WEIGHT ELIMINATION IN SERRE-TYPE CONJECTURES

DANIEL LE, BAO V. LE HUNG, and BRANDON LEVIN

Abstract

We prove the weight elimination direction of the Serre weight conjectures as formulated by Herzig for forms of $U(n)$ which are compact at infinity and split at places dividing p in generic situations. That is, we show that all modular weights for a mod p Galois representation are contained in the set predicted by Herzig. Under some additional hypotheses, we also show modularity of all the "obvious" weights.

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1. Introduction

Let p be a prime. In 1973, Serre conjectured that every irreducible odd 2-dimensional $\overline{\mathbb{F}}_p$-representation $\overline{r}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ comes from a modular form. He later refined the conjecture into the strong form which asserts that every such $\overline{r}$ arises from a modular form of a specific minimal weight and prime-to-p level determined by the local properties of $\overline{r}$ (see [38]). The recipe for the minimal weight is more subtle than the minimal level and, as Serre suggested at the time, reflects the deeper structures of a "mod p Langlands philosophy." The landmark proof of Serre's original conjecture due to Kisin and Khare–Wintenberger relies crucially on knowing that the weak form implies the strong form.

The first comprehensive conjecture for Hilbert modular forms is due to Buzzard–Diamond–Jarvis (BDJ) in [5]. The weight $k \geq 2$ is replaced by the notion of *Serre weights*, irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ in this case. Furthermore, there is no

DUKE MATHEMATICAL JOURNAL
Vol. 168, No. 13, © 2019 DOI 10.1215/00127094-2019-0015
Received 17 February 2017. Revision received 17 January 2019.
First published online 10 September 2019.
2010 *Mathematics Subject Classification*. Primary 11F80; Secondary 11F33, 11F75, 20C33.

longer a notion of minimal weight; rather, BDJ define a collection of Serre weights for which a given $\overline{r}$ should be modular. The weight part (weak $\implies$ strong) of the BDJ conjecture is now a theorem due to the work of many people (see [16]–[18], [36]).

Building on the work of Ash–Doud–Pollack in [1] and others, Herzig in [23] formulated a vast generalization of the weight part of Serre's conjecture to tame n-dimensional Galois representations which was further extended by Gee–Herzig–Savitt in [15]. In our earlier work [31] and [32] with S. Morra, we establish the weak implies strong conjecture for tame 3-dimensional Galois representations (and for definite unitary groups unramified at p). However, results for $n > 3$ were limited to a few partial results (see [2], [13], [14]). In this paper, we establish the weight elimination direction of the weight part of Serre's conjecture for n-dimensional Galois representations in generic situations, namely, the set of modular weights is a subset of the set of weights predicted by [23].

To describe these conjectures, let F be an imaginary CM number field unramified at p. Let F^+ be the maximal totally real subfield. Assume that $F^+ \neq \mathbb{Q}$, and assume that all primes of F^+ above p split in F. Let G be a unitary group over F^+ which is isomorphic to $U(n)$ at each infinite place and split at each prime above p. Let $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be an irreducible odd continuous representation.

A *global Serre weight* is an irreducible representation V of $\mathrm{GL}_n(\mathcal{O}_{F^+,p})$, which are all of the form $\bigotimes_{v|p} V_v$ with V_v an irreducible representation of $\mathrm{GL}_n(k_v)$, where k_v is the residue field of F^+ at v. In Definition 4.2.5, we define what it means for $\overline{r}$ to be *modular* of weight V. Roughly speaking, this means the Hecke eigensystem associated to $\overline{r}$ appears in a space of automorphic forms for G of weight V. For each place v, fix a place $\widetilde{v}$ of F dividing v, and define $\overline{\rho}_v := \overline{r}|_{\mathrm{Gal}(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})}$. We now state the main theorem.

THEOREM 1.0.1
Let $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be an irreducible odd representation. Assume that p is unramified in F, and assume that, for all places $v \mid p$ of F^+, $\overline{\rho}_v$ is $(6n-2)$-generic (see Definition 2.2.5). Then,

$$\overline{r} \text{ is modular of weight } \bigotimes_{v|p} V_v \implies V_v \in W^?(\overline{\rho}_v^{\mathrm{ss}}) \text{ for all } v \mid p,$$

where $W^?(\overline{\rho}_v^{\mathrm{ss}})$ is defined by [23].

The set $W^?(\overline{\rho}_v^{\mathrm{ss}})$ is an explicit collection of irreducible representations of $\mathrm{GL}_n(k_v)$ given by a representation-theoretic recipe. Conjecture 1.1 of [23] predicts that the reverse implication should also be true when $\overline{\rho}_v$ is semisimple for all v's (assuming $\overline{r}$ is modular). In fact, we prove a partial converse which shows the modularity of a subset of the predicted weights as discussed below (Theorem 1.0.4).

Remark 1.0.2

(1) When $n = 3$ and p splits completely in F^+, this result is due to [11], [25], [35], and [34].

(2) When $\overline{\rho}_v$ is not semisimple, there is no explicit conjecture, but one expects there to be a strict subset $W^?(\overline{\rho}_v) \subset W^?(\overline{\rho}_v^{\mathrm{ss}})$ which predicts the modular weights. When $n = 3$, this will be taken up in [33].

(3) Our methods are purely local, and so at least with some technical assumptions a version of Theorem 1.0.1 should hold in other global setups as well, for example, unitary Shimura varieties.

A key feature of the BDJ conjecture and a motivation for the generalizations is the relation between the weight recipe and p-adic Hodge theory properties of the local representation $\overline{\rho}_v$. Let $K/\mathbb{Q}_p$ be a finite unramified extension, and consider $\overline{\rho} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$. When $n = 2$, the weight recipe is in terms of the existence of the crystalline lifts of $\overline{\rho}$ in small (between $[0, p]$) Hodge–Tate weights. In Herzig's conjecture, one expects (at least for $\overline{\rho}$ semisimple) that $W^?(\overline{\rho})$ should also be predicted by the existence of crystalline lifts in small weights (see [15, Section 1.5 or Section 5] for a detailed discussion). However, the range of "small" Hodge–Tate weights is now $[0, (n-1)p]$ and determining reductions of n-dimensional crystalline representations in this range is still well beyond the current technology in p-adic Hodge theory.

We consider another local problem, namely, reductions of tamely crystalline representations with fixed (parallel) Hodge–Tate weights $\eta := (n-1, n-2, \ldots, 0)$. A *tamely crystalline* representation is a $\overline{\mathbb{Q}}_p$-representation of G_K which becomes crystalline when restricted to G_L for L/K a tame extension. The descent from L/K is then encoded in a tame inertial type τ. For short, we will call these representations of type (η, τ). For generic τ, we give a complete description of the semisimple Galois representations which are reductions mod p of representations of type (η, τ) (see Theorems 3.2.1 and 4.4.3).

If ρ is a Galois stable lattice in a representation of type (η, τ), then by deep results in integral p-adic Hodge theory due to Kisin in [28] building on work of Breuil, one can associate a semilinear algebra object, a Kisin module $\mathfrak{M}$ with Hodge type η and descent data of type τ. A. Caraiani and the third author in [7] construct a moduli stack of Kisin modules $Y^{\eta,\tau}$ with Hodge type $\leq \eta$ and tame inertial type τ. An upper bound on which $\overline{\rho}$ can arise as the reduction of a representation of type (η, τ) comes from a description of the special fiber of $Y^{\eta,\tau}$. To describe the special fiber, [7] relates $Y^{\eta,\tau}$ to a local model $M(\eta)$ constructed by Pappas and Zhu in [37]. The special fiber of $M(\eta)$ is a closed subscheme of an affine flag variety. The coherence conjecture of Pappas and Rapoport, proved by Zhu in [40], allows [37] to describe the special fiber

as a union of affine Schubert cells resolving a deep and long-standing question in the subject.

By classifying mod p Kisin modules of type (η, τ), we arrive at a combinatorial upper bound for the semisimple $\overline{\rho}$ that arise as the reduction of a representation of type (η, τ) in terms of a subset $\mathrm{Adm}(\eta) \subset \widetilde{W}$ of the extended affine Weyl group of GL_n called the η-admissible elements originally introduced by Kottwitz and Rapoport. More precisely, we assign to any semisimple $\overline{\rho}$ a relative position $\widetilde{w}(\overline{\rho}, \tau)^* \in \widetilde{W}$ called the (*dual*) *shape* (Definitions 2.1.2, 3.2.11, and 3.2.19). (Technically, the shape is associated with a Kisin module with descent data, but, in generic situations, there is a unique Kisin module which corresponds to $\overline{\rho}$.)

THEOREM 1.0.3 (Theorem 4.4.3)
A $(6n-2)$-generic semisimple Galois representation $\overline{\rho}$ has a lift of type (η, τ) if and only if $\widetilde{w}(\overline{\rho}, \tau)^* \in \mathrm{Adm}(\eta)$.

We first use the "only if" direction of Theorem 1.0.3 to reduce weight elimination (Theorem 1.0.1) to a representation theory/combinatorics problem. For simplicity, assume that $K = \mathbb{Q}_p$. If $F(\lambda)$ is a Serre weight for $\mathrm{GL}_n(\mathbb{F}_p)$ with p-restricted highest weight λ which is not in $W^?(\overline{\rho})$, one has to exhibit a tame $\mathrm{GL}_n(\mathcal{O}_K)$-type $\sigma(\tau)$ such that $F(\lambda)$ is a Jordan–Hölder factor of the reduction $\overline{\sigma}(\tau)$ and $\widetilde{w}(\overline{\rho}, \tau)^*$ is not η-admissible. Initially, this might seem daunting, since there are many types which contain $F(\lambda)$. In fact, it suffices to consider only the types which "obviously" contain $F(\lambda)$, namely, the Deligne–Lusztig representations $R_s(\widetilde{w}_h \cdot \lambda + \eta)$ for $s \in W(\mathrm{GL}_n)$ (see Section 4 for undefined notation). Precisely, we show that if, for all $s \in W(\mathrm{GL}_n)$, the shape of $\overline{\rho}$ relative to $R_s(\widetilde{w}_h \cdot \lambda + \eta)$ is η-admissible, then $F(\lambda) \in W^?(\overline{\rho})$. The argument uses alcove geometry to relate admissibility to a description of $W^?(\overline{\rho})$ in terms of dominant p-restricted alcoves and linkage due to Herzig. In turn, we use this relationship together with Theorem 1.0.4 below and global arguments to establish the "if" direction of Theorem 1.0.3.

We now discuss our second main theorem, which represents partial progress toward the other direction of the weight part of Serre's conjecture. Among the predicted weights, there is a distinguished subset $W_{\mathrm{obv}}(\overline{\rho}) \subset W^?(\overline{\rho})$ called *obvious* Serre weights. Obvious weights are defined precisely in Definition 7.1.3 of [15], but roughly correspond to the Hodge–Tate weights (with an η-shift) in which $\overline{\rho}$ has a crystalline lift which is the direct sum of inductions of characters from unramified extensions. When $n = 2$, there are only obvious weights, so the naive generalization of the weight part of Serre's conjecture would be that $W_{\mathrm{obv}}(\overline{\rho})$'s are exactly the modular weights. Despite their name, the modularity of these weights is by no means obvious. However, they are more easily accessed via automorphy lifting techniques. For example, [14]

obtains essentially complete results on the modularity of ordinary obvious weights in the ordinary setting. When $\overline{\rho}$ is a generic direct sum of characters, there are $(n!)^{[K:\mathbb{Q}_p]}$ obvious weights of which only $n!$ are ordinary. In [2], Barnet-Lamb, Gee, and Geraghty prove the modularity of the obvious weights when $n = 3$ and obtain partial results when $n > 3$. Using Theorem 1.0.1 and a generalization of part of [31] on potentially crystalline deformation rings, we extend these results to GL_n.

THEOREM 1.0.4
Let $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be an irreducible representation satisfying Taylor–Wiles conditions. Assume that p is unramified in F, and assume that, for all places $v \mid p$, $\overline{\rho}_v$ is semisimple and $(6n-2)$-generic. If $\overline{r}$ is modular of any obvious weight, then $\overline{r}$ is modular of all obvious weights.

Before giving an overview of the paper, we summarize the relationship between the methods and results of this paper and those of [31], which are for $n = 3$. The results about shapes of mod p Kisin modules and the triviality of the Kisin variety under genericity conditions in [31, Sections 2–3] generalize directly to the n-dimensional setting. Establishing these generalizations is enough to prove the only-if direction of Theorem 1.0.3. The only-if direction is the necessary local input to establish weight elimination (Theorem 1.0.1).

To prove Theorem 1.0.4, we compute certain potentially crystalline deformations, which requires generalizing arguments of [31, Sections 4–5]. The deformation rings are computed explicitly in [31] for all shapes. Here, we compute the deformation rings only for shapes of the form translation by a permutation of $(0, 1, \dots, n-1)$. (When $n = 3$, this corresponds to the shapes $\alpha\beta\alpha\gamma$ and $\beta\gamma\alpha\gamma$ appearing in tables at the end of [31].) For these special shapes, the deformation rings turn out to be formally smooth, and this is the necessary local input to prove Theorem 1.0.4.

We now give an overview of the paper. In Section 2, we begin with some background on affine Weyl groups, Serre weights, tame types, and inertial local Langlands. In Section 3, we prove the main local results in p-adic Hodge theory. Sections 3.1–3.3 are direct generalizations to n-dimensional Galois representations of results of [31] and [32] on Kisin modules with descent data. In Section 3.4, we show that if $\overline{\rho}$ has special shape with respect to τ, then the potentially crystalline deformation ring of type (η, τ) is formally smooth and we deduce the existence of potentially diagonalizable lifts.

The main theorems appear in Section 4. Weight elimination is in Section 4.2. The modularity of the obvious weights is in Section 4.3, and in Section 4.4, we complete the proof of Theorem 1.0.3, the local reduction problem using global input.

1.1. Notation and conventions

Fix $n \geq 2$. Let p be a prime with $p > n$. Fix a finite unramified extension $K/\mathbb{Q}_p$ of degree f. Let k denote the residue field of K of cardinality $q = p^f$. Let $\mathcal{O}_K := W(k)$ be the ring of integers of K. We denote the arithmetic Frobenius automorphism on $\mathcal{O}_K$ by φ, which acts as raising to the pth power on the residue field. Let $G_K = \mathrm{Gal}(\overline{K}/K)$. Let I_K denote the inertia subgroup, and let W_K denote the Weil group.

Let $E/\mathbb{Q}_p$ be finite extension assumed to be sufficiently large such that, for any unramified extension K'/K of degree the order of an element of $S_n \times S_n$, E contains a copy of K'. Let $\mathcal{O}$ be the ring of integers of E with uniformizer ϖ and residue field $\mathbb{F}$. We fix an embedding σ_0 of K into E (equivalently an embedding k into $\mathbb{F}$). Define $\sigma_j = \sigma_0 \circ \varphi^{-j}$.

For $r \geq 1$, we fix a compatible system of $(p^{rf} - 1)$st roots $\varpi_r = (-p)^{\frac{1}{p^{rf}-1}} \in \overline{K}$. The choice of root ϖ_1 defines a character $\omega_{\varpi_1} : I_K \to \mathcal{O}_K^\times$. Using our choice of embedding σ_0, we get a fundamental character of niveau f:

$$\omega_f := \sigma_0 \circ \omega_{\varpi_1} : I_K \to \mathcal{O}^\times.$$

We fix once and for all a sequence $\underline{p} := (p_n)_{n\in\mathbb{N}}$, where $p_n \in \overline{\mathbb{Q}}_p$ verify $p_{n+1}^p = p_n$ and $p_0 = -p$. We let $K_\infty := \bigcup_{n\in\mathbb{N}} K(p_n)$ and $G_{K_\infty} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/K_\infty)$.

Let $\underline{G} = \mathrm{Res}_{k/\mathbb{F}_p}\mathrm{GL}_n$. Let $T \subset \mathrm{GL}_n$ be the diagonal torus, and let $\underline{T} = \mathrm{Res}_{k/\mathbb{F}_p} T$ be a maximal torus of $\underline{G}$. Let $\underline{Z} \subset \underline{T}$ denote the center of $\underline{G}$. Let $W(\underline{G}) = W(\mathrm{GL}_n)^{\mathrm{Hom}(k,\mathbb{F})}$ denote the Weyl group of $\underline{G}$. Similarly, let $X^*(\underline{T})$ be the (geometric) characters of $\underline{T}$, which is equipped with an action of Frobenius π. We have an action of π on $W(\underline{G})$ by the formula $\pi(w)(\pi(v)) = \pi(w(v))$. There are isomorphisms $W(\underline{G}) \cong W(\mathrm{GL}_n)^f$ and $X^*(\underline{T}) \cong X^*(T)^f$ where the jth entry corresponds to the embedding σ_j. Under this identification, the action of π is such that if $v = (v_j) \in X^*(\underline{T})$, then $\pi(v)_j = v_{j-1}$. Let $\Lambda_R \subset X^*(T)$ (resp., $\underline{\Lambda}_R \subset X^*(\underline{T})$) denote the root lattice for GL_n (resp., $\underline{G}$).

Let W_a (resp., $\widetilde{W}$) denote the (resp., extended) affine Weyl group for GL_n. Similarly, we will use $\underline{W}_a \cong W_a^f$ (resp., $\underline{\widetilde{W}} \cong \widetilde{W}^f$) to denote the (resp., extended) affine Weyl group of $\underline{G}$. Recall that

$$W_a = \Lambda_R \rtimes W(\mathrm{GL}_n), \qquad \widetilde{W} = X^*(T) \rtimes W(\mathrm{GL}_n)$$

and similarly for $\underline{W}_a$ and $\underline{\widetilde{W}}$. We use $t_\nu \in \widetilde{W}$ to denote translation by $\nu \in X^*(T)$. The action of π on $X^*(\underline{T})$ and $W(\underline{G})$ extends naturally to $\underline{\widetilde{W}}$.

Let $\underline{R}^+ \subset \underline{R}$ (resp., $\underline{R}^{+,\vee} \subset \underline{R}^\vee$) denote the subset of positive roots (resp., positive coroots) in the set of roots (resp., coroots) with respect to the upper triangular Borel subgroup in each embedding. Define dominant (co)characters with respect to this choice of positive roots.

We fix an isomorphism $X^*(T) \cong \mathbb{Z}^n$ in the standard way, where the standard ith basis element $\varepsilon_i = (0, \dots, 1, \dots, 0)$ (with the 1 in the ith position) of the right-hand side corresponds to extracting the ith diagonal entry of a diagonal matrix. Dually we get a standard isomorphism $X_*(T) \cong \mathbb{Z}^n$, and let $\{\varepsilon_i^\vee\}$ denote the dual basis. Let $\eta_0 = (n-1, n-2, \dots, 0) \in X^*(T)$ be a fixed lift of the half sum of positive roots for GL_n. Define $\eta = (\eta_0, \eta_0, \dots, \eta_0) \in X^*(\underline{T})$. In the paper, sometimes we will consider simultaneously the group $\underline{G} = \mathrm{Res}_{k/\mathbb{F}_p}\mathrm{GL}_n$ for multiple k's. In such a situation, the symbol η will be used for the above element for any of the groups that appear, and it will always be clear which group it occurs in. Note that $\eta_j = \eta_0$ for any j. We will always denote the duality pairing between a free $\mathbb{Z}$-module and its dual (e.g., $X^*(T)$ and $X_*(T)$) by $\langle \cdot, \cdot \rangle$.

For any ring S, we define $M_n(S)$ to be the set of $n \times n$ matrix with entries in S. If $\alpha = \varepsilon_i - \varepsilon_j$ is a root of GL_n, we also call the (i, j)th entry of a matrix $X \in M_n(S)$ the αth entry. We will make use of both notations X_{ij} and X_α for this entry.

