{G_{\mathbb{Q}_p^2}}^{G_{\mathbb{Q}_p}}(\omega_2^2\chi)$ exactly on $v_p(\mathcal{L} - 1) < \frac{1}{2}$ (see [17, Theorem 5.0.5]). Thus, the bound $v_p(\mathcal{L}) < \frac{1}{2}$ from Theorem 4.1 produces too large a region of $\mathcal{L}$-invariants, whereas (4.10) produces a region too small. For the interested reader, Guerberoff and Park also determined, for any $\mathcal{L}$, the restriction of $\overline{V}_{3,\mathcal{L}}$ to the inertia subgroup. The restriction to inertia was recently removed by Chitrao, Ghate, and Yasuda using a completely different method. See [10, Theorem 1.3]. Thus we have a complete picture of $\overline{V}_{3,\mathcal{L}}$. It would be amusing to understand if that picture can be recovered from the method here.

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