If P is a statement, the symbol $\delta_P \in \{0, 1\}$ takes value 1 if P is true, and 0 if P is false. If W is a de Rham representation of G_K over E, then for each $\kappa \in \mathrm{Hom}(K, E)$, we write $\mathrm{HT}_\kappa(W)$ for the multiset of Hodge–Tate weights labeled by embedding κ normalized such that the p-adic cyclotomic character has Hodge–Tate weight $\{1\}$ for every κ. For $\mu = (\mu_j) \in X^*(\underline{T})$, we say that an n-dimensional representation W has Hodge–Tate weights μ if

$$\mathrm{HT}_{\sigma_j}(W) = \{\mu_{1,j}, \mu_{2,j}, \dots, \mu_{n,j}\}.$$

An inertial type is a representation $\tau : I_K \to \mathrm{GL}_n(E)$ with open kernel and which extends to W_K. We say that an n-dimensional potentially semistable representation $\rho : G_K \to \mathrm{GL}_n(E)$ has type (μ, τ) if ρ has Hodge–Tate weights μ and the Weil–Deligne representation $\mathrm{WD}(\rho)$ restricted to I_K is isomorphic to τ. Note that this differs from the conventions of [15] via a shift by η.

Let $\mathrm{Art}_K : K^\times \to W_K^{\mathrm{ab}}$ denote the Artin map normalized so that uniformizers correspond to geometric Frobenius elements. For τ an inertial type, we use $\sigma(\tau)$ to denote the finite-dimensional smooth irreducible $\overline{\mathbb{Q}_p}$-representation of $\mathrm{GL}_n(\mathcal{O}_K)$ associated to τ by the "inertial local Langlands correspondence" (see Section 2.3). In fact, in all situations, $\sigma(\tau)$ will be defined over E. If V is a finite-length representation, then we use $\mathrm{JH}(V)$ to denote the set of Jordan–Hölder factors.

2. Background

2.1. Affine Weyl group

Fix the dominant base alcove for the apartment of (GL_n, T) which defines a Bruhat order on W_a denoted by $\leq$. If Ω is the stabilizer of the base alcove, then $\widetilde{W} = W_a \rtimes \Omega$

and so $\widetilde{W}$ inherits a Bruhat order in the standard way: for $\widetilde{w}_1, \widetilde{w}_2 \in W_a$ and $\widetilde{w} \in \Omega$, $\widetilde{w}_1\widetilde{w} \leq \widetilde{w}_2\widetilde{w}$ if and only if $\widetilde{w}_1 \leq \widetilde{w}_2$, and elements in different right W_a-cosets are incomparable. We also have the natural generalization $\underline{\Omega}$ for $\underline{G}$ and a Bruhat order on $\underline{\widetilde{W}}$. We now recall the definition of the admissible set as introduced by Kottwitz and Rapoport.

Definition 2.1.1
Let $\lambda_0 \in X^*(T)$. Then define

$$\mathrm{Adm}(\lambda_0) := \left\{\widetilde{w} \in \widetilde{W} \;\middle|\; \widetilde{w} \leq t_{w(\lambda_0)} \text{ for some } w \in W(\mathrm{GL}_n)\right\}.$$

Similarly, if $\lambda = (\lambda_j) \in X^*(\underline{T})$, then define $\mathrm{Adm}(\lambda) = \prod_j \mathrm{Adm}(\lambda_j) \subset \underline{\widetilde{W}}$.

When working on the Galois side, it is natural to work with the partially ordered group $\underline{\widetilde{W}}^\vee$ (resp., $\widetilde{W}^\vee$), which is identified with $\underline{\widetilde{W}}$ (resp., $\widetilde{W}$) as a group, but whose Bruhat order, also denoted by $\leq$, is defined by the *antidominant* base alcove. For any character $\mu \in X^*(\underline{T})$, define the subset $\mathrm{Adm}^\vee(\mu) \subset \underline{\widetilde{W}}^\vee$ as in Definition 2.1.1.

Definition 2.1.2
Define a bijection $\widetilde{w} \mapsto \widetilde{w}^*$ between $\underline{\widetilde{W}}^\vee$ and $\underline{\widetilde{W}}$ as follows:
(1) for $w = (w_j) \in W(\underline{G})$, define $w^* := (w_j^*) \in W(\underline{G})$ by $w_j^* = w_{f-1-j}^{-1}$;
(2) for $\nu = (\nu_j) \in X^*(\underline{T})$, define $\nu^* := (\nu_j^*) \in X^*(\underline{T})$ by $\nu_j^* = \nu_{f-1-j}$;
(3) for $\widetilde{w} = wt_\nu \in \underline{\widetilde{W}}^\vee$, define $\widetilde{w}^* \in \underline{\widetilde{W}}$ by $\widetilde{w}^* := t_{\nu^*}w^*$.
Note that $\widetilde{w} \mapsto \widetilde{w}^*$ is an antihomomorphism. By specializing to the case $f = 1$, we obtain a bijective antihomomorphism between $\widetilde{W}^\vee$ and $\widetilde{W}$.

We now record a few basic lemmas for later.

LEMMA 2.1.3
We have $\widetilde{w}_1 \leq \widetilde{w}_2$ in $\widetilde{W}^\vee$ if and only if $\widetilde{w}_1^* \leq \widetilde{w}_2^*$ in $\widetilde{W}$.

Proof
Suppose that $\widetilde{w}_2$ has a reduced expression $(\prod_{\alpha \in I} s_\alpha)\tau$, where each s_α is an affine reflection along a wall of the antidominant base alcove and τ stabilizes the antidominant base alcove. Then $\widetilde{w}_2^*$ is the product

$$\Big(\prod_{\alpha \in I} {}^{\tau^*}s_\alpha^*\Big)\tau^*, \tag{2.1}$$

where ${}^{\tau^*}s_\alpha^* = \tau^* s_\alpha^* (\tau^*)^{-1}$. (The order of factors indexed by I should of course be reversed from the reduced factorization of $\widetilde{w}_2$.) It is easy to check that τ^* is in Ω

and that each ${}^{\tau^*}s_\alpha^*$ is an affine reflection along a wall of the dominant base alcove. From this, we see that $\ell(\widetilde{w}_2^*) \leq \ell(\widetilde{w}_2)$. (Note that the lengths are with respect to different sets of generating reflections.) By symmetry, we see that $\ell(\widetilde{w}_2^*) = \ell(\widetilde{w}_2)$ so that (2.1) is a reduced expression. Since $\widetilde{w}_1 \leq \widetilde{w}_2$ in $\widetilde{W}^\vee$ if and only if $\widetilde{w}_1$ has a reduced expression $(\prod_{\alpha\in J} s_\alpha)\tau$, where J is some subsequence of I, and similarly for $\widetilde{w}_1^*$, the result follows. □

LEMMA 2.1.4
For $\widetilde{w} \in \underline{\widetilde{W}}^\vee$, we have $\widetilde{w} \in \mathrm{Adm}^\vee(\mu)$ if and only if $\widetilde{w}^* \in \mathrm{Adm}(\mu^*)$.

Proof
This follows from Lemma 2.1.3. □

LEMMA 2.1.5
Let $\lambda \in X^*(\underline{T})$ be a dominant weight. If $t_\nu s \in \mathrm{Adm}(\lambda)$, then

$$\max\{|\langle \nu, \alpha^\vee\rangle| \mid \alpha^\vee \in \underline{R}^\vee\} \leq \max\{|\langle \lambda, \alpha^\vee\rangle| \mid \alpha^\vee \in \underline{R}^\vee\}.$$

Proof
We reduce immediately to the case $f = 1$. By a result of Kottwitz–Rapoport (see Theorem 3.2 in [19]), $t_\nu s$ is λ-permissible, which means in particular that ν is in the convex hull of the Weyl group orbit of λ. The claim is that the inequality holds for any ν in the convex hull. For this, we can replace ν with the dominant representative in its Weyl group orbit. Convexity then says that $\lambda - \nu$ is an $\mathbb{R}$-linear combination of positive roots where all of the coefficients are nonnegative. If $\alpha_h^\vee$ is the highest positive coroot, then for any positive coroot α, we have

$$\langle \nu, \alpha^\vee\rangle \leq \langle \nu, \alpha_h^\vee\rangle.$$

It suffices then to observe that $\langle \nu, \alpha_h^\vee\rangle \leq \langle \lambda, \alpha_h^\vee\rangle$, which follows from that fact that $\alpha_h^\vee$ is dominant. □

Recall that *p-alcoves* of $\underline{G}$ are defined to be the connected components of

$$\big(X^*(\underline{T}) \otimes \mathbb{R}\big) \backslash \{x \mid \alpha^\vee(x + \eta) = pm\}_{\alpha^\vee \in \underline{R}^\vee, m\in\mathbb{Z}}.$$

Define the collection of p-restricted dominant weights

$$X_1(\underline{T}) = \{\lambda \in X^*(\underline{T}) \mid 0 \leq \langle \lambda, \alpha^\vee\rangle \leq p-1 \text{ for all simple positive coroots } \alpha^\vee\}.$$

A p-alcove C is called *p-restricted* if $C \cap X^*(\underline{T}) \subset X_1(\underline{T})$. We say that $\lambda \in X_1(\underline{T})$ is *regular p-restricted* if furthermore $\langle \lambda, \alpha^\vee\rangle < p-1$ for all simple positive coroots $\alpha^\vee$.

Recall also that

$$X^0(\underline{T}) = \left\{\lambda \in X^*(\underline{T}) \;\middle|\; \langle \lambda, \alpha^\vee \rangle = 0 \text{ for all coroots } \alpha^\vee\right\}.$$

Definition 2.1.6
Define the dot action of $\widetilde{W}$ on $X^*(\underline{T}) \otimes \mathbb{R}$ by

$$\widetilde{w} \cdot x = (wt_\nu) \cdot x = w(x + \eta + p\nu) - \eta.$$

In the literature, this is often thought of as an action of $W(\underline{G}) \rtimes pX^*(\underline{T})$, but it will be convenient to include the p-scaling in the definition. Recall that the group $\underline{W}_a$ acts simply transitively on the collection of p-alcoves. Let $\underline{C}_0$ denote the dominant base p-alcove, that is, the alcove containing $\underline{0}$.

Definition 2.1.7
Let $\lambda \in X^*(\underline{T})$ be a weight. We say that λ lies m-deep in its alcove if there exist integers $n_\alpha \in \mathbb{Z}$ such that $pn_\alpha + m < \langle \lambda + \eta, \alpha^\vee \rangle < p(n_\alpha + 1) - m$ for all positive coroots $\alpha^\vee \in \underline{R}^{\vee,+}$.

For example, a dominant weight $\lambda = (\lambda_j) \in X^*(\underline{T})$ is m-deep in $\underline{C}_0$ if $m < \langle \lambda_j + \eta_0, \alpha_j^\vee \rangle < p - m$ for all $j = 0, \ldots, f-1$ and all positive coroots $\alpha_j^\vee \in R^{\vee,+}$.

2.2. Tame types and Serre weights
We begin with some setup. An *inertial type* $\tau : I_K \to \mathrm{GL}_n(E)$ is a representation with open kernel which extends to the Weil group of K. An inertial type is *tame* if it factors through tame inertia. All our tame types will be defined over $\mathcal{O}$.

Tame inertial types have a combinatorial description which we will now recall (see [23, (6.15)] or [15, Definition 8.2.2]). Let $(w, \mu) \in W(\underline{G}) \times X^*(\underline{T})$. As in [23, (4.1)] (see also the paragraph preceding [15, Definition 10.1.12]), for any $(\nu, \sigma) \in X^*(\underline{T}) \rtimes W(\underline{G})$, define

$$^{(\nu,\sigma)}(w, \mu) = \big(\sigma w \pi(\sigma)^{-1}, \sigma(\mu) + p\nu - \sigma w \pi(\sigma)^{-1} \pi(\nu)\big), \tag{2.2}$$

and we write $(w, \mu) \sim (w', \mu')$ if there exists (ν, σ) such that $^{(\nu,\sigma)}(w, \mu) = (w', \mu')$.

Let r be the order of an element of S_n. For any such r, we choose an embedding σ_0' of the unramified extension K'/K of degree r into E extending σ_0. Let $e' = p^{fr} - 1$, let $e = p^f - 1$, and let $f' = fr$. Using our choice of e'th root of $(-p)$ in Section 1.1, we get a fundamental character $\omega_{f'} : I_K \to \mathcal{O}^\times$ such that $\omega_{f'}^{\frac{e'}{e}} = \omega_f$. The following describes all isomorphism classes of tame inertial types for K.

Definition 2.2.1
Define an inertial type $\tau(w,\mu) : I_K \to \mathrm{GL}_n(\mathcal{O})$ as follows. If $w = (s_0, \dots, s_{f-1})$, then set $s_\tau = s_0 s_{f-1} s_{f-2} \cdots s_1 \in W(\mathrm{GL}_n)$ and $\boldsymbol{\alpha} \in X^*(\underline{T})$ such that $\boldsymbol{\alpha}_0 = \mu_0$, $\boldsymbol{\alpha}_j = s_1^{-1} s_2^{-1} \dots s_j^{-1}(\mu_j)$ for $1 \le j \le f-1$. Let r denote the order of s_τ. Then,

$$\tau(w,\mu) \stackrel{\mathrm{def}}{=} \bigoplus_{1 \le i \le n} \omega_{f'}^{\sum_{0 \le k \le r-1} \mathbf{a}^{(0)}_{s_\tau^k(i)} p^{fk}} \tag{2.3}$$

where $\mathbf{a}^{(0)} := \sum_{j=0}^{f-1} \boldsymbol{\alpha}_j p^j \in \mathbb{Z}^n$. Note that $(w,\mu) \sim ((s_\tau, 1, \dots, 1), \boldsymbol{\alpha})$ and $\tau(w,\mu) \cong \tau((s_\tau, 1, \dots, 1), \boldsymbol{\alpha})$ by construction.

For any $\mathcal{O}$-valued inertial type τ, we use $\overline{\tau} : I_K \to \mathrm{GL}_n(\mathbb{F})$ to denote the reduction to the residue field. Note that, since $\omega_{f'}$ is the Teichmüller lift of its reduction to $\mathbb{F}$, for tame inertial types, $\overline{\tau}$ determines τ.

We say that a pair $(w,\mu) \in W(\underline{G}) \times X^*(\underline{T})$ is *good* if $(T_w, \theta_{w,\mu})$ is maximally split (see [15, Section 9.2] for the definitions of $(T_w, \theta_{w,\mu})$ and maximally split). This definition is consistent with [23, Definition 6.19] by [23, Proposition 6.20]. As in [15, Section 9.2], which follows [26], we attach a Deligne–Lusztig representation to a *good* pair $(w,\mu) \in W(\underline{G}) \times X^*(\underline{T})$, which we denote by $R_w(\mu)$. For any tame representation $\overline{\tau} : I_K \to \mathrm{GL}_n(\mathbb{F})$ which extends to G_K, there is an associated E-valued $\mathrm{GL}_n(\mathcal{O}_K)$-representation $V(\overline{\tau})$ defined in [15, Proposition 9.2.1]. (In [15, Proposition 9.2.1], it is denoted $V_\phi(\overline{\tau})$.) By [15, Proposition 9.2.3] if $\overline{\tau} \cong \overline{\tau}(w,\mu)$, then

$$V(\overline{\tau}) = R_w(\mu). \tag{2.4}$$

Remark 2.2.2
The condition that (w,μ) is good guarantees that the Deligne–Lusztig representation $R_w(\mu)$ is a genuine representation (and not only virtual; see [15, Proposition 9.2.1]). The genericity condition defined below will guarantee that $R_w(\mu)$ is in fact (absolutely) irreducible over E.

LEMMA 2.2.3
Suppose that $\mu - \eta \in X^*(\underline{T})$ is in alcove $\underline{C}_0$. Then (w,μ) is good for any $w \in W(\underline{G})$.

Proof
Let (T_w^*, s) be the F^*-stable maximal torus of $\underline{G}^*$ and semisimple element $s \in T_w^{*F^*}$ corresponding to $(T_w, \theta_{w,\mu})$ as in [15, Section 9.2(ii)]. As in the proof of [15, Lemma 10.1.10], let s' be $(g^*_{F^*(w^{-1})})^{-1} s F(g^*_{F^*(w^{-1})})$. (Recall that $(g^*_{F^*(w^{-1})})^{-1} F(g^*_{F^*(w^{-1})}) \in N(\underline{T}^*)$ represents $F^*(w^{-1})$ as in [15, Section 9.2(ii)].) By the proof of [15, Lemma 10.1.10], if the Weyl group of T_w^* in $Z_{\underline{G}^*}(s)$, which

is isomorphic to $\mathrm{Stab}_W(s')$, is trivial, then the claim follows. Suppose that s_α is in $\mathrm{Stab}_W(s')$ (which is generated by reflections). Let d be the order of $w\pi$ as an automorphism of $X^*(\underline{T})$. (In particular, f divides d.) Then by the proof of [15, Lemma 10.1.10],

$$\sum_{i=0}^{d-1} p^i \langle \mu, (w\pi)^i \alpha^\vee \rangle \tag{2.5}$$

is divisible by $p^d - 1$. Since $\mu - \eta$ is in $\underline{C}_0$, this divisibility forces $\langle \mu, (w\pi)^i \alpha^\vee \rangle$ to be $p-1$ for all i or $1-p$ for all i. Thus, either $(w\pi)^i \alpha^\vee$ are highest coroots in $\underline{R}^\vee$ for all i, or they are lowest coroots for all i. (Note that $\underline{R}^\vee$ has exactly f highest coroots.) This in turn implies that $(w\pi)^i \alpha^\vee = \pi^i \alpha^\vee$ for all $0 \le i \le d-1$, by comparing the unique nonzero component on both sides. Since $d \ge f$, this implies that w fixes $\pi^i \alpha$ for all i and that $s_{\pi^i \alpha} \in \mathrm{Stab}_W(s')$ for all i. We conclude that $\mathrm{Stab}_W(s')$ is $\langle s_{\pi^i \alpha} \rangle_i$ and centralizes w. From this, we see that $Z_{\underline{G}^*}(s)$ is isomorphic to $\mathrm{Res}_{k/\mathbb{F}_p} \mathrm{GL}_2 \times \underline{T}'$ for some torus $\underline{T}'$, and T_w^* is $\underline{T}_2 \times \underline{T}'$, where $\underline{T}_2 \subset \mathrm{Res}_{k/\mathbb{F}_p} \mathrm{GL}_2$ is a maximally split torus. Then by definition, $(T_w, \theta_{w,\mu})$ is maximally split. □

PROPOSITION 2.2.4

Let (w, μ) and (w', μ') be in $W(\underline{G}) \times X^*(\underline{T})$. If $(w, \mu) \sim (w', \mu')$, then the tame inertial types $\tau(w, \mu)$ and $\tau(w', \mu')$ are isomorphic. If $\mu - \eta$ and $\mu' - \eta$ are in alcove $\underline{C}_0$ and the tame inertial types $\tau(w, \mu)$ and $\tau(w', \mu')$ are isomorphic, then $(w, \mu) \sim (w', \mu')$.

Proof

The first part follows from a direct computation. For the second part, $(T_w, \theta_{w,\mu})$ and $(T_{w'}, \theta_{w',\mu'})$ are maximally split by Lemma 2.2.3. The second part now follows from [15, Proposition 9.2.1]. □

Definition 2.2.5

Let τ be a tame inertial type.

(1) Define τ to be *m-generic* if there is an isomorphism $\tau \cong \tau(s, \lambda + \eta)$ for some $s \in W(\underline{G})$ and $\lambda \in X^*(\underline{T})$ which is m-deep in alcove $\underline{C}_0$.

(2) Define $\overline{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F})$ to be m-generic if $\overline{\rho}^{\mathrm{ss}}|_{I_K} \cong \overline{\tau}(s, \lambda + \eta)$ for $\lambda \in X^*(\underline{T})$ which is m-deep in alcove $\underline{C}_0$.

(3) We say that τ (resp., $\overline{\rho}$) is *generic* if it is n-generic (resp., $3n$-generic).

(4) A *lowest alcove presentation* of τ is a pair $(s, \mu) \in W(\underline{G}) \times X^*(\underline{T})$ where $\mu \in \underline{C}_0$ such that $\tau \cong \tau(s, \mu + \eta)$ (which by definition exists exactly when τ is 0-generic).

Remark 2.2.6
The results in Section 3.2 hold for τ which are n-generic and $\overline{\rho}$ which are $2n$-generic. For Section 3.4, we will use $(2n-1)$-generic to control the monodromy condition. In most of Section 4, $\overline{\rho}$ will be $(6n-2)$-generic because of representation-theoretic input and a reliance on [12] to eliminate weights near the alcove walls. For example, Proposition 4.1.3 requires the type to be $2n$-generic, which combined with [12] forces $\overline{\rho}$ to be $(6n-2)$-generic. A more careful analysis would likely improve this bound.

Remark 2.2.7
Since the bounds for genericity do not depend on p, as p gets large, most semisimple representations will be generic.

Concretely, τ is m-generic if there exists an isomorphism $\tau \cong \tau(s, \lambda + \eta)$ with $\lambda_j + \eta_j = (a_{1,j}, a_{2,j}, \ldots, a_{n,j})$ such that, for all j,

$$m < |a_{i,j} - a_{k,j}| < p - m, \quad \text{for all } 1 \leq i \neq k \leq n.$$

Remark 2.2.8
The notion of generic here is slightly stronger than that of [31] when $n = 3$ and [12] in general. Comparing with Definition 2.1 in [31] and Definition 2 in [12], we see that if τ is m-generic as in Definition 2.2.5, then it is m-generic in the sense of [31] and [12]. If it is m-generic in the sense of [31] and [12], then it is $(m-1)$-generic in the sense of Definition 2.2.5. The difference being that the first inequality above is strict. Both [31] and [12] state genericity as a condition on a presentation as in (2.3), that is, as a condition on the $\boldsymbol{\alpha}_j$'s. To compare the two definitions, note that if (s, μ) is a lowest alcove presentation, then $\boldsymbol{\alpha}_j$ is in the Weyl group orbit of $\mu_j + \eta_j$.

Definition 2.2.9
We say that a tame inertial type τ is *regular* if the characters appearing in τ as in (2.3) are pairwise distinct.

Note that 1-generic implies regular, but regular is a weaker condition.

Definition 2.2.10
A *Serre weight* is an irreducible representation of $\mathrm{GL}_n(\mathbb{F}_{p^f})$. If $\lambda \in X_1(\underline{T})$ is a p-restricted dominant weight, then $F(\lambda)$ denotes the associated Serre weight which is the socle of the (dual) Weyl module, $W(\lambda)$, with highest weight λ. A Serre weight V is *regular* if $V \cong F(\lambda)$ for a regular p-restricted weight λ (see discussion after Lemma 2.1.5).

We recall that the map $\lambda \mapsto F(\lambda)$ induces a bijection between $X_1(\underline{T})/(p-\pi)X^0(\underline{T})$ and the set of isomorphism classes of Serre weights (see [15, Lemma 9.2.4]).

Let w_0 denote the longest elements of $W(\underline{G})$. Recall the self-bijection on regular p-restricted weights defined by $\lambda \mapsto w_0 \cdot (\lambda - p\eta)$ which induces a map $\mathcal{R}$ on regular Serre weights (see [15, p. 54]). If we let $\widetilde{w}_h := w_0 t_{-\eta}$, then

$$\mathcal{R}\big(F(\lambda)\big) = F\big(w_0 \cdot (\lambda - p\eta)\big) = F(\widetilde{w}_h \cdot \lambda).$$

Note that $\widetilde{w}_h \cdot \underline{C}_0$ is the highest p-restricted p-alcove.

We are now ready to state the Serre weight recipe following [23] and [15].

Definition 2.2.11 (see [15, Definition 9.2.5])
Let $\overline{\rho}$ be a generic semisimple n-dimensional representation of $\mathrm{Gal}(\overline{K}/K)$. Then,

$$W^?(\overline{\rho}) := \big\{\mathcal{R}\big(F(\lambda)\big) : F(\lambda) \text{ is a Jordan–Hölder constituent of } \overline{V}(\overline{\rho}|_{I_K})\big\},$$

where $\overline{V}(\overline{\rho}|_{I_K})$ denotes the semisimplification of a reduction modulo ϖ of $V(\overline{\rho}|_{I_K})$. Define $W_{\mathrm{obv}}(\overline{\rho}) \subset W^?(\overline{\rho})$ as in [15, Definition 7.1.3].

We give another characterization of the obvious weights.

PROPOSITION 2.2.12
Let $\overline{\rho}$ be generic and semisimple. Then for $\lambda \in X_1(\underline{T})$, $F(\lambda) \in W_{\mathrm{obv}}(\overline{\rho})$ if and only if $\overline{\rho}|_{I_K} \cong \overline{\tau}(w, \lambda + \eta)$ for some $w \in W(\underline{G})$.

Proof
See [15, Proposition 9.3.7]. □

COROLLARY 2.2.13
Let $\overline{\rho}$ be n-generic and semisimple. Then $|W_{\mathrm{obv}}(\overline{\rho})| = (n!)^f$.

Proof
Since $\overline{\rho}$ is n-generic, $\overline{\rho}|_{I_K} \cong \overline{\tau}(s, \mu + \eta)$ for some μ which is n-deep in $\underline{C}_0$. We define a map

$$W(\underline{G}) \to W_{\mathrm{obv}}(\overline{\rho}),$$
$$\sigma \mapsto F(\mu'),$$

where $(\sigma s \pi(\sigma)^{-1}, \mu' + \eta) = {}^{(\nu,\sigma)}(s, \mu + \eta)$ and $\nu \in X^*(\underline{T})$ is the unique element up to $X^0(\underline{T})$ such that $(t_\nu \sigma) \cdot \underline{C}_0$ is p-restricted. Explicitly,

$$\mu' = \sigma(\mu+\eta) + p\nu - \sigma s\pi(\sigma)^{-1}\pi(\nu) - \eta = (t_\nu\sigma)\cdot\mu - \sigma s\pi(\sigma)^{-1}\pi(\nu). \quad (2.6)$$

The inequality $|\langle \nu, \alpha^\vee\rangle| \le |\langle \eta, \alpha^\vee\rangle| < n$ from Lemma 2.1.5 coupled with the fact that μ is n-deep in $\underline{C}_0$ implies that μ' is in the alcove $(t_\nu\sigma)\cdot\underline{C}_0$. By Proposition 2.2.12, $F(\mu')$ is in $W_{\mathrm{obv}}(\overline{\rho})$, so that our map lands in the claimed set.

We now prove surjectivity. Consider any pair (ν,σ), and let μ' be such that $(\sigma s\pi(\sigma)^{-1}, \mu'+\eta) = {}^{(\nu,\sigma)}(s, \mu+\eta)$. Assume that μ' is p-restricted. Any obvious weight is of this form by Propositions 2.2.4 and 2.2.12. We claim then that $(t_\nu\sigma)\cdot\underline{C}_0$ is p-restricted and so (ν,σ) is one of the pairs above. Since μ' is p-restricted, for any coroot $\alpha^\vee$,

$$0 \le \left|\langle \mu'+\eta, \alpha^\vee\rangle\right| = \left|\left\langle \sigma(\mu+\eta) + p\nu - \sigma s\pi(\sigma)^{-1}\pi(\nu), \alpha^\vee\right\rangle\right| \le p(n-1).$$

Since μ is n-deep in alcove $\underline{C}_0$, we have $n < |\langle \sigma(\mu+\eta), \alpha^\vee\rangle| < p-n$. We conclude that

$$\left|\left\langle p\nu - \sigma s\pi(\sigma)^{-1}\pi(\nu), \alpha^\vee\right\rangle\right| < n(p-1).$$

Set $M = \max_{\alpha^\vee}\{|\langle \nu, \alpha^\vee\rangle|\}$. We deduce that $(p-1)M < n(p-1)$ and so $M < n$. Since $M < n$ and $(t_\nu\sigma)\cdot\mu$ is n-deep in its alcove, we deduce that μ' lies in the alcove $(t_\nu\sigma)\cdot\underline{C}_0$. Since μ' is p-restricted so is $(t_\nu\sigma)\cdot\underline{C}_0$.

For injectivity, suppose that ${}^{(\nu,\sigma)}(s,\mu+\eta) \equiv {}^{(\nu',\sigma')}(s,\mu+\eta) \mod (p-\pi)X^0(\underline{T})$. Then restricting the part in $X^*(\underline{T})$ to $\underline{Z}$, we see that $(p-\pi)\nu|_{\underline{Z}} \equiv (p-\pi)\nu'|_{\underline{Z}} \mod (p-\pi)nX^*(\underline{Z})$. Since $p-\pi$ acts injectively on $X^*(\underline{Z})$, we deduce that $\nu|_{\underline{Z}} \equiv \nu'|_{\underline{Z}} \mod nX^*(\underline{Z})$, so after modifying ν' by an element in $X^0(\underline{T})$ we can now assume that $\nu - \nu' \in \underline{\Lambda}_R$.

As observed at the beginning of the proof, the fact that μ is n-deep in $\underline{C}_0$ implies that the part in $X^*(\underline{T})$ of ${}^{(\nu,\sigma)}(s,\mu+\eta)$ lies in alcove $t_\nu\sigma\cdot\underline{C}_0$. Thus, the above equality implies an equality of alcoves $(t_\nu\sigma)\cdot\underline{C}_0 = (t_{\nu'}\sigma')\cdot\underline{C}_0$. Combining this with the decomposition $\widetilde{\underline{W}} = \underline{W}_a \rtimes \underline{\Omega}$ shows that $\nu = \nu'$ and $\sigma = \sigma'$, thus giving what we want. □

Definition 2.2.14

We say that $F(\lambda) \in W_{\mathrm{obv}}(\overline{\rho})$ is the obvious weight *associated* to $w \in W(\underline{G})$ if w maps to $F(\lambda)$ in the bijection from the proof of Corollary 2.2.13. (Note that this depends on the choice of (s,μ) such that $\overline{\rho}|_{I_K} \cong \overline{\tau}(s,\mu+\eta)$ and μ is n-deep in C_0.)

PROPOSITION 2.2.15

Assume that τ is d-generic with $d \ge 1$. Then for any lowest alcove presentation (s',μ') of τ, μ' lies $(d-1)$-deep in alcove $\underline{C}_0$. Furthermore, if (s,μ) is a fixed lowest alcove presentation of τ, then the collection of lowest alcove presentations is given by

$$\left\{(s',\mu') \in W(\underline{G}) \times X^*(\underline{T}) \;\middle|\; (s',\mu'+\eta) = {}^{(\nu,\sigma)}(s,\mu+\eta), t_\nu\sigma \in \underline{\Omega}\right\}.$$

Proof

The proof is similar to that of Corollary 2.2.13. Fix a presentation (s,μ) with μ d-deep in $\underline{C}_0$. For any other presentation (s',μ'), $(s',\mu'+\eta) = {}^{(\nu,\sigma)}(s,\mu+\eta)$ by Proposition 2.2.4 so

$$\mu' = (t_\nu\sigma)\cdot\mu - \sigma s\pi(\sigma)^{-1}\pi(\nu)$$

as in (2.6). Since $\mu' \in \underline{C}_0$, for all $\alpha^\vee$,

$$0 < \left|\left\langle \sigma(\mu+\eta) + p\nu - \sigma s\pi(\sigma)^{-1}\pi(\nu), \alpha^\vee\right\rangle\right| \leq p-1.$$

The same argument in Corollary 2.2.13 shows that if $M = \max_{\alpha^\vee}\{|\langle \nu,\alpha^\vee\rangle|\}$, then $M<2$.

Thus, if μ is d-deep in $\underline{C}_0$, then $\mu' = (t_\nu\sigma)\cdot\mu - \sigma s\pi(\sigma)^{-1}\pi(\nu)$ is $(d-1)$-deep in $(t_\nu\sigma)\cdot\underline{C}_0$. Since μ' is in $\underline{C}_0$, we see that $t_\nu\sigma \in \underline{\Omega}$ and μ' is $(d-1)$-deep in $\underline{C}_0$. We have thus shown that all lowest alcove presentations occur in the set described in the statement of the proposition.

Finally, we observe that any pair (s',μ') in the set described in the proposition indeed gives a lowest alcove presentation. This is because if $t_\nu\sigma \in \underline{\Omega}$, then $|\langle \nu,\alpha^\vee\rangle| \leq 1$. Using that μ is at least d-deep in $\underline{C}_0$ and $d \geq 1$, we conclude that μ' is also in $\underline{C}_0$. □

PROPOSITION 2.2.16

Let λ be a weight which is d-deep in a p-restricted alcove $\underline{C}$ with $d \geq n$. Then for any $s \in W(\underline{G})$ and any lowest alcove presentation (s',μ') of $\tau(s,\lambda+\eta)$, μ' is $(d-n+1)$-deep in $\underline{C}_0$. Moreover, $\tau(s,\lambda+\eta)$ is at least $(d-n+1)$-generic.

Proof

The same argument as in Corollary 2.2.13 shows that $(s',\mu'+\eta) = {}^{(\nu,\sigma)}(s,\lambda+\eta)$ with $(t_\nu\sigma)\cdot\underline{C} = \underline{C}_0$. The result now follows because $|\langle \nu,\alpha^\vee\rangle| \leq n-1$ for all coroots $\alpha^\vee$. □

2.3. Inertial local Langlands

In this section, we establish some simple instances of the inertial local Langlands following [11, Section 2.4]. We let $B_n \subset \mathrm{GL}_n$ be the Borel subgroup consisting of upper triangular matrices. Fix an isomorphism $i : \overline{\mathbb{Q}}_p \cong \mathbb{C}$. As above let $K \subset \overline{\mathbb{Q}}_p$ be the unramified extension of degree f. Let $\mathrm{rec}_{K,\mathbb{C}}$ denote the local Langlands correspondence for $\mathrm{GL}_n(K)$ of [21]. Using i, define a local Langlands correspondence rec_K over $\overline{\mathbb{Q}}_p$ such that $i\circ\mathrm{rec}_K = \mathrm{rec}_{K,\mathbb{C}}\circ i$. We first recall the existence statement.

THEOREM 2.3.1
Let τ be an inertial type. Then there is a finite-dimensional smooth irreducible $\overline{\mathbb{Q}}_p$-representation $\sigma(\tau)$ of $\mathrm{GL}_n(\mathcal{O}_K)$ such that if π is any irreducible smooth $\overline{\mathbb{Q}}_p$-representation of $\mathrm{GL}_n(K)$, then $\pi|_{\mathrm{GL}_n(\mathcal{O}_K)}$ contains a unique copy of $\sigma(\tau)$ as a subrepresentation if and only if $\mathrm{rec}_K(\pi)|_{I_K} \cong \tau$ and $N = 0$ on $\mathrm{rec}_K(\pi)$. If π is an irreducible smooth $\overline{\mathbb{Q}}_p$-representation of $\mathrm{GL}_n(K)$ such that

$$\mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_K)}(\mathrm{Ind}_{B_n(k)}^{\mathrm{GL}_n(k)} \boxtimes_{i=1}^n \chi_i, \pi) \neq 0,$$

then $\mathrm{rec}_K(\pi)|_{I_K} \cong \bigoplus_{i=1}^n \chi_i \circ \mathrm{Art}_K^{-1}|_{I_K}$.

Proof
The first part is [6, Theorem 3.7]. We now prove the second part. One proves as in the proof of [11, Proposition 2.4.1(ii)] that π is a subrepresentation of $\mathrm{n-Ind}_{B_n(K)}^{\mathrm{GL}_n(K)} \delta_{B_n}^{-1/2} \boxtimes_{i=1}^n \widetilde{\chi}_i$, for some $\widetilde{\chi}_i$ extending χ_i as in [11, Proposition 2.4.1(ii)]. Then the result follows as in [11, Proposition 2.4.1(ii)] (though $N(\pi)$ is not necessarily zero in our context). □

What we will need is an explicit $\sigma(\tau)$ in the case when τ is a tame inertial type. Let K'/K denote an unramified extension of degree r with residue fields k'/k. A character $\theta : k'^{\times} \to \overline{\mathbb{Q}}_p^{\times}$ is *primitive* if all its $\mathrm{Gal}(k'/k)$-conjugates are distinct. Following [23, Section 4], let $\kappa(\theta) = (-1)^{r-1} R_{T_w}^{\theta}$ denote the cuspidal representation of $\mathrm{GL}_r(k)$ parameterized by θ.

PROPOSITION 2.3.2
Let $\theta : k'^{\times} \to \overline{\mathbb{Q}}_p^{\times}$ be a primitive character. Let $\tau = \bigoplus_{i=0}^{r-1} (\theta \circ \mathrm{Art}_{K'}^{-1})^{(i)}|_{I_K}$, where $\chi^{(i)}$ denotes the ith Frobenius twist. Then $\sigma(\tau)$ can be taken to be $\kappa(\theta)$ interpreted as a $\mathrm{GL}_r(\mathcal{O}_K)$-representation via the reduction map $\mathrm{GL}_r(\mathcal{O}_K) \to \mathrm{GL}_r(k)$.

Proof
See [11, Proposition 2.4.1(i)]. □

Definition 2.3.3
Let $n = \sum_{j=1}^{\ell} r_j$ be a partition of n. For each j, let θ_j be a primitive character for the extension of degree r_j of k. Define $\mathrm{PInd}(\kappa(\theta_1), \ldots, \kappa(\theta_\ell))$ to be the parabolic induction to $\mathrm{GL}_n(k)$ of $\bigotimes_j \kappa(\theta_j)$ as a representation of the rational points of a parabolic $P \supset B$ with Levi subgroup $\prod_j \mathrm{GL}_{r_j}$.

PROPOSITION 2.3.4
Let $\tau = \bigoplus_{j=1}^{\ell} \tau_j$, where τ_j is a cuspidal inertial type associated to primitive char-

acters θ_j of degree r_j as in Proposition 2.3.2. Assume that the cuspidal types $\kappa(\theta_j)$ are pairwise distinct. Then $\sigma(\tau)$ can be taken to be $\mathrm{PInd}(\kappa(\theta_1),\dots,\kappa(\theta_\ell))$.

Proof
This follows from Section 6 of [37], where $\sigma(\tau)$ is constructed as $\sigma_{\mathcal{P}}(\lambda)$ for maximal $\mathcal{P}$ (see also Section 3.6 of [6]). In the case of principal series, see also Proposition 2.4.1(ii) in [11]. □

COROLLARY 2.3.5
Let $\tau = \tau(w,\mu)$ be a regular tame inertial type (Definition 2.2.9). Then $\sigma(\tau)$ can be taken to be $R_w(\mu)$.

Proof
As in Definition 2.2.1, we immediately reduce to the case in which $w = (s_\tau, \mathrm{id}, \dots, \mathrm{id})$. The condition of being regular corresponds to the pairwise distinctness condition in Proposition 2.3.4. Finally, we use Lemma 4.7 of [23] to relate the parabolic induction in Proposition 2.3.4 to the Deligne–Lusztig representation $R_w(\mu)$. □

From now on, for any regular tame inertial type, we let $\sigma(\tau)$ be as in Corollary 2.3.5.

Remark 2.3.6
Note that, for a regular tame inertial type τ, by Corollary 2.3.5 and (2.4)

$$V(\overline{\tau}) \cong \sigma(\tau).$$

3. Local results

In this section, we prove the main results on reductions of potentially crystalline representations which will be used for weight elimination in Section 4.1.

3.1. Étale φ-modules

In this section, we consider étale φ-modules associated to affine Weyl group elements and determine their corresponding semisimple Galois representations. The key result is Proposition 3.1.2, which relates the Galois representation to the tame types defined in Section 2.2.

Let $\mathcal{O}_{\mathcal{E}}$ denote the p-adic completion of $\mathfrak{S}[\frac{1}{v}]$, where $\mathfrak{S} := \mathcal{O}_K[\![v]\!]$ is endowed with a Frobenius morphism φ extending the Frobenius on $\mathcal{O}_K$ such that $\varphi(v) = v^p$. Let R be a local, complete Noetherian $\mathcal{O}$-algebra with finite residue field. By base change, the ring $\mathcal{O}_{\mathcal{E}}\widehat{\otimes}_{\mathbb{Z}_p}R$ is naturally endowed with a Frobenius endomorphism φ, and we write $\Phi\text{-Mod}^{\text{ét}}(R)$ for the category of étale $(\varphi, \mathcal{O}_{\mathcal{E}}\widehat{\otimes}_{\mathbb{Z}_p}R)$-modules. Its

objects are finite-type projective modules $\mathcal{M}$ over $\mathcal{O}_{\mathcal{E}} \widehat{\otimes}_{\mathbb{Z}_p} R$, endowed with a Frobenius semilinear endomorphism $\phi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ inducing an isomorphism on the pullback: $\mathrm{id} \otimes_{\varphi} \phi_{\mathcal{M}} : \varphi^*(\mathcal{M}) \xrightarrow{\sim} \mathcal{M}$.

Since K_∞/K is totally wildly ramified, the subgroup G_{K_∞} of G_K projects surjectively to the tame quotient of G_K. Hence, the restriction map

$$\mathrm{Rep}^t_{\mathbb{F}}(G_K) \to \mathrm{Rep}_{\mathbb{F}}(G_{K_\infty})$$

is fully faithful, where Rep^t denotes the subcategory of tame representations. We use $\mathrm{Rep}^t_{\mathbb{F}}(G_{K_\infty})$ to denote the essential image of this map and will often implicitly identify these representations of G_{K_∞} with their canonical extensions to G_K. Note that this essential image contains exactly representations of G_{K_∞} which are trivial on $G_{K_\infty} \cap G_{K^t}$, where K^t is the maximal tamely ramified extension of K.

For any complete local Noetherian $\mathcal{O}$-algebra R with finite residue field, by the theory of norm fields, there is an antiequivalence of categories (see [30, Lemma 1.2.7] for a version with coefficients)

$$\mathbb{V}^* : \Phi\text{-Mod}^{\text{ét}}(R) \to \mathrm{Rep}_R(G_{K_\infty}).$$

If K' is finite unramified extension of K, let $K'_\infty = K_\infty \otimes_K K'$, and we can similarly consider the category of étale φ-modules over K' denoted $\Phi\text{-Mod}^{\text{ét}}_{K'}(R)$ together with the antiequivalence $\mathbb{V}^*_{K'} : \Phi\text{-Mod}^{\text{ét}}_{K'}(R) \to \mathrm{Rep}_R(G_{K'_\infty})$.

For any $(\mathcal{M}, \phi_{\mathcal{M}}) \in \Phi\text{-Mod}^{\text{ét}}(R)$, we decompose $\mathcal{M} = \bigoplus_j \mathcal{M}^{(j)}$ over embeddings $\sigma_j : W(k) \to \mathcal{O}$ with the induced maps $\phi^{(j)}_{\mathcal{M}} : \mathcal{M}^{(j)} \to \mathcal{M}^{(j+1)}$. We are now ready to define "semisimple" étale φ-modules. We fix an embedding $\widetilde{W}^\vee = X^*(T) \rtimes W(\mathrm{GL}_n) \hookrightarrow N_{\mathrm{GL}_n}(T)(\mathbb{F}((v)))$ given by $\mu \mapsto v^\mu$ and identifying $W(\mathrm{GL}_n)$ with the subgroup of permutation matrices. Here, for $\mu = (a_1, \dots, a_n)$ we define v^μ to be the diagonal matrix with entries v^{a_i}. (We interpret $X^*(T)$ as the group of cocharacters of the dual torus, and v^μ is the associated cocharacter evaluated at v.)

Definition 3.1.1
For any $\widetilde{w} = (\widetilde{w}_j) \in \underline{\widetilde{W}}^\vee$ and $D = (D_j) \in \underline{T}(\mathbb{F})$, define $\mathcal{M}(\widetilde{w}, D) \in \Phi\text{-Mod}^{\text{ét}}(\mathbb{F})$ to be the free module over $\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F}$ of rank n such that the Frobenius $\phi^{(j)}_{\mathcal{M}}$ is given by the matrix $D_j \widetilde{w}_j \in N_{\mathrm{GL}_n}(T)(\mathbb{F}((v)))$ (with respect to the standard basis). Set $\mathcal{M}(\widetilde{w}) := \mathcal{M}(\widetilde{w}, \mathrm{Id})$.

PROPOSITION 3.1.2
Let $\mathcal{M}(\widetilde{w}, D)$ be as in Definition 3.1.1 with $\widetilde{w} = st_\mu \in \underline{\widetilde{W}}^\vee$. Then, $\mathbb{V}^*(\mathcal{M}(\widetilde{w}, D))$ lies in $\mathrm{Rep}^t_{\mathbb{F}}(G_{K_\infty})$ and

$$\mathbb{V}^*\big(\mathcal{M}(\widetilde{w}, D)\big)|_{I_K} \cong \overline{\tau}(s^*, \mu^*).$$

In particular, the restriction to inertia does not depend on D.

Proof

Assume that $(s_1^*, \mu_1^*) \sim (s_2^*, \mu_2^*)$ via conjugation by (ν^*, σ^*). Then for any $D_1 \in \underline{T}(\mathbb{F})$, there exists a $D_2 \in \underline{T}(\mathbb{F})$ such that φ-conjugation by $t_{\pi^{-1}(\nu)}\sigma$ induces an isomorphism $\mathcal{M}(s_1 t_{\mu_1}, D_1) \cong \mathcal{M}(s_2 t_{\mu_2}, D_2)$ by (2.2) . For any $\widetilde{w}$, by an appropriate conjugation, we can assume $\widetilde{w}_j = 1$ for $j \neq f-1$. Let $\widetilde{w}_{f-1} = s_0^{-1} t_{\mu_0}$ with $\mu_0 = (a_1, \dots, a_n) \in \mathbb{Z}^n$, and let r denote the order of s_0. Consider the base change

$$\mathcal{M}' := \mathcal{M}(\widetilde{w}, D) \otimes_{W(k)} W(k'),$$

where k'/k is a finite extension of degree r. Let $\phi_{\mathcal{M}'}$ denote the Frobenius on $\mathcal{M}'$. A straightforward computation as in [32, Lemma 3.3.3] shows that there exists a basis (e_i) for $(\mathcal{M}')^{(0)}$ (the piece corresponding to the fixed embedding $\sigma_0' : W(k') \to \mathcal{O}$) such that

$$\phi_{\mathcal{M}'}^{fr}(e_i) = d_i v^{\sum_{m=0}^{r-1} a_{s_0^{m+1}(i)} p^{fm}} e_i$$

for some scalars d_i determined by D. Following Proposition 2.1.7 in [8] and using our choice of embedding $\sigma_0' : W(k') \to \mathcal{O}$, one can determine $\mathbb{V}^*_{K'}(\mathcal{M}')$ from $\phi_{\mathcal{M}'}^{fr}$. If $K'_\infty = K_\infty \otimes_{W(k)} W(k')$, then there are unramified characters ξ_i such that

$$\mathbb{V}^*_{K'}(\mathcal{M}') \cong \Big(\bigoplus_{i=1}^{n} \xi_i \omega_{rf}^{\sum_{m=0}^{r-1} p^{fm} a_{s_0^{m+1}(i)}}\Big)\Big|_{G_{K'_\infty}}.$$

Hence, $\mathbb{V}^*(\mathcal{M}(\widetilde{w}, D))$ is tame, and by comparison with (2.3),

$$\mathbb{V}^*\big(\mathcal{M}(\widetilde{w}, D)\big)|_{I_K} \cong \overline{\tau}\big((s_0, 1, \dots, 1), (\mu_0, 0, \dots, 0)\big). \qquad \square$$

3.2. Semisimple Kisin modules

In this section, we generalize [31] from GL_3 to GL_n and study reductions of Kisin modules with descent. For the convenience of the reader, we first state the main theorem which is used for weight elimination in Section 4. The theorem will be a consequence of Theorem 3.2.26 about reductions of Kisin modules with descent data. The proof appears at the end of the subsection.

THEOREM 3.2.1

Let $\tau \cong \tau(s, \mu + \eta)$ be 1-generic with lowest alcove presentation (s, μ), and let

$\lambda \in X^*(\underline{T})$. Let $\overline{\rho}$ be the reduction of a potentially crystalline representation of type (λ, τ). Assume that either (1) τ is a principal series type and $\overline{\rho}^{\mathrm{ss}}$ is a direct sum of characters or (2) $\lambda = \eta$ and τ is generic. Then there exists $(w, \nu) \in W(\underline{G}) \times X^*(\underline{T})$ such that

$$\overline{\rho}^{\mathrm{ss}}|_{I_K} \cong \overline{\tau}(w, \nu + \eta) \qquad \textit{and} \qquad s^{-1} t_{\nu-\mu} w \in \mathrm{Adm}(\lambda).$$

Remark 3.2.2
The element $s^{-1} t_{\nu-\mu} w$ in Theorem 3.2.1(2) is later defined to be $\widetilde{w}^*(\overline{\rho}, \tau)$ in Definition 4.1.6.

We continue to use the notation of Section 2.2. Let $\tau : I_K \to \mathrm{GL}_n(E)$ be a tame inertial type. We will assume throughout that τ is 1-generic and fix a lowest alcove presentation (s, μ) where $\mu \in \underline{C}_0$ (i.e., $\tau \cong \tau(s, \mu + \eta)$).

If $s = (s_0, \dots, s_{f-1})$ and $\mu = (\mu_j)_{0 \le j \le f-1} \in X^*(\underline{T})$, we take $s_\tau \overset{\text{def}}{=} s_0 s_{f-1} s_{f-2} \cdots s_1 \in W(\mathrm{GL}_n)$ and $\boldsymbol{\alpha}_{(s,\mu)} \in X^*(\underline{T})$ such that $\boldsymbol{\alpha}_{(s,\mu),j} = s_1^{-1} s_2^{-1} \cdots s_j^{-1}(\mu_j + \eta_j)$ for $1 \le j \le f-1$ and $\boldsymbol{\alpha}_{(s,\mu),0} = \mu_0 + \eta_0$. Let r denote the order of s_τ and $f' = rf$. As in Definition 2.2.1, $\tau(s, \mu + \eta) \cong \tau((s_\tau, 1, \dots, 1), \boldsymbol{\alpha}_{(s,\mu)})$, and concretely,

$$\tau \cong \bigoplus_{i=1}^{n} \chi_i \quad \text{with } \chi_i := \omega_{f'}^{\sum_{0 \le k \le r-1} \mathbf{a}^{(0)}_{(s,\mu),s_\tau^k(i)} p^{fk}}, \tag{3.1}$$

where $\mathbf{a}^{(0)}_{(s,\mu)} = \sum_{j=0}^{f-1} \boldsymbol{\alpha}_{(s,\mu),j} p^j \in \mathbb{Z}^n$ (compare with (2.3)). By fixing a choice of lowest alcove presentation, we also fix the order of the characters χ_i as above.

Remark 3.2.3
In [31] and [32], the notion of lowest alcove presentation does not appear. Everything is written for presentations of the form $\tau((s_\tau, 1, \dots, 1), \boldsymbol{\alpha}_{(s,\mu)})$ (see, e.g., the beginning of Section 2.1, Section 6.1 of [31], or Section 3.1 of [32]). In the notation of Section 6.1 of [31] and Section 3.1 of [32], $\boldsymbol{\alpha}_{(s,\mu),j} = (a_{1,j}, a_{2,j}, a_{3,j})$. If

$$s_{\mathrm{or}} := (s_1^{-1} s_2^{-1} \cdots s_{f-1}^{-1}, s_1^{-1} s_2^{-1} \cdots s_{f-2}^{-1}, \dots, s_1^{-1}, 1) \in W(\underline{G}),$$

then $s^*_{\mathrm{or}}(\boldsymbol{\alpha}_{(s,\mu)}) = \mu + \eta$, and conjugation by $(\underline{0}, s^*_{\mathrm{or}})$ changes one presentation to the other. The element s_{or} is called the *orientation* of $\boldsymbol{\alpha}_{(s,\mu)}$ (see Definition 2.6 and equation (2.2) in [31]).

Remark 3.2.4
Comparing (3.1) with the equation at the beginning of [31, Section 2.1] or [32, Section 3.1], the exponents differ by a minus sign. This is because of a dual that appears

in Definition 3.2.5 which makes everything consistent. See Remark 3.2.7 for more details.

We continue to write K (resp., K') for the unramified extension of $\mathbb{Q}_p$ of degree f (resp., $f' \overset{\text{def}}{=} fr$). If $r = 1$, we say that τ is a *principal series type*. Otherwise, we write τ' for the base change of τ to K'/K (which is just τ considered as a principal series type for $G_{K'}$). We record the relevant data for τ'. Define $\boldsymbol{\alpha}'_{(s,\mu)} \in X^*(T)^{\mathrm{Hom}(k',\mathbb{F})} \cong X^*(T)^{f'}$ (using the fixed choice of embedding σ'_0) by

$$\boldsymbol{\alpha}'_{(s,\mu),j+kf} := s_\tau^{-k}(\boldsymbol{\alpha}_{(s,\mu),j}) \quad \text{for } 0 \le j \le f-1, 0 \le k \le r-1.$$

If $\tau_{K'}(w', \mu')$ is the analogous construction of tame types over K' for $(w', \mu') \in (W(\mathrm{GL}_n) \times X^*(T))^{\mathrm{Hom}(k',\mathbb{F})}$, then $\tau' \cong \tau_{K'}(1, \boldsymbol{\alpha}'_{(s,\mu)})$ by direct comparison using (3.1). The *orientation* $s'_{\mathrm{or}} \in W(\mathrm{GL}_n)^{f'}$ of $\boldsymbol{\alpha}'_{(s,\mu)}$ in the sense of Definition 2.6 in [31] is given by (compare with [31, Proposition 6.1])

$$s'_{\mathrm{or},j+kf} := s_\tau^{k+1} s_{\mathrm{or},j} \quad \text{for } 0 \le j \le f-1, 0 \le k \le r-1. \tag{3.2}$$

Note that $\boldsymbol{\alpha}'_{(s,\mu),j+kf}$ is in the $W(\mathrm{GL}_n)$-orbit of $\mu_j + \eta_j$. If (s,μ) is a lowest alcove presentation with μ m-deep in alcove $\underline{C}_0$, then for all coroots $\alpha^\vee$ of GL_n and $0 \le j' \le f'-1$,

$$\begin{aligned} m < \left|\langle \boldsymbol{\alpha}_{(s,\mu),j'}, \alpha^\vee \rangle\right| < p - m \qquad &\text{and} \\ (s'_{\mathrm{or},j'})^{-1}(\boldsymbol{\alpha}'_{(s,\mu),f'-1-j'}) = \mu_{f-1-j} + \eta_{f-1-j} \quad &\text{is dominant,} \end{aligned} \tag{3.3}$$

where $j \equiv j' \bmod f$.

Define $L' := K'(\varpi_r) = K'((-p)^{\frac{1}{p^{rf}-1}})$, and let $\Delta' := \mathrm{Gal}(L'/K') \subset \Delta := \mathrm{Gal}(L'/K)$. Note that τ defines an $\mathcal{O}$-valued representation of Δ'. For any complete local Noetherian $\mathcal{O}$-algebra with residue field $\mathbb{F}'$ finite over $\mathbb{F}$, let $\mathfrak{S}_{L',R} := (W(k') \otimes_{\mathbb{Z}_p} R)[\![u']\!]$. We endow $\mathfrak{S}_{L',R}$ with an action of Δ as follows. For any τ in Δ', $\tau(u') = \frac{\tau(\varpi_r)}{\varpi_r} u'$ and τ acts trivially on the coefficients. If $\sigma \in \mathrm{Gal}(L'/\mathbb{Q}_p)$ is the lift of Frobenius on $W(k')$ which fixes ϖ_r, then σ^f generates $\mathrm{Gal}(K'/K)$ acting in the natural way on $W(k')$ and trivially on both u' and R. Set $v = (u')^{p^{rf}-1}$, and note that

$$(\mathfrak{S}_{L',R})^{\Delta=1} = \big(W(k) \otimes_{\mathbb{Z}_p} R\big)[\![v]\!].$$

As usual, $\varphi : \mathfrak{S}_{L',R} \to \mathfrak{S}_{L',R}$ acts as σ on $W(k')$, acts trivially on R, and sends u' to $(u')^p$.

For any positive integer h, let $Y^{[0,h],\tau}(R)$ be the category of Kisin modules over L' with tame descent of type τ and height in $[0,h]$ as defined in Section 3 of [7] if

τ is a principal series type. For other types, we refer to Section 6 of [31] for further background.

Definition 3.2.5
An element $(\mathfrak{M}, \phi_{\mathfrak{M}}, \{\widehat{g}\}) \in Y^{[0,h],\tau}(R)$ is a Kisin module $(\mathfrak{M}, \phi_{\mathfrak{M}})$ over $\mathfrak{S}_{L',R}$ ([31, Definition 2.3]) with height less than h together with a semilinear action of Δ which commutes with $\phi_{\mathfrak{M}}$ such that, for each $0 \leq j \leq f' - 1$,

$$\mathfrak{M}^{(j)} \mod u' \cong \tau^{\vee} \otimes_{\mathcal{O}} R$$

as Δ'-representations. In particular, the semilinear action induces an isomorphism $(\sigma^f)^*(\mathfrak{M}) \cong \mathfrak{M}$ (see [31, Section 6.1]) as elements of $Y^{[0,h],\tau'}(R)$.

Remark 3.2.6
As explained in [31, Section 6.1], the data of an extension of the action of Δ' to an action of Δ is equivalent to the choice of an isomorphism $(\sigma^f)^*(\mathfrak{M}) \cong \mathfrak{M}$ satisfying an appropriate cocycle condition. We will use both points of view interchangeably.

Remark 3.2.7
The appearance of $\tau^{\vee}$ in the definition is due to the fact that we are using the contravariant functors to Galois representations to be consistent with [31] and [32] as opposed to the covariant versions which appear in [7] and [11]. In [31], we did not use the notation $\tau^{\vee}$. Instead, we included it in our description of descent data by having a minus sign in the equation before [31, Definition 2.1]. The notion of a Kisin module with tame descent data of type τ here is consistent with what appears in [31, Definition 2.1].

Recall that we have fixed a lowest alcove presentation (s, μ) with $\mu \in \underline{C}_0$. Definitions 3.2.11 and 3.2.8 as well as the matrix of partial Frobenius $A^{(j)}$ below depend on the choice of presentation (see Remark 3.2.12).

Recall the following definition.

Definition 3.2.8
For any complete local Noetherian $\mathcal{O}$-algebra R, an eigenbasis β for $\mathfrak{M} \in Y^{[0,h],\tau}(R)$ is a (ordered) basis $\beta^{(j')} = (f_1^{(j')}, f_2^{(j')}, \ldots, f_n^{(j')})$ of $\mathfrak{M}^{(j')}$ for each $0 \leq j' < f'$ such that Δ' acts on $f_i^{(j')}$ via the character χ_i^{-1} from (3.1) and such that $(\sigma^f)^*(\beta) = \beta$ (see [31, Definition 2.8] and [32, Definition 3.1.1] for details).

Note that, since the order of Δ' is prime to p and $\mathcal{O}$ is assumed to be sufficiently large, eigenbases exist for any $\mathfrak{M} \in Y^{[0,h],\tau}(R)$ when R is a complete local

Noetherian $\mathcal{O}$-algebra as above. Given $\mathfrak{M} \in Y^{[0,h],\tau}(R)$ together with an eigenbasis β, the matrix of the partial Frobenius $\phi^{(j')}_{\mathfrak{M},s'_{\mathrm{or},j'+1}(n)}$ with respect to β is defined as in Definition 2.11 of [31]. Namely, let $\phi^{(j')}_{\mathfrak{M},s'_{\mathrm{or},j'+1}(n)} : \varphi^*(\mathfrak{M})^{(j')}_{\chi_{s'_{\mathrm{or},j'+1}(n)}} \to \mathfrak{M}^{(j'+1)}_{\chi_{s'_{\mathrm{or},j'+1}(n)}}$ be the Frobenius map on the $\chi_{s'_{\mathrm{or},j'+1}(n)}$ isotypic pieces of $\mathfrak{M}^{(j'+1)}$ and $\varphi^*(\mathfrak{M})^{(j')}$, respectively. For any $0 \le j' \le f' - 1$, set

$$\mathbf{a}'^{(j')}_{(s,\mu)} = \sum_{i=0}^{f'-1} \boldsymbol{\alpha}'_{(s,\mu),-j'+i}\, p^i, \tag{3.4}$$

where $-j' + i$ is taken modulo f'.

If $\beta^{(j'+1)} = (f_1^{(j'+1)}, \dots, f_n^{(j'+1)})$, then as in Lemma 2.9 in [31],

$$\left\{(u')^{\mathbf{a}'^{(j'+1)}_{(s,\mu),i} - \mathbf{a}'^{(j'+1)}_{(s,\mu),s'_{\mathrm{or},j'+1}(n)}} f_i^{(j'+1)}\right\}_{i=1}^n$$

is a basis of $\mathfrak{M}^{(j'+1)}_{\chi_{s'_{\mathrm{or},j'+1}(n)}}$. Similarly, if $\beta^{(j')} = (f_1^{(j')}, \dots, f_n^{(j')})$, then

$$\left\{(u')^{\mathbf{a}'^{(j'+1)}_{(s,\mu),i} - \mathbf{a}'^{(j'+1)}_{(s,\mu),s'_{\mathrm{or},j'+1}(n)}} \otimes f_i^{(j')}\right\}_{i=1}^n$$

is a basis for $\varphi^*(\mathfrak{M})^{(j')}_{\chi_{s'_{\mathrm{or},j'+1}(n)}}$. We order these bases such that the u'-multiple of $f_{s'_{\mathrm{or},j'+1}(i)}$ is the ith basis vector. Note that the orientation s'_{or} is chosen such that, for all $i < k$,

$$p^{f'} - 1 > \mathbf{a}'^{(j'+1)}_{(s,\mu),s'_{\mathrm{or},j'+1}(i)} - \mathbf{a}'^{(j'+1)}_{(s,\mu),s'_{\mathrm{or},j'+1}(k)} > 0, \tag{3.5}$$

so that all the exponents which appear are positive. The inequalities are strict because τ is regular (since τ is 1-generic).

The matrix of the j'th partial Frobenius with respect to these bases ordered as above will be denoted by $A^{(j')} = \mathrm{Mat}_\beta(\phi^{(j')}_{\mathfrak{M},s'_{\mathrm{or},j'+1}(n)})$. We stress that the notion of eigenbasis and the definition of $A^{(j')}$ depend on the chosen presentation of τ, as well as our choice of the ordering of the characters in τ. By our requirement that β is σ^f-invariant, $A^{(j')}$ only depends on $j' \bmod f$. We also observe that the height condition implies $v^h (A^{(j')})^{-1} \in M_n(R[\![v]\!])$.

For any $\mathcal{O}$-algebra R, define

- $\mathcal{J}(R) := \{M \in \mathrm{GL}_n(R[\![v]\!]) \mid M \mod v \text{ is upper triangular}\}$;
- $\mathcal{J}_1(R) := \{M \in \mathrm{GL}_n(R[\![v]\!]) \mid M \mod v \text{ is upper triangular unipotent}\}$;
- for any $m \ge 1$, $\mathcal{D}_m(R) := \{M \in \mathrm{GL}_n(R[\![v]\!]) \mid M \mod v^m \text{ is diagonal}\}$.

For any $M \in \mathrm{Mat}_n(R((u')))$ and $g \in \mathrm{GL}_n(R((u')))$, define

$$\mathrm{Ad}(g)(M) := gMg^{-1}. \tag{3.6}$$

We can now record the effect of changing the eigenbasis β on the matrices $A^{(j)}$, which is the generalization of Propositions 2.15 and 2.16 in [31].

PROPOSITION 3.2.9

Let R be a complete local Noetherian $\mathcal{O}$-algebra. Let $\mathfrak{M} \in Y^{[0,h],\tau}(R)$ together with two eigenbases $\beta^{(j)} := (f_1^{(j)}, f_2^{(j)}, \dots, f_n^{(j)})$ and $\beta'^{(j)} := (f_1'^{(j)}, f_2'^{(j)}, \dots, f_n'^{(j)})$ be related by

$$(f_1'^{(j)}, f_2'^{(j)}, \dots, f_n'^{(j)})D^{(j)} = (f_1^{(j)}, f_2^{(j)}, \dots, f_n^{(j)})$$

with $D^{(j)} \in \mathrm{GL}_n(R[\![u']\!])$. Let us write $A^{(j)} \overset{\mathrm{def}}{=} \mathrm{Mat}_\beta(\phi^{(j)}_{\mathfrak{M},s'_{\mathrm{or},j+1}(n)})$ and $A'^{(j)} \overset{\mathrm{def}}{=} \mathrm{Mat}_{\beta'}(\phi^{(j)}_{\mathfrak{M},s'_{\mathrm{or},j+1}(n)})$ as above. Set $I^{(j)} \overset{\mathrm{def}}{=} \mathrm{Ad}((s'_{\mathrm{or},j})^{-1}(u')^{-\mathbf{a}'^{(j)}_{(s,\mu)}})(D^{(j)}) \in \mathcal{I}(R)$, which only depends on $j \mod f$. Then for all $0 \le j \le f'-1$,

$$A'^{(j)} = I^{(j+1)}A^{(j)}\big(s_j^*(I^{(j),\varphi})(s_j^*)^{-1}\big),$$

where

$$I^{(j),\varphi} \overset{\mathrm{def}}{=} v^{\mu_j^*+\eta_j^*}\big(\varphi(I^{(j)})^{-1}\big)v^{-\mu_j^*-\eta_j^*}.$$

Furthermore, if μ is m-deep in alcove $\underline{C}_0$, then $I^{(j),\varphi} \in \mathcal{D}_{m+1}(R)$.

Proof

The formula for change of basis only depends on $\mathfrak{M}$ as a Kisin module over L' for the principal series type τ'. The fact that $I^{(j)}$ only depends on $j \mod f$ follows from the fact that β is fixed by σ^f. The rest of the proof is the same as in Proposition 2.15 of [31], but we note that the s_j's which appear in Proposition 2.15 of [31] are called $s'_{\mathrm{or},j}$'s here. We use that $(s'_{\mathrm{or},j+1})^{-1}s'_{\mathrm{or},j} = s_j^*$ by (3.2) and Remark 3.2.3. Also, we use that, for $0 \le j \le f-1$, $(s'_{\mathrm{or},j})^{-1}(\boldsymbol{\alpha}'_{(s,\mu),f-1-j}) = \mu_{f-1-j} + \eta_{f-1-j}$ (see (3.3)). That $I^{(j)} \in \mathcal{I}(R)$ follows from (3.5), which follows from the fact that the characters appearing in τ are distinct (see Proposition 4.6 in [7]). The fact that $I^{(j),\varphi} \in \mathcal{D}_{m+1}(R)$ is straightforward (compare with Proposition 2.16 in [31]). □

Remark 3.2.10

In the situation of Proposition 3.2.9, we call the tuple $(I^{(j)}) \in \mathcal{I}(R)^{f'}$ the change of basis tuple from β to β'. It satisfies $I^{(j)} = I^{(k)}$ if $j \equiv k \mod f$. Conversely, any tuple in $\mathcal{I}(R)^{f'}$ with this property is the change of basis tuple from β to another

eigenbasis β'. (This uses our running assumption that τ is 1-generic.) In other words, given β, the data of an eigenbasis β' is the same as the data of the tuple $(I^{(j)})$.

Recall (from [7, Definition 5.5]) the notion of shape.

Definition 3.2.11
If τ is a principal series type, the *shape* of a Kisin module $\mathfrak{M} \in Y^{[0,h],\tau}(\mathbb{F}')$ is the element $\widetilde{w} = (\widetilde{w}_0, \widetilde{w}_1, \dots, \widetilde{w}_{f-1}) \in \underline{\widetilde{W}}^\vee = (\widetilde{W}^\vee)^{\mathrm{Hom}(k,\mathbb{F})}$ such that, for any eigenbasis β and any $0 \le j \le f-1$, the matrix $A^{(j)} = \mathrm{Mat}_\beta(\phi^{(j)}_{\mathfrak{M},s'_{\mathrm{or},j+1}(n)})$ lies in $\mathcal{I}(\mathbb{F}')\widetilde{w}_j\mathcal{I}(\mathbb{F}')$. (Recall the fixed inclusion $\widetilde{W}^\vee \hookrightarrow N_{\mathrm{GL}_n}(T)(\mathbb{F}((v)))$ before Definition 3.1.1.)

For a nonprincipal series type τ, we define the shape via base change as in [31, Definition 6.10]. By definition, an element $\mathfrak{M} \in Y^{[0,h],\tau}(\mathbb{F}')$ consists of an element $\mathfrak{M}' \in Y^{[0,h],\tau'}(\mathbb{F}')$ together with an isomorphism $(\sigma^f)^*(\mathfrak{M}') \cong \mathfrak{M}'$ satisfying the cocycle condition as in [31, Definition 6.3]. By the principal series type case, we have the shape of $\mathfrak{M}'$, which is an element $\widetilde{w}' = (\widetilde{w}'_0, \widetilde{w}'_1, \dots, \widetilde{w}'_{f'-1}) \in (\widetilde{W}^\vee)^{\mathrm{Hom}(k',\mathbb{F})}$. By the isomorphism $(\sigma^f)^*(\mathfrak{M}') \cong \mathfrak{M}'$ and our requirement that eigenbases are compatible with it, the components of $\widetilde{w}'$ corresponding to two embeddings $k' \hookrightarrow \mathbb{F}$ are equal if they restrict to the same embedding $k \hookrightarrow \mathbb{F}$. In our numbering, this gives $\widetilde{w}'_j = \widetilde{w}'_{j+f}$. We then *define* the shape $\widetilde{w}$ of $\mathfrak{M}$ as the element $\widetilde{w} = (\widetilde{w}'_0, \widetilde{w}'_1, \dots, \widetilde{w}'_{f-1}) \in \underline{\widetilde{W}}^\vee = (\widetilde{W}^\vee)^{\mathrm{Hom}(k,\mathbb{F})}$.

Remark 3.2.12
Proposition 3.2.9 shows that the shape of a Kisin module is well defined. The shape of a Kisin module depends mildly on the choice of a lowest alcove presentation of τ (and the associated ordered characters χ_i in (3.1)). For a different choice of presentation, the shape changes by an outer automorphism of $\underline{W}_a$ coming from the action of the fundamental group (see [31, Corollary 2.24]). Everything we do depends on the choice of a lowest alcove presentation, and we will always fix it at the outset before talking about objects such as $A^{(j)} = \mathrm{Mat}_\beta(\phi^{(j)}_{\mathfrak{M},s'_{\mathrm{or},j+1}(n)})$, and so on.

Remark 3.2.13
The shape (Definition 3.2.19) is a kind of relative position between the two tame representations $\overline{\rho}^{ss}$ and $\overline{\tau}$. The shape is closely related to the geometry of the potentially crystalline deformation ring of type (η, τ) as studied in [31].

Recall the functor from Section 6.1 of [31]:

$$T^*_{dd} : Y^{[0,h],\tau}(R) \to \mathrm{Rep}_R(G_{K_\infty}).$$

Let $\lambda \in X^*(\underline{T})$ be effective, that is, $\lambda_j = (a_{i,j})$ with $a_{i,j} \geq 0$. We will need finer control of the shape of the Kisin module in the case when $\overline{\rho}$ is semisimple. For any λ that is effective, we have a closed substack $Y^{\lambda,\tau} \subset Y^{[0,h],\tau}$ constructed in [7, Proposition 5.2] (see also [31, Section 3.1]). Then for any finite extension $\mathbb{F}'/\mathbb{F}$, $Y^{\lambda,\tau}(\mathbb{F}')$ is the full subgroupoid of $Y^{[0,h],\tau}(\mathbb{F}')$ (for any sufficiently large h) consisting of Kisin modules whose shapes lie in $\mathrm{Adm}^\vee(\lambda)$ by [7, Proposition 5.4].

Definition 3.2.14
Let $\overline{\mathfrak{M}} \in Y^{[0,h],\tau}(\mathbb{F}')$, where $\mathbb{F}'/\mathbb{F}$ is a finite extension. We say that $\overline{\mathfrak{M}}$ is *semisimple* of shape $\widetilde{w} = (\widetilde{w}_j) \in \underline{\widetilde{W}}^\vee$ if there exists an eigenbasis β of $\overline{\mathfrak{M}}$ such that

$$A^{(j)} = \mathrm{Mat}_\beta(\phi^{(j)}_{\overline{\mathfrak{M}},s'_{\mathrm{or},j+1}(n)}) \in T\big(\mathbb{F}'[\![v]\!]\big)\widetilde{w}_j$$

for $0 \leq j \leq f' - 1$.

Remark 3.2.15
Since the set of monomial matrices (i.e., matrices that have at most one nonzero entry in each row and column) in $\mathcal{J}(\mathbb{F}')\widetilde{w}_j\mathcal{J}(\mathbb{F}')$ is exactly $T(\mathbb{F}'[\![v]\!])\widetilde{w}_j$, the above condition is equivalent to $A^{(j)}$ being a monomial matrix.

PROPOSITION 3.2.16
If $\overline{\mathfrak{M}}$ *is* semisimple *of shape* $\widetilde{w} = (\widetilde{w}_j) \in \underline{\widetilde{W}}^\vee$*, then there exists an eigenbasis* β *such that*

$$A^{(j)} \in T(\mathbb{F}')\widetilde{w}_j \quad \textit{for } 0 \leq j \leq f - 1.$$

Proof
By definition, there exists an eigenbasis β such that $A^{(j)} = D_j\widetilde{w}_j$ for $D_j \in T(\mathbb{F}'[\![v]\!])$. Let $\overline{D}_j = D_j \mod v$. For $0 \leq j \leq f' - 1$, set $I^{(j+1)} = \overline{D}_j D_j^{-1} \in \mathcal{J}(\mathbb{F}')$ with j considered mod f. Then $(I^{(j)}) \in \mathcal{J}(\mathbb{F}')^{f'}$ defines an f'-tuple as in Remark 3.2.10. By Proposition 3.2.9, there is an eigenbasis β_1 for $\overline{\mathfrak{M}}$ such that the matrix of partial Frobenius with respect to β_1 is

$$A_1^{(j)} = I^{(j+1)}A^{(j)}s_j^*(I^{(j),\varphi})^{-1}(s_j^*)^{-1} = \overline{D}_j\widetilde{w}_j s_j^*(I^{(j),\varphi})^{-1}(s_j^*)^{-1}.$$

Since $I^{(j)} \equiv 1 \mod v$ and is an element of $T(\mathbb{F}'[\![v]\!])$, $s_j^*(I^{(j),\varphi})^{-1}(s_j^*)^{-1} \equiv 1 \mod v^p$ and is an element of $T(\mathbb{F}'[\![v]\!])$. We conclude that $A_1^{(j)} = D_{1,j}\widetilde{w}_j$, where $D_{1,j} \equiv \overline{D}_j \mod v^p$. Repeating this process, we can inductively construct a sequence of eigenbases β_m such that the matrix of partial Frobenius with respect to β_m has the form $D_{m,j}\widetilde{w}_j$, where $D_{m,j} \equiv \overline{D}_j \mod v^{p^m}$. The sequence β_m converges to an eigenbasis with the desired property. □

COROLLARY 3.2.17
Let (s,μ) be a lowest alcove presentation for τ. If $\overline{\mathfrak{M}} \in Y^{[0,h],\tau}(\mathbb{F}')$ is semisimple of shape $\widetilde{w} \in \underline{\widetilde{W}}^\vee$, then $T^*_{dd}(\overline{\mathfrak{M}})$ is semisimple and (after extending coefficients)

$$T^*_{dd}(\overline{\mathfrak{M}})|_{I_K} \cong \mathbb{V}^*\big(\mathcal{M}(\widetilde{w}s^* t_{\mu^*+\eta^*})\big)|_{I_K} \cong \overline{\tau}(w, \nu+\eta),$$

where $\widetilde{w}s^* t_{\mu^*} = w^* t_{\nu^*}$.

Proof
The second isomorphism is from Proposition 3.1.2. This first isomorphism follows from a direct computation of the étale φ-module $\overline{\mathcal{M}} = (\overline{\mathfrak{M}}[1/u'])^{\Delta=1}$ as in [32, Proposition 3.1.2, Corollary 3.1.3]. We briefly go through the main points.

Let $\beta = (\beta^{(j)})$ be an eigenbasis for $\overline{\mathfrak{M}}$ as in Proposition 3.2.16. We start by considering a basis $\widetilde{\beta}'$ for $\overline{\mathcal{M}}' := (\overline{\mathfrak{M}}[1/u'])^{\Delta'=1}$ as follows: for $0 \le j' \le f'-1$, if $\beta^{(j')} = (f_1^{(j')}, \dots, f_n^{(j')})$, define $\widetilde{\beta}'^{,(j')} = ((u')^{\mathbf{a}'^{(j')}_{(s,\mu),1}} f_1^{(j')}, \dots, (u')^{\mathbf{a}'^{(j')}_{(s,\mu),n}} f_n^{(j')})$, which is a basis for $\overline{\mathcal{M}}'^{(j')}$. The matrix for $\phi^{(j')}_{\overline{\mathcal{M}}'} : \overline{\mathcal{M}}'^{(j')} \to \overline{\mathcal{M}}'^{(j'+1)}$ with respect to $\widetilde{\beta}'$ is given by

$$s'_{\mathrm{or},j'+1} A^{(j')} (s'_{\mathrm{or},j'+1})^{-1} (u')^{p\mathbf{a}'^{(j')}_{(s,\mu)} - \mathbf{a}'^{(j'+1)}_{(s,\mu)}}.$$

Since $p\mathbf{a}'^{(j')}_{(s,\mu)} - \mathbf{a}'^{(j'+1)}_{(s,\mu)} = (p^{f'}-1)\boldsymbol{\alpha}'_{(s,\mu),f'-1-j'}$, this is the same as

$$s'_{\mathrm{or},j'+1} A^{(j')} (s'_{\mathrm{or},j'+1})^{-1} v^{\boldsymbol{\alpha}'_{(s,\mu),f'-1-j'}}.$$

Define $\widetilde{\beta}$ by $\widetilde{\beta}^{(j')} = \widetilde{\beta}'^{(j')} s'_{\mathrm{or},j'}$. Let $j' = j + if$ for $0 \le j \le f-1$. Then the matrix for $\phi^{(j')}_{\overline{\mathcal{M}}'}$ with respect to $\widetilde{\beta}$ is given by

$$A^{(j')} (s'_{\mathrm{or},j'+1})^{-1} s'_{\mathrm{or},j'} v^{(s'_{\mathrm{or},j'})^{-1}(\boldsymbol{\alpha}'_{(s,\mu),f'-1-j'})} = A^{(j')} s_j^* v^{\mu_j^*+\eta_j^*}$$

using (3.2) and (3.3). Since $(\sigma^f)^*(\widetilde{\beta}^{(j')}) = \widetilde{\beta}^{(j'-f)}$, this descends to a basis of $\overline{\mathcal{M}} := (\overline{\mathfrak{M}}[1/u'])^{\Delta=1} = (\overline{\mathcal{M}}')^{\sigma^f=1}$ such that Frobenius $\phi^{(j)}_{\overline{\mathcal{M}}} : \overline{\mathcal{M}}^{(j)} \to \overline{\mathcal{M}}^{(j+1)}$ is given by

$$A^{(j)} s_j^* v^{\mu_j^*+\eta_j^*} = D_j \widetilde{w}_j s_j^* v^{\mu_j^*+\eta_j^*}$$

for $D = (D_j) \in T(\mathbb{F}')^f$ using Proposition 3.2.16. Thus, $\overline{\mathcal{M}} \cong \mathcal{M}(\widetilde{w}s^* t_{\mu^*+\eta^*}, D)$ (Definition 3.1.1). □

Before proving the main theorems of the section, we show that, in generic situations, the Kisin module which gives rise to $\overline{\rho}$ is necessarily unique (if one exists); that is, the Kisin variety is trivial.

PROPOSITION 3.2.18

Assume that τ is generic.

(1) *Let $\overline{\mathfrak{M}}, \overline{\mathfrak{M}}' \in Y^{\eta,\tau}(\mathbb{F}')$ for some finite extension $\mathbb{F}'/\mathbb{F}$. If $T^*_{dd}(\overline{\mathfrak{M}}) \cong T^*_{dd}(\overline{\mathfrak{M}}')$, then $\overline{\mathfrak{M}} \cong \overline{\mathfrak{M}}'$.*

(2) *Let $\overline{\mathfrak{M}} \in Y^{\eta,\tau}(\mathbb{F}')$, and let $\overline{\rho} := T^*_{dd}(\overline{\mathfrak{M}})$. Define a groupoid*

$$\mathfrak{t}_{\overline{\mathfrak{M}}} = \left\{(\mathfrak{M}, \delta_0) \;\middle|\; \mathfrak{M} \in Y^{\eta,\tau}\left(\mathbb{F}'[\varepsilon]/\varepsilon^2\right), \delta_0 : \mathfrak{M}/\varepsilon\mathfrak{M} \xrightarrow{\sim} \overline{\mathfrak{M}}\right\}.$$

Then the functor T^*_{dd} induces a fully faithful functor

$$\begin{aligned} T^*_{\tan} : \mathfrak{t}_{\overline{\mathfrak{M}}} &\to \mathrm{Rep}_{\mathbb{F}'[\varepsilon]/\varepsilon^2}(G_{K_\infty})_{\overline{\rho}} \\ &= \left\{(\rho, \gamma_0) \;\middle|\; \rho \in \mathrm{Rep}_{\mathbb{F}'[\varepsilon]/\varepsilon^2}(G_{K_\infty}), \gamma_0 : \rho \mod \varepsilon \xrightarrow{\sim} \overline{\rho}\right\}. \end{aligned}$$

Proof

We fix a lowest alcove presentation (s, μ) of τ such that μ is n-deep in alcove $\underline{C}_0$ to perform all calculations. For part (1), since the G_{K_∞}-representations are isomorphic, we have $\overline{\mathfrak{M}}[1/u'] \cong \overline{\mathfrak{M}}'[1/u']$ as étale φ-modules with descent datum. We pick two eigenbases β, β' of $\overline{\mathfrak{M}}, \overline{\mathfrak{M}}'$, and let $(D^{(j)}) \in (\mathrm{GL}_n(\mathbb{F}'((u'))))^{f'}$ be the f'-tuple which expresses the basis β' in terms of β as in Proposition 3.2.9. Note that $D^{(j)} = D^{(j+f)}$, since our eigenbases are compatible with the action of σ^f. The same computation in the proof of [31, Theorem 3.2] with 2 replaced everywhere by $n-1$ (see Remark 2.2.8 for the comparison of the genericity in the present paper with that of [31]) now shows that $D^{(j)} \in \mathrm{GL}_n(\mathbb{F}'[\![u']\!])$ for all j.

Part (2) is similar to part (1). The argument in the proof of [31, Proposition 3.4] adapts to our situation. □

Definition 3.2.19

Assume that τ is generic with a chosen lowest alcove presentation (s, μ). If there exists $\overline{\mathfrak{M}} \in Y^{\eta,\tau}(\mathbb{F})$ such that $T^*_{dd}(\overline{\mathfrak{M}}) \cong \overline{\rho}|_{G_{K_\infty}}$, then define $\widetilde{w}(\overline{\rho}, \tau) \in \mathrm{Adm}^\vee(\eta)$ to be the shape of $\overline{\mathfrak{M}}$. This is well defined by Proposition 3.2.18.

A key input for weight elimination is the following.

THEOREM 3.2.20

Assume that either (1) τ is a regular principal series type or (2) $\lambda = \eta$ and τ is generic. If $\overline{\rho}$ has a potentially crystalline lift with type (λ, τ) where λ is effective, then there is a Kisin module $\overline{\mathfrak{M}} \in Y^{\lambda,\tau}(\mathbb{F})$ such that $T^*_{dd}(\overline{\mathfrak{M}}) \cong \overline{\rho}|_{G_{K_\infty}}$.

Proof

If τ is a principal series type, this is direct consequence of [7, Proposition 5.4 and

Corollary 5.18] (also see [7, Remark 5.6], which compares the stratification of the moduli of the Kisin module with Definition 3.2.11). Proposition 5.4 in [7] crucially uses results about local models from [37].

For case (2), by the case (1), there exists a Kisin module $\overline{\mathfrak{M}}' \in Y^{\eta,\tau'}(\mathbb{F})$ with $T^*_{dd,K'}(\overline{\mathfrak{M}}') = \overline{\rho}|_{G_{K'_\infty}}$, where $T^*_{dd,K'}$ is the analogous functor over K'. Since $\overline{\rho}$ extends to G_{K_∞}, we have an isomorphism $\iota : (\sigma^f)^*(\overline{\mathfrak{M}}'[1/u']) \cong \overline{\mathfrak{M}}'[1/u']$ (satisfying an appropriate cocycle condition; see [31, Section 6.1]). Since τ is generic, by the proof of Proposition 3.2.18, $\iota((\sigma^f)^*(\overline{\mathfrak{M}}')) = \overline{\mathfrak{M}}'$ inside $\overline{\mathfrak{M}}'[1/u']$. Thus, $\overline{\mathfrak{M}}'$ defines an element of $Y^{\eta,\tau}(\mathbb{F})$ and hence has shape in $\mathrm{Adm}^\vee(\eta)$. □

Remark 3.2.21

Our definition of $\mathrm{Adm}^\vee(\lambda)$ (Definition 2.1.1) is in terms of the Bruhat order on the (extended) affine Weyl group for GL_n using our choice of antidominant base alcove for the standard apartment. The Bruhat order which appears in [7, Section 5] comes from the closure relation on the affine flag variety with respect to the standard Iwahori subgroup, the subgroup $\mathcal{I}(\mathbb{F})$ of matrices which are upper triangular mod v. The subgroup $\mathcal{I}(\mathbb{F})$ is the stabilizer of the antidominant base alcove; hence the Bruhat order (and hence the admissible set as well) in [7, Section 5] is the same as ours.

The following proposition gives us control on $Y^{\eta,\tau}(\mathbb{F}')$.

PROPOSITION 3.2.22

Assume that τ is generic. Let $\overline{\mathfrak{M}} \in Y^{[0,n-1],\tau}(\mathbb{F}')$, let β be an eigenbasis of $\overline{\mathfrak{M}}$, and let $A^{(j)} = \mathrm{Mat}_\beta(\phi^{(j)}_{\overline{\mathfrak{M}},s'_{\mathrm{or},j+1}(n)})$. Then the assignment $\beta' \mapsto (\mathrm{Mat}_{\beta'}(\phi^{(j)}_{\overline{\mathfrak{M}},s'_{\mathrm{or},j+1}(n)}))_{0\le j<f'}$ defines a bijection between the set of eigenbases β' such that $\beta' = \beta \bmod u'$ and the set tuples of the form $(X_j A^{(j)})_{0\le j<f'}$ such that

- *$X_j \in \mathcal{I}_1(\mathbb{F}')$ for all j;*
- *$X_j = X_k$ if $j \equiv k \bmod f$.*

Proof

Throughout this proof, we adopt the same notation as in Proposition 3.2.9: we let $(D^{(j)})$ be the tuple of matrices expressing β in terms of β', from which we get the matrices $I^{(j)}$, $I^{(j),\varphi}$. Observe that the condition $\beta' \equiv \beta \bmod u'$ is equivalent to $D^{(j)} \equiv 1 \bmod u'$ and also equivalent to $I^{(j)} \in \mathcal{I}_1(\mathbb{F}')$. If these equivalent conditions hold, then $I^{(j),\varphi} \equiv 1 \bmod v^{n+1}$, since τ is assumed to be generic. We also observe that if $j \equiv k \bmod f$, then the jth and kth components of any tuple of matrices that we work with are equal.

Since $\overline{\mathfrak{M}} \in Y^{[0,n-1],\tau}(\mathbb{F}')$ we have $v^{n-1}(A^{(j)})^{-1} \in M_n(\mathbb{F}'[\![v]\!])$. This gives us a bound on the denominators of $(A^{(j)})^{-1}$.

First we check that our assignment is actually defined, in the sense that the collection $A'^{(j)} = \mathrm{Mat}_{\beta'}(\phi^{(j)}_{\overline{\mathfrak{M}},s'_{\mathrm{or},j+1}(n)})$ associated to β' is really of the prescribed form. By Proposition 3.2.9 we have

$$X_j = A'^{(j)}(A^{(j)})^{-1} = I^{(j+1)}A^{(j)}\big(s_j^*(I^{(j),\varphi})(s_j^*)^{-1}\big)(A^{(j)})^{-1}.$$

It is clear that $X_j = X_k$ if $j \equiv k \bmod f$. As observed above, $I^{(j),\varphi} \equiv 1 \bmod v^{n+1}$; hence we can write $s_j^*(I^{(j),\varphi})(s_j^*)^{-1} = 1 + v^{n+1}Y_j$ with $Y_j \in M_n(\mathbb{F}'[\![v]\!])$. Since $v^{n-1}(A^{(j)})^{-1} \in M_n(\mathbb{F}'[\![v]\!])$, we get

$$X_j = A'^{(j)}(A^{(j)})^{-1} = I^{(j+1)}\big(1 + v^{n+1}A^{(j)}Y_j(A^{(j)})^{-1}\big) \in \mathcal{J}_1(\mathbb{F}')$$

as desired.

Next, we show that our assignment is surjective. Thus we are given (X_j) with $X_j \in \mathcal{J}_1(\mathbb{F}')$, and we need to solve the system of equations

$$X_j = I^{(j+1)}A^{(j)}\big(s_j^*(I^{(j),\varphi})(s_j^*)^{-1}\big)(A^{(j)})^{-1}$$

with $I^{(j)} \in \mathcal{J}_1(\mathbb{F}')$. To do this, we carry out the same limiting procedure as in the proof of Lemma 2.20 of [31], using $I^{(j),\varphi} \equiv 1 \bmod v^{n+1}$ and $v^{n-1}(A^{(j)})^{-1} \in M_n(\mathbb{F}'[\![v]\!])$ to establish convergence to a solution.

Finally, we show that our assignment is injective. This means that we have to show that if there are two collections $(I^{(j)}), (I'^{(j)}) \in \mathcal{J}_1(\mathbb{F}')^{f'}$ corresponding to eigenbases β', β'' such that

$$I^{(j+1)}A^{(j)}\big(s_j^*(I^{(j),\varphi})(s_j^*)^{-1}\big)(A^{(j)})^{-1} = I'^{(j+1)}A^{(j)}\big(s_j^*(I'^{(j),\varphi})(s_j^*)^{-1}\big)(A^{(j)})^{-1},$$

then $I^{(j)} = I'^{(j)}$ for all j. By replacing β with β'' and $A^{(j)}$ with $A''^{(j)}$, we reduce to the case when $I'^{(j)} = 1$. Thus, it suffices to show that if

$$I^{(j+1)}A^{(j)}\big(s_j^*(I^{(j),\varphi})(s_j^*)^{-1}\big)(A^{(j)})^{-1} = 1,$$

then $I^{(j)} = 1$ for all j.

Indeed, by the observations at the beginning of the proof,

$$A^{(j)}(s_j^*(I^{(j),\varphi})(s_j^*)^{-1})(A^{(j)})^{-1} \equiv 1 \bmod v^2$$

for all j; thus, $I^{(j)} \equiv 1 \bmod v^2$ for all j. Suppose that we already have $I^{(j)} \equiv 1 \bmod v^{\delta}$ for some $\delta \geq 2$ and for all j. Then $I^{(j),\varphi} \equiv 1 \bmod v^{p\delta-n-1}$, and hence $A^{(j)}(s_j^*(I^{(j),\varphi})(s_j^*)^{-1})(A^{(j)})^{-1} \equiv 1 \bmod v^{p\delta-2n}$ for all j. Hence, also $I^{(j)} \equiv 1 \bmod v^{p\delta-2n}$ for all j. Since $p\delta - 2n > \delta$ (the existence of a generic τ implies $p > 2n+1$), this shows that $I^{(j)} \equiv 1$ mod arbitrarily high powers of v for all j. This shows $I^{(j)} = 1$ for all j. □

We now discuss the notion of a gauge basis, which provides certain normal forms for Kisin modules. For a root $\alpha = \epsilon_i - \epsilon_j$ of GL_n, recall that the (i,j)th entry of an $n \times n$ matrix A is also called the αth entry and is denoted by A_α. For any statement $\star$, define $\delta_\star$ to be 1 if $\star$ is true and 0 if $\star$ is false.

For any $\widetilde{w} = wt_\nu \in \widetilde{W}^\vee$ and any ring R, we define the subset $U_{\widetilde{w}}(R) \subset \mathrm{GL}_n(R((v)))$ to be the set of $X \in \mathrm{GL}_n(R((v)))$ satisfying the following conditions.

- The diagonal entries of X are in $R^\times$.
- For any root α of GL_n, the αth entry X_α is of the form $\sum_i a_i v^i \in R((v))$, where $a_i = 0$ unless $\delta_{\alpha<0} \le i < -\langle \nu, \alpha^\vee \rangle + \delta_{w(\alpha)<0}$. In particular, $X_\alpha = 0$ unless $\delta_{\alpha<0} < -\langle \nu, \alpha^\vee \rangle + \delta_{w(\alpha)<0}$.

By a standard computation with affine root groups, the natural map

$$\widetilde{w} U_{\widetilde{w}}(\mathbb{F}') \to \mathcal{I}_1(\mathbb{F}') \backslash \mathcal{I}(\mathbb{F}') \widetilde{w} \mathcal{I}(\mathbb{F}')$$

is a bijection, for any extension $\mathbb{F}'$ of $\mathbb{F}$.

The following definition generalizes Definition 2.22 of [31].

Definition 3.2.23
Let $\overline{\mathfrak{M}} \in Y^{\eta,\tau}(\mathbb{F}')$ with shape $(\widetilde{w}_j)$. A *gauge basis* β for $\overline{\mathfrak{M}}$ is an eigenbasis for $\overline{\mathfrak{M}}$ such that $A^{(j)} = \mathrm{Mat}_\beta(\phi^{(j)}_{\mathfrak{M}, s'_{\mathrm{or}, j+1}(n)})$ belongs to $\widetilde{w}_j U_{\widetilde{w}_j}(\mathbb{F}')$ for all $0 \le j < f$.

Example 3.2.24
When $n = 3$, the list of η-admissible elements $\mathrm{Adm}^\vee(\eta)$ is given in [31, Table 1]. For each $\widetilde{w} \in \mathrm{Adm}^\vee(\eta)$ up to outer automorphism, [31, Table 4] lists the set $\widetilde{w}_j U_{\widetilde{w}_j}(\mathbb{F})$ which consists of the possible matrices of partial Frobenii for pairs $(\overline{\mathfrak{M}}, \overline{\beta})$ with shape $\widetilde{w}$, where $\overline{\beta}$ is a gauge basis.

For τ generic, Proposition 3.2.22 shows that gauge bases exist and are unique up to scaling by the subgroup of $T(\mathbb{F}')^{f'}$ consisting of tuples whose jth and kth entries are the same for $j \equiv k \bmod f$.

Remark 3.2.25
Assume that τ is generic. If $\overline{\mathfrak{M}} \in Y^{\eta,\tau}(\mathbb{F}')$ is semisimple of shape $\widetilde{w} = (\widetilde{w}_j)$, then Proposition 3.2.16 shows that there is an eigenbasis β with the property that $A^{(j)} = \mathrm{Mat}_\beta(\phi^{(j)}_{\mathfrak{M}, s'_{\mathrm{or}, j+1}(n)})$ belongs to $T(\mathbb{F}')\widetilde{w}_j = \widetilde{w}_j T(\mathbb{F}') \subset \widetilde{w}_j U_{\widetilde{w}_j}(\mathbb{F}')$. Such an eigenbasis is therefore a gauge basis, and we deduce that the matrices of partial Frobenii with respect to any gauge basis have this form. In particular, they are all monomial matrices. Conversely, if there is a gauge basis for which the matrices of partial Frobenii is monomial, then $\overline{\mathfrak{M}} \in Y^{\eta,\tau}(\mathbb{F}')$ is semisimple, by Remark 3.2.15.

THEOREM 3.2.26

Let $\overline{\rho}: G_K \to \mathrm{GL}_n(\mathbb{F})$ be a semisimple representation, and let τ be a 1-generic type. Assume that either (1) $\overline{\rho}$ is a direct sum of characters, or (2) $\lambda = \eta$ and τ is generic. If there exists a Kisin module $\overline{\mathfrak{M}} \in Y^{\lambda,\tau}(\mathbb{F})$ such that $T^*_{dd}(\overline{\mathfrak{M}}) \cong \overline{\rho}|_{G_{K_\infty}}$, then there exist a finite extension $\mathbb{F}'/\mathbb{F}$ and a semisimple Kisin module $\overline{\mathfrak{M}}' \in Y^{\lambda,\tau}(\mathbb{F}')$ with shape in $\mathrm{Adm}^\vee(\lambda)$ (see Definition 3.2.14) such that (after extending scalars) $T^*_{dd}(\overline{\mathfrak{M}}') \cong \overline{\rho}|_{G_{K_\infty}}$. Furthermore, we can take $\mathbb{F}' = \mathbb{F}$ in case (2).

Proof

Let $\mathcal{M}_{dd} = \overline{\mathfrak{M}}[1/u']$. It is an étale φ-module over L' with descent datum to K.

We first treat the case where $\overline{\rho}$ is a direct sum of characters. Since $\mathbb{V}^*_{dd}$ is an equivalence of categories and $\overline{\rho}$ is a direct sum of characters, $\mathcal{M}_{dd} = \bigoplus_{i=1}^n \mathcal{M}_i$, where each $\mathcal{M}_i$ has rank 1, is stable under $\phi_{\mathcal{M}_{dd}}$ and the descent datum.

Let $Y^{\lambda,\tau}_{\mathcal{M}_{dd}}$ be the Kisin variety parameterizing lattices in $\mathcal{M}_{dd}$ which lie in $Y^{\lambda,\tau}$, that is, which have shape in $\mathrm{Adm}^\vee(\lambda)$. When τ is principal series type, $Y^{\lambda,\tau}_{\mathcal{M}_{dd}}$ is defined as in Definition 3.1 in [31] and is shown there to be a projective scheme over $\mathbb{F}$. In general, we define $Y^{\lambda,\tau}_{\mathcal{M}_{dd}}$ to be the closed subscheme of fixed points on $Y^{\lambda,\tau'}_{\mathcal{M}_{dd}}$ for the natural action of σ^f. Note that by assumption $Y^{\lambda,\tau}_{\mathcal{M}_{dd}}$ is nonempty.

The torus $T = \mathbb{G}_m^n$ acts on $\mathcal{M}_{dd}$ by scaling individually in each factor of the above decomposition. As a consequence, we get an algebraic action of T on the projective variety $Y^{\lambda,\tau}_{\mathcal{M}_{dd}}$. Any such action has a fixed point (possibly after passing to a finite extension $\mathbb{F}'$ of $\mathbb{F}$). Let $\overline{\mathfrak{M}}' \subset Y^{\lambda,\tau}_{\mathcal{M}_{dd}}(\mathbb{F}')$ be a T-fixed point. Let $\psi_i : T \to \mathbb{G}_m$ denote projection onto the ith coordinate, and set $\overline{\mathfrak{M}}'_i := (\overline{\mathfrak{M}}')^{\psi_i}$. Then

$$\overline{\mathfrak{M}}' = \bigoplus_{i=1}^n \overline{\mathfrak{M}}'_i. \tag{3.7}$$

Since the T-action commutes with $\phi_{\mathcal{M}_{dd}}$ and Δ, each $\overline{\mathfrak{M}}'_i$ is stable under both; hence $\overline{\mathfrak{M}}'_i$ is in fact a rank 1 Kisin module with descent datum. Any choice of eigenbasis which respects this decomposition shows that $\overline{\mathfrak{M}}'$ is semisimple. Because $\overline{\mathfrak{M}}'$ is in $Y^{\lambda,\tau}(\mathbb{F}')$, it is semisimple with an admissible shape $\widetilde{w} \in \mathrm{Adm}^\vee(\lambda)$.

Now suppose that τ is generic, but $\overline{\rho}$ is not necessarily a direct sum of characters. In this case, $\overline{\mathfrak{M}}$ is the unique element of $Y^{\eta,\tau}(\mathbb{F})$ such that $T^*_{dd}(\overline{\mathfrak{M}}) \cong \overline{\rho}|_{G_{K_\infty}}$ by Proposition 3.2.18. We pick an unramified extension $\breve{K}/K'$ such that $\overline{\rho}|_{G_{\breve{K}}}$ is a direct sum of characters. Let $\breve{\tau}$ be the base change of τ to $\breve{K}$, and let $\breve{\overline{\mathfrak{M}}} = \overline{\mathfrak{M}} \otimes_{W(k')} W(\breve{k})$ be the base change of the Kisin module $\overline{\mathfrak{M}}$ to $\breve{K}L'$. Since $\breve{\tau}$ is still generic by Lemma 3.3.1, $\breve{\overline{\mathfrak{M}}}$ is the unique lattice in $\breve{\overline{\mathfrak{M}}}[1/u']$ which belongs to $Y^{\eta,\breve{\tau}}(\mathbb{F})$. By the above argument, the set of such Kisin lattices must have a semisimple element (pos-

sibly after extending $\mathbb{F}$); thus $\overline{\breve{\mathfrak{M}}}$ is semisimple. Fix a gauge basis β of $\overline{\mathfrak{M}}$, and let $\breve{\beta}$ be the induced eigenbasis of $\overline{\breve{\mathfrak{M}}}$. It is easy to check that it is a gauge basis of $\overline{\breve{\mathfrak{M}}}$. By Remark 3.2.25, the matrices of partial Frobenii of $\overline{\breve{\mathfrak{M}}}$ with respect to $\breve{\beta}$ are monomial; hence the same is true for the matrices of partial Frobenii of $\overline{\mathfrak{M}}$ with respect to β. This shows that $\overline{\mathfrak{M}} \in Y^{\eta,\tau}(\mathbb{F})$ is semisimple. □

Remark 3.2.27

In the proof of Theorem 3.2.26, while Δ' acts on $(\overline{\mathfrak{M}}'_1)^{(j)} \mod u'$ through one of the characters of τ, it need not be the same character for each j. This is why, even though $\phi_{\overline{\mathfrak{M}}}$ is "diagonal," the individual $A^{(j)}$'s need not be diagonal, but only monomial.

Proof of Theorem 3.2.1

If $\overline{\rho}$ is the reduction of a potentially crystalline representation of type (λ, τ), by a standard argument (see, e.g., [12, Lemma 5]), $\overline{\rho}^{\mathrm{ss}}$ is also the reduction (after possible extending scalars) of a potentially crystalline representation of type (λ, τ). So, without loss of generality, we assume that $\overline{\rho}$ is semisimple. After twisting, we may assume that λ is effective. Then, by Theorem 3.2.20 combined with Theorem 3.2.26, after possibly replacing $\mathbb{F}$ by a finite extension, there exists a semisimple Kisin module $\overline{\mathfrak{M}} \in Y^{\lambda,\tau}(\mathbb{F})$ of shape $\widetilde{w} \in \mathrm{Adm}^{\vee}(\lambda)$ such that $\overline{\rho}|_{G_{K_\infty}} \cong T^*_{dd}(\overline{\mathfrak{M}})$.

If we write

$$\widetilde{w} s^* t_{\mu^*+\eta^*} = w^* t_{\nu^*+\eta^*}$$

for $(w, \nu) \in W(\underline{G}) \times X^*(\underline{T})$, then, by Corollary 3.2.17,

$$\overline{\rho}^{\mathrm{ss}}|_{I_K} \cong \overline{\tau}(w, \nu + \eta).$$

Furthermore, by Lemma 2.1.4,

$$\widetilde{w}^* = s^{-1} t_{\nu-\mu} w \in \mathrm{Adm}(\lambda).$$ □

3.3. Genericity conditions

LEMMA 3.3.1

Let K'/K be an unramified extension. If τ is an m-generic tame inertial type, then $\tau|_{I_{K'}}$ is also m-generic. Similarly, if $\overline{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F})$ is an m-generic representation, then $\overline{\rho}|_{G_{K'}}$ is m-generic.

Proof

Let r be the degree of K'/K. Let $\tau_{K'}$ denote the analogous construction as in Definition 2.2.1 with K replaced by K' (using the compatible $(p^{r'f} - 1)$-system

of roots of $-p$ from Section 1.1 for all r' and an embedding $\sigma_0' : K' \to E$ extending σ_0). Then, for $(w', \mu') \in W(\underline{G})^r \times X^*(\underline{T})^r$, there is a tame inertial type $\tau_{K'}(w', \mu') : I_{K'} \to \mathrm{GL}_n(\mathcal{O})$. For $(w, \mu) \in W(\underline{G}) \times X^*(\underline{T})$, let $\iota(w, \mu) = ((w, w, \dots, w), (\mu, \mu, \dots, \mu)) \in W(\underline{G})^r \times X^*(\underline{T})^r$. Then,

$$\tau(w, \mu)|_{I_{K'}} \cong \tau_{K'}\big(\iota(w, \mu)\big). \tag{3.8}$$

This can be checked by direct computation. Alternatively, one can appeal to Proposition 3.1.2, which says, in particular, that $\mathbb{V}^*(\mathcal{M}(w^* t_{\mu^*}, D))|_{I_K} \cong \overline{\tau}(w, \mu)$ for any $D \in \underline{T}(\mathbb{F})$. If k' is the residue field of k, then restricting to $G_{K'}$ corresponds to tensoring with k' on the étale φ-module side. We see immediately that

$$\begin{aligned} &\mathcal{M}(w^* t_{\mu^*}, D) \otimes_k k' \\ &\quad = \mathcal{M}\big((w^* t_{\mu^*}, w^* t_{\mu^*}, \dots, w^* t_{\mu^*}), D'\big) \quad \text{for } D' = (D, \dots, D) \in \underline{T}(\mathbb{F})^r \end{aligned}$$

and so $\overline{\tau}(w, \mu)|_{I_K} \cong \overline{\tau}_{K'}(\iota(w, \mu))$. Applying Teichmüller lifts yields (3.8).

Let $\underline{G}' = \mathrm{Res}_{k'/k} \underline{G}$. Let $\underline{C}_0'$ be the analogous lowest alcove with k replaced by k' viewed as a subset of $X^*(\underline{T})^r \otimes \mathbb{R}$. Let $\tau \cong \tau(s, \lambda + \eta)$ with λ m-deep in $\underline{C}_0$. Then clearly $\iota(s, \lambda)$ is a lowest alcove presentation with $\iota(\lambda)$ m-deep in $\underline{C}_0'$. The argument for $\overline{\rho}$ is the same. □

PROPOSITION 3.3.2

Let τ be a 1-generic tame inertial type. Assume that $\overline{\rho}$ is semisimple and m-generic where $m > n$. If $\overline{\rho}$ is the reduction of a potentially crystalline representation of type (η, τ), then τ is $(m - n)$-generic. In particular, if $\overline{\rho}$ is $2n$-generic, then τ is generic.

Proof

Let (s, μ) be a lowest alcove presentation for τ. First, assume that $\overline{\rho}$ is a direct sum of characters, and assume that τ is a principal series. By Theorem 3.2.1, we have

$$\overline{\rho}|_{I_K} \cong \overline{\tau}(w, \mu' + \eta)$$

with $s^{-1} t_{\mu' - \mu} w \in \mathrm{Adm}(\eta)$. Define $\nu = \mu' - \mu$. By Lemma 2.1.5, for any $\alpha^\vee \in \underline{R}^\vee$ we have

$$\big|\langle \nu, \alpha^\vee \rangle\big| \leq n - 1. \tag{3.9}$$

Since $\overline{\rho}$ is m-generic, there exists $(\nu', \sigma) \in X^*(\underline{T}) \times W(\underline{G})$ such that

$$\sigma(\mu' + \eta) + p\nu' - \sigma w \pi(\sigma)^{-1} \pi(\nu') - \eta$$

is m-deep in alcove $\underline{C}_0$. Hence, for any $\alpha^\vee \in \underline{R}^\vee$,

$$m < \left|\left\langle \sigma(\mu' + \eta) + p\nu' - \sigma w\pi(\sigma^{-1})\pi(\nu'), \alpha^\vee \right\rangle\right| < p - m. \tag{3.10}$$

Since μ is in $\underline{C}_0$,

$$\left|\langle \mu' + \eta, \sigma^{-1}\alpha^\vee\rangle\right| = \left|\langle \mu + \eta + \nu, \sigma^{-1}\alpha^\vee\rangle\right| < p + n - 1.$$

It follows that

$$\left|\left\langle p\nu' - \sigma w\pi(\sigma^{-1})\pi(\nu'), \alpha^\vee \right\rangle\right| < 2p + n - m - 1$$

for all $\alpha^\vee \in \underline{R}^\vee$. Let $M = \max_{\alpha^\vee \in \underline{R}^\vee}\{|\langle \nu', \alpha^\vee\rangle|\}$. By choosing $\alpha^\vee$ so that $|\langle \nu', \alpha^\vee\rangle| = M$ in the above inequality, we get $(p-1)M < 2p + n - m - 1$; hence, $M < 2$. Thus, we have shown $|\langle \nu', \alpha^\vee\rangle| \le 1$ for all $\alpha^\vee \in \underline{R}^\vee$. Hence, for all $\alpha^\vee \in \underline{R}^\vee$,

$$\left|\left\langle \sigma w\pi(\sigma^{-1})\pi(\nu'), \alpha^\vee \right\rangle\right| \le 1. \tag{3.11}$$

Since

$$\sigma(\mu' + \eta) + p\nu' - \sigma w\pi(\sigma^{-1})\pi(\nu') = \sigma(\mu + \eta) + \sigma(\nu) + p\nu' - \sigma w\pi(\sigma^{-1})\pi(\nu'),$$

inequalities (3.9) (with $\alpha^\vee$ replaced by $\sigma^{-1}\alpha^\vee$), (3.10), (3.11), and the equality $\mu' = \mu + \nu$ together imply that

$$m - n < \left|\left\langle \sigma(\mu + \eta) + p\nu', \alpha^\vee \right\rangle\right| = \left|\langle \mu + \eta, \sigma^{-1}\alpha^\vee\rangle + p\langle \nu', \alpha^\vee\rangle\right| < p - m + n$$

for all coroots $\alpha^\vee$. It follows that, for any coroot $\alpha^\vee$, the integer $\langle \mu + \eta, \alpha^\vee\rangle$ is not congruent modulo p to any integer between $n - m$ and $m - n$. But since $\mu \in \underline{C}_0$, for any *positive* coroot $\alpha^\vee$ we also have that $0 < \langle \mu + \eta, \alpha^\vee\rangle < p$, and thus we in fact have

$$m - n < \langle \mu + \eta, \alpha^\vee\rangle < p - m + n.$$

This shows that τ is $(m-n)$-generic.

Finally, if $\overline{\rho}$ is not the direct sum of characters or τ is not a principal series, let K'/K be an unramified extension over which both become true. By Lemma 3.3.1, $\overline{\rho}|_{G_{K'}}$ is m-generic. We will use notation from the proof of Lemma 3.3.1. By (3.8), $\iota(s, \mu)$ is a lowest alcove presentation for $\tau|_{I_{K'}}$. The same argument as above with K replaced by K' shows that $\iota(\mu)$ is $(m-n)$-deep in $\underline{C}'_0$. Hence, μ is $(m-n)$-deep in $\underline{C}_0$. □

3.4. Potential diagonalizability

Let $\overline{\rho}$ be a tame representation. The goal of this section is to show that, for certain sufficiently generic tame types τ, all potentially crystalline lifts of $\overline{\rho}$ of type (η, τ) are potentially diagonalizable in the sense of [3]. The main theorem is the following.

THEOREM 3.4.1
Let $\tau \cong \tau(s,\mu)$ be a tame type with a lowest alcove presentation $(s,\mu-\eta)$ such that $\mu-\eta$ is $(2n-1)$-deep in $\underline{C}_0$. Let $\overline{\rho}: G_K \to \mathrm{GL}_n(\mathbb{F})$ be a semisimple representation. Assume that there exists $\overline{\mathfrak{M}} \in Y^{\eta,\tau}(\mathbb{F})$ of shape $(t_{w_j(\eta_0)})_j$, where $w_j \in W(\mathrm{GL}_n)$ and such that $T^*_{dd}(\overline{\mathfrak{M}}) \cong \overline{\rho}|_{G_{K_\infty}}$. Then the framed potentially crystalline deformation ring $R^{\eta,\tau}_{\overline{\rho}}$ is nonzero and formally smooth.

Remark 3.4.2
The proof of Theorem 3.4.1 uses the techniques introduced in [31] for $n=3$. When $n=3$, the shapes $t_{w_j(\eta_0)}$ as in the theorem correspond to the shapes $\alpha\beta\alpha\gamma$ and $\beta\gamma\alpha\gamma$ (and their cyclic permutations) in the tables at the end of [31]. In [31, Table 6], the reader can see that the deformation ring for these shapes is formally smooth.

For the rest of the section, we will be in the setting of the theorem. By Theorem 3.2.26 and the uniqueness of $\overline{\mathfrak{M}}$ (see Theorem 3.2.18), $\overline{\mathfrak{M}}$ is semisimple. We fix a gauge basis $\overline{\beta}$ of $\overline{\mathfrak{M}}$. By Remark 3.2.25, for all j, the matrices of the jth partial Frobenius with respect to $\overline{\beta}$ have the form

$$\overline{A}^{(j)} = \overline{D}_j v^{w_j(\eta_0)} \tag{3.12}$$

with $\overline{D}_j \in T(\mathbb{F})$.

We will need the following result, which gives a normal form for deformations of the pair $(\overline{\mathfrak{M}}, \overline{\beta})$

PROPOSITION 3.4.3
Let R be complete local Noetherian $\mathcal{O}$-algebra with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{F}$, and let $\tau \cong \tau(s,\mu)$ be a type with $\mu-\eta$ n-deep in $\underline{C}_0$. Let $\mathfrak{M} \in Y^{[0,n-1],\tau}(R)$ be such that $\mathfrak{M} \otimes \mathbb{F} \cong \overline{\mathfrak{M}}$. Then there exists an eigenbasis β lifting $\overline{\beta}$ such that the matrices of partial Frobenii $(A^{(j)})_{0\le j\le f'-1}$ with respect to β satisfy the following degree bounds:

$$A^{(j)}_{ik} \in v^{\delta_{i>k}} R[v] \text{ and has degree less than } \delta_{i\ge k} + \left\langle w_j(\eta_0), \varepsilon_k^\vee \right\rangle.$$

(Note that automatically $A^{(j)}_{ik} \in v^{\delta_{i>k}} R[\![v]\!]$). Furthermore, such β is uniquely determined up to scaling by the group $\{(t_j) \in \ker(T(R) \to T(\mathbb{F}))^{f'} \mid t_j = t_k \text{ for } j \equiv k \bmod f\}$.

Example 3.4.4
Let $n=3$, let $f=1$, and let τ be a generic principal series type. Let $\overline{\mathfrak{M}}$ be a Kisin module with shape $t_{(1,2,0)} = \beta\gamma\alpha\gamma$ in the notation of [31], and choose a gauge

basis $\overline{\beta}$. Proposition 3.4.3 says that any lift $\mathfrak{M}$ with height in $[0, n-1]$ has an eigenbasis β lifting $\overline{\beta}$ such that the matrix $A^{(0)}$ has polynomial entries with degrees

$$\begin{pmatrix} 1 & \leq 1 & < 0 \\ \leq 1 & 2 & < 0 \\ \leq 1 & \leq 2 & 0 \end{pmatrix},$$

where the entries below the diagonal are divisible by v (compare with [31, Table 5] where degree bounds are given for all admissible shapes).

Remark 3.4.5

Our method of proof for Proposition 3.4.3 can be adapted easily (with more burdensome notation) to treat *semisimple* Kisin modules $\overline{\mathfrak{M}}$ of more general shapes. On the other hand, the generalization to the situation where $\overline{\mathfrak{M}}$ is not semisimple requires more work. We only treat the case covered in Proposition 3.4.3 in this paper as this is all that we need. We leave the generalizations to future work.

Proof of Proposition 3.4.3

The proof is a straightforward generalization of the arguments in [31, Section 4]. As in [31, Section 4], we introduce a semivaluation on R given by $v_R(r) = \max\{k \in \mathbb{N} \mid k \geq 0, r \in \mathfrak{m}^k\}$ and $v_R(0) = \infty$. For $P = \sum_i r_i v^i \in R[\![v]\!]$, define $d(P) = \min_i((n+3)v_R(r_i) + i)$. For a matrix X with entries in $R[\![v]\!]$, define $d(X)$ to be the minimum of $d(X_{ij})$, where the X_{ij}'s are the entries of X, and for a tuple of matrices $(X_j)_j \in M_n(R[\![v]\!])^{f'}$ we define $d((X_j)_j) = \min_j d(X_j)$. Note that in all cases d takes values in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$. We have:

- $d(a+b) \geq \min\{d(a), d(b)\}$ for a, b both in either $R[\![v]\!]$, $M_n(R[\![v]\!])$, or $M_n(R[\![v]\!])^{f'}$;
- $d(ab) \geq d(a) + d(b)$ for a, b both in either $R[\![v]\!]$ or $M_n(R[\![v]\!])$.

On any of the spaces $R[\![v]\!]$, $M_n(R[\![v]\!])$, or $M_n(R[\![v]\!])^{f'}$, the function $\|a\| = 2^{-d(a)}$ defines a norm, which is furthermore submultiplicative in the first two cases. Thus, each of these spaces is endowed with a metric topology, which is easily checked to be complete.

For each $0 \leq j \leq f' - 1$, we define the truncation operator $\mathfrak{Tr}_j : M_n(R[\![v]\!]) \to M_n(R[\![v]\!])$ as follows: for $X \in M_n(R[\![v]\!])$,

- if $i < k$, then $\mathfrak{Tr}_j(X)_{ik}$ is the sum of the terms in $X_{ik} \in R[\![v]\!]$ of degree at least $\langle w_j(\eta_0), \varepsilon_k^\vee \rangle$;
- if $i \geq k$, then $\mathfrak{Tr}_j(X)_{ik}$ is the sum of the terms in $X_{ik} \in R[\![v]\!]$ of degree greater than $\langle w_j(\eta_0), \varepsilon_k^\vee \rangle$.

In other words $\mathfrak{Tr}_j(X)$ kills off precisely the part of X that satisfies the degree bounds on $A^{(j)}$ in the conclusion. It is clearly an idempotent additive map. We observe that

our degree bounds are chosen precisely so that the image $\mathfrak{Tr}_j$ is the subspace of X such that $Xv^{-w_j(\eta_0)} \in M_n(R[\![v]\!])$ is integral and is furthermore upper triangular nilpotent mod v.

Note also, since $d(vP) = 1 + d(P)$ for $P \in R[\![v]\!]$, we have

$$d\big(\mathfrak{Tr}_j(X)v^{-w_j(\eta_0)}\big) \geq d\big(\mathfrak{Tr}_j(X)\big) - n + 1 \geq d(X) - n + 1. \tag{3.13}$$

We also define $\mathfrak{Tr} : M_n(R[\![v]\!])^{f'} \to M_n(R[\![v]\!])^{f'}$ by $\mathfrak{Tr}((X_j)_j) = (\mathfrak{Tr}_j(X_j))_j$.

We will show that, for any given eigenbasis β of $\mathfrak{M}$ lifting $\overline{\beta}$, there is a unique β' lifting $\overline{\beta}$ such that $\beta' \equiv \beta \bmod u'$, and β' satisfies the conclusion of the proposition. This proves the proposition, since the set of all possible β's mod u' forms a torsor for the group $\{(t_j) \in \ker(T(R) \to T(\mathbb{F}))^{f'} \mid t_j = t_k \text{ for } j \equiv k \bmod f\}$.

We now fix an eigenbasis β lifting $\overline{\beta}$. Our strategy will be to interpret the problem of finding β' as finding a fixed point for a certain mapping on a complete subspace of $M_n(R[\![v]\!])^{f'}$. We then show that this mapping is contracting on this subspace, and the proposition follows by the contraction mapping theorem.

By Remark 3.2.10, prescribing any other eigenbasis β' of $\mathfrak{M}$ is the same as prescribing a change of basis tuple $(I^{(j)})_j \in \mathcal{J}(R)^{f'}$ such that $I^{(j)}$ depends only on $j \bmod f$. The condition that β' also lifts $\overline{\beta}$ is equivalent to $I^{(j)} \equiv 1 \bmod \mathfrak{m}$, and the condition that $\beta' \equiv \beta \bmod u'$ is equivalent to $I^{(j)} \in \mathcal{J}_1(R)$. Thus, the tuple $(X_j)_j = (I^{(j)} - 1)_j$ satisfies

- X_j depends only on $j \bmod f$;
- $X_j \equiv 0 \bmod \mathfrak{m}$;
- X_j is upper triangular nilpotent mod v.

This leads us to define the subspace $V \subset M_n(R[\![v]\!])^{f'}$ consisting of tuples satisfying all these conditions. Clearly V is stable under componentwise addition and is easily seen to be a closed subspace and, hence, is also complete.

Let $(A^{(j)})_j$ be the tuple of matrices of partial Frobenii with respect to β. Since β lifts $\overline{\beta}$, $\overline{A}^{(j)}$ has the form given in (3.12); thus, we can decompose

$$A^{(j)} = D_j v^{w_j(\eta_0)} + M_j$$

with $D_j \in T(R)$ and $M_j \in M_n(\mathfrak{m}[\![v]\!])$. We can and will assume that this decomposition has been chosen so that D_j and M_j only depend on $j \bmod f$.

By Proposition 3.2.9, our problem of finding β' now reduces to finding $(X_j)_j \in V$ such that for all j

$$\mathfrak{Tr}_j\big((1 + X_{j+1})\big)A^{(j)}\,\mathrm{Ad}(s_j^* v^{\mu_j^*})\big(\varphi\big((1 + X_j)^{-1}\big)\big) = 0 \tag{3.14}$$

by Proposition 3.2.9. (Recall that $\mathrm{Ad}(g)(M) := gMg^{-1}$.) To lighten the notation, we put $Y_j = Y_j(X_j) = \mathrm{Ad}(s_j^* v^{\mu_j^*})(\varphi((1 + X_j)^{-1}))$ and think of it as a function in X_j. We now rewrite the left-hand side of the above equation as

$$\begin{aligned}
&\mathfrak{T}\mathfrak{r}_j\big((1+X_{j+1})(D_j v^{w_j(\eta_0)}+M_j)\big)+\mathfrak{T}\mathfrak{r}_j\big((1+X_{j+1})A^{(j)}(Y_j-1)\big)\\
&\quad=\mathfrak{T}\mathfrak{r}_j(D_j v^{w_j(\eta_0)})+\mathfrak{T}\mathfrak{r}_j(X_{j+1}D_j v^{w_j(\eta_0)})\\
&\qquad+\mathfrak{T}\mathfrak{r}_j\big((1+X_{j+1})M_j\big)+\mathfrak{T}\mathfrak{r}_j\big((1+X_{j+1})A^{(j)}(Y_j-1)\big)\\
&\quad=X_{j+1}D_j v^{w_j(\eta_0)}+\mathfrak{T}\mathfrak{r}_j\big((1+X_{j+1})M_j\big)+\mathfrak{T}\mathfrak{r}_j\big((1+X_{j+1})A^{(j)}(Y_j-1)\big),
\end{aligned}$$

where the last equality is due to the fact that $X_{j+1}\in M_n(R[\![v]\!])$ is upper triangular nilpotent mod v.

Thus (3.14) is equivalent to $(X_j)_j$ being a fixed point of the map $H: V\to M_n(R[\![v]\!])^{f'}$ given by

$$\begin{aligned}
H\big((X_j)_j\big)=\big(-\big(&\mathfrak{T}\mathfrak{r}_{j-1}\big((1+X_j)M_{j-1}\big)\\
&+\mathfrak{T}\mathfrak{r}_{j-1}\big((1+X_j)A^{(j-1)}\big(Y_{j-1}(X_{j-1})-1\big)\big)\big)v^{-w_{j-1}(\eta_0)}(D_{j-1})^{-1}\big)_j.
\end{aligned}$$

Note that the assumption $\mu-\eta$ is n-deep in $\underline{C}_0$ implies that $(Y_j-1)_j\in V$, so that each expression that gets truncated is indeed in the domain of definition of the appropriate truncation operator.

Clearly $H((X_j)_j)$ satisfies the first property defining V. Now $H((X_j)_j)$ satisfies the second property defining V, since truncation operators preserve the property of being 0 mod $\mathfrak{m}$, and $M_j=0$ mod $\mathfrak{m}$ since $\overline{A}^{(j)}$ satisfied the correct degree bounds. Finally, the description of the image of $\mathfrak{T}\mathfrak{r}_j$ implies that $H((X_j)_j)$ satisfies the third property defining V. Thus H maps V to V.

The proof of the proposition is complete once we have the following.

LEMMA 3.4.6
We have

$$d\big(H(a)-H(b)\big)\geq d(a-b)+1$$

for $a,b\in V$.

Proof
Put $a=(X_j)_j$ and $b=(X_j+\Delta_j)_j$. Put $\delta=d(a-b)=\min_j d(\Delta_j)$.

On the $(j+1)$th component, we have

$$\begin{aligned}
&\big(H(a)_{j+1}-H(b)_{j+1}\big)D_j v^{w_j(\eta_0)}\\
&\quad=\mathfrak{T}\mathfrak{r}_j(\Delta_{j+1}M_j)+\mathfrak{T}\mathfrak{r}_j\big((1+X_{j+1}+\Delta_{j+1})A^{(j)}\big(Y_j(X_j+\Delta_j)-1\big)\big)\\
&\qquad-\mathfrak{T}\mathfrak{r}_j\big((1+X_{j+1})A^{(j)}\big(Y_j(X_j)-1\big)\big)\\
&\quad=\mathfrak{T}\mathfrak{r}_j(\Delta_{j+1}M_j)+\mathfrak{T}\mathfrak{r}_j\big(\Delta_{j+1}A^{(j)}\big(Y_j(X_j+\Delta_j)-1\big)\big)
\end{aligned}$$

$$+ \mathfrak{T}\mathfrak{r}_j\big((1+X_{j+1})A^{(j)}\big(Y_j(X_j+\Delta_j)-Y_j(X_j)\big)\big)$$

For the first term, since $M_j \in M_n(\mathfrak{m}[\![v]\!])$, $d(M_j) \geq n+3$. Thus

$$d\big(\mathfrak{T}\mathfrak{r}_j(\Delta_{j+1}M_j)\big) \geq d(\Delta_{j+1}) + d(M_j) \geq \delta + n + 3.$$

For the second term, as observed before, $Y_j(X_j+\Delta_j)-1 \in M_n(\mathfrak{m}[\![v]\!])$, and we similarly have

$$d\big(\mathfrak{T}\mathfrak{r}_j\big(\Delta_{j+1}A^{(j)}\big(Y_j(X_j+\Delta_j)-1\big)\big)\big) \geq \delta + n + 3.$$

For the third term, we observe

$$\begin{aligned}
&d\big(\mathfrak{T}\mathfrak{r}_j\big((1+X_{j+1})A^{(j)}\big(Y_j(X_j+\Delta_j)-Y_j(X_j)\big)\big)\big)\\
&\quad\geq d\big(Y_j(X_j+\Delta_j)-Y_j(X_j)\big)\\
&\quad= d\big(-\mathrm{Ad}(s_j^* v^{\mu_j^*})\big(\varphi\big((1+X_j+\Delta_j)^{-1}\big)\varphi(\Delta_j)\varphi\big((1+X_j)^{-1}\big)\big)\big)\\
&\quad\geq d\big(\mathrm{Ad}(s_j^* v^{\mu_j^*})\big(\varphi(\Delta_j)\big)\big) = d\big(\mathrm{Ad}(v^{\mu_j^*})\big(\varphi(\Delta_j)\big)\big),
\end{aligned}$$

where the second inequality is due to the fact that $\mathrm{Ad}(s_j^* v^{\mu_j^*})(\varphi(X)) \in M_n(R[\![v]\!])$ for $X \in \mathcal{J}_1(R)$. (This uses that $\mu-\eta$ is 0-deep in alcove $\underline{C}_0$.) For the diagonal entries, we have

$$d\big(\big(\mathrm{Ad}(v^{\mu_j^*})\big(\varphi(\Delta_j)\big)\big)_{ii}\big) = d\big(\varphi(\Delta_j)_{ii}\big) \geq p-1+d\big((\Delta_j)_{ii}\big) \geq p-1+\delta \geq \delta+n$$

by the observation that $d(\varphi(P)) = p + d(\varphi(P/v)) \geq p + d(P/v) = p-1+d(P)$ for $P \in vR[\![v]\!]$. For the αth entry where α is a root of GL_n, we have

$$d\big(\big(\mathrm{Ad}(v^{\mu_j^*})\big(\varphi(\Delta_j)\big)\big)_\alpha\big) = d\big(\varphi(\Delta_j)_\alpha\big) + \langle \mu_j^*, \alpha^\vee\rangle.$$

By the above observation, the fact that $(\Delta_j)_\alpha \in vR[\![v]\!]$ for $\alpha < 0$, and the fact that $\mu-\eta$ is n-deep in alcove $\underline{C}_0$, we conclude that for all roots α

$$d\big(\big(\mathrm{Ad}(v^{\mu_j^*})\big(\varphi(\Delta_j)\big)\big)_\alpha\big) \geq \delta + n.$$

Thus,

$$d\big(\mathfrak{T}\mathfrak{r}_j\big((1+X_{j+1})A^{(j)}\big(Y_j(X_j+\Delta_j)-Y_j(X_j)\big)\big)\big) \geq \delta + n.$$

Putting everything together, we obtain

$$\begin{aligned}
d\big(H(a)_{j+1}-H(b)_{j+1}\big) &\geq d\big(\big(H(a)_{j+1}-H(b)_{j+1}\big)D_j v^{w_j(\eta_0)}\big) - n + 1\\
&\geq \delta + n - n + 1 = \delta + 1
\end{aligned}$$

as desired. □

Thus we deduce that the map $H : V \to V$ is a contraction mapping. In particular, H has a unique fixed point in V, which is what we wanted. □

We call a basis as above a *gauge basis* (lifting $\overline{\beta}$) of the deformation $\mathfrak{M}$ of $\overline{\mathfrak{M}}$. This is consistent with [31, Definition 4.15]. Since we have fixed the data $(\overline{\mathfrak{M}}, \overline{\beta})$, we will suppress the dependence on $\overline{\beta}$.

For each $0 \le j < f'$, the deformation problem that assigns to each Artinian $\mathcal{O}$-algebra A with residue field $\mathbb{F}$ the set of matrices $A_j \in M_n(A[\![v]\!])$ lifting $\overline{A}^{(j)}$ and satisfying the degree bounds in Proposition 3.4.3 is clearly representable by a complete local Noetherian $\mathcal{O}$-algebra R_j, which is a formal power series ring over $\mathcal{O}$. It carries the universal matrix A_j^{univ}.

For any Artinian $\mathcal{O}$-algebra A with residue field $\mathbb{F}$, let $D^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}(A)$ be the category of pairs $(\mathfrak{M}_A, \beta_A)$ deforming $(\overline{\mathfrak{M}}, \overline{\beta})$, where $\mathfrak{M}_A \in Y^{\eta,\tau}(A)$ and β_A is a gauge basis of $\mathfrak{M}_A$. We would like to give an "explicit" presentation for $D^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}$ as in [31, Theorem 4.17]. Define $R^{\mathrm{expl}}_{w_j(\eta_0)}$ to be the quotient of the formal power series ring R_j above by the height $\le \eta$ relations:

- $\det A_j^{\mathrm{univ}} = x_j^*(v+p)^{n(n-1)/2}$ where $x_j^* \in (R_j[\![v]\!])^\times$;
- $(v+p)^{k(k-1)/2}$ divides each $k \times k$ minor of A_j^{univ}. Note that the condition that $(v+p)^l$ divides $P \in R_j[\![v]\!]$ can be expressed as

$$P|_{v=-p} = 0, \qquad \frac{d}{dv}P\Big|_{v=-p} = 0, \qquad \ldots, \qquad \Big(\frac{d}{dv}\Big)^{l-1} P\Big|_{v=-p} = 0,$$

which are algebraic conditions in the coefficients of P.

PROPOSITION 3.4.7
The functor $D^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}$ is representable by the complete local Noetherian $\mathcal{O}$-algebra

$$R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}} = \widehat{\bigotimes}_{0\le j<f} (R^{\mathrm{expl}}_{w_j(\eta_0)})^{p\text{-flat, red}}. \tag{3.15}$$

Proof
By Proposition 3.4.3, there exists a closed immersion $D^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}} \hookrightarrow \mathrm{Spf}\,\widehat{\otimes}_{0\le j<f} R_j$ (note that $Y^{\eta,\tau}$ is closed in $Y^{[0,n-1],\tau}$) so $D^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}$ is representable by a quotient $R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}$ of $\widehat{\bigotimes}_{0\le j<f} R_j$. By [7, Theorem 5.3], $Y^{\eta,\tau}$ is equisingular to the local model $M(\eta)$. Since the addition of a gauge basis is formally smooth, $R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}$ is p-flat and reduced. It suffices then to compare $\overline{\mathbb{Q}}_p$-points of $R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}$ and $\widehat{\bigotimes}_{0\le j<f}(R^{\mathrm{expl}}_{w_j(\eta_0)})^{p\text{-flat, red}}$.

Let F/E be a finite extension with ring of integers $\mathcal{O}_F$. Let $x : \widehat{\bigotimes}_{0\le j<f} R_j \to \mathcal{O}_F$ with associated Kisin module $\mathfrak{M}_x$ and the matrix of partial Frobenii given by

$A_{j,x}$. By Theorem 5.13 in [7], $\mathfrak{M}_x$ lies in $Y^{\eta,\tau}(\mathcal{O}_F)$ and, hence, $D^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}(\mathcal{O}_F)$ if and only if $\mathfrak{M}_x[1/p]$ has p-adic Hodge type $\leq \eta$. In our notation, $\mathfrak{M}_x[1/p]$ has p-adic Hodge type $\leq \eta$ if, for each j, the filtration on $\mathfrak{M}^{(j)}_{x,\chi_{s'_{\mathrm{or}}(n)}}[1/p]$ is of type μ_j with $\mu_j \leq \eta_0$. The filtration is induced by the partial Frobenius with matrix $A_{j,x}$ and so this is equivalent to the condition that the elementary divisors of $A_{j,x}$ as a matrix over $F[\![v+p]\!]$ are $(v+p)^{\mu_j}$ for each $0 \leq j < f$. Thus, $\mathfrak{M}_x$ lies in $D^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}(\mathcal{O}_F)$ if and only if the elementary divisors of $A_{j,x}$ as a matrix over $F[\![v+p]\!]$ are bounded by $(v+p)^{\eta_0}$ for each $0 \leq j < f$. But this condition is exactly the divisibility condition on the minors and the determinant condition on $A_{j,x}$ imposed by the relations defining $R^{\mathrm{expl}}_{w_j(\eta_0)}$. □

Let $(\mathfrak{M}^{\mathrm{univ}}, \beta^{\mathrm{univ}})$ be the universal pair living over $R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}$.

PROPOSITION 3.4.8

(1) *Over $R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}$, the universal matrices of partial Frobenii of $\mathfrak{M}^{\mathrm{univ}}$ with respect to β^{univ} have the form*

$$A^{(j),\mathrm{univ}} = D^{\mathrm{univ}}_j (v+p)^{w_j(\eta_0)} U^{(j),\mathrm{univ}}$$

for $0 \leq j < f$, where

- *$D^{\mathrm{univ}}_j \in T(R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}})$ lifts $\overline{D}_j$;*
- *$w_j^{-1} U^{(j),\mathrm{univ}} w_j$ is lower triangular unipotent, and for any root $\alpha = \varepsilon_i - \varepsilon_k$ of GL_n, its αth entry is a polynomial with topologically nilpotent coefficients of the form $v^{\delta_{w_j(\alpha)<0}} f^{(j)}_\alpha(v)$, where $\deg f^{(j)}_\alpha(v) < -\langle \alpha^\vee, \eta_0 \rangle = i - k$.*

(2) *$R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}$ is the formal power series ring over $\mathcal{O}$ generated by the coefficients $X^{(j),l}_\alpha$ of $f^{(j)}_\alpha$ (where $0 \leq j < f$, $\alpha < 0$ is a negative root of GL_n, $0 \leq l < -\langle \alpha^\vee, \eta_0 \rangle$) and the variables $c^{(j)}_{ii} = (D^{\mathrm{univ}}_j)_{ii} - [(\overline{D}_j)_{ii}]$ (where $1 \leq i \leq n$, $0 \leq j < f$, and $[\cdot]$ denotes the Teichmüller lift).*

Example 3.4.9

In the situation of Example 3.4.4, that is, for $n = 3$, $f = 1$, τ a generic principal series type, and $\overline{\mathfrak{M}}$ having shape $t_{(1,2,0)} = \beta\gamma\alpha\gamma$, the proposition asserts that the universal deformation of $(\overline{\mathfrak{M}}, \overline{\beta})$ living over $R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}$ has matrix of Frobenius of the form

$$\begin{pmatrix} (v+p)c^*_{11} & (v+p)c_{12} & 0 \\ 0 & (v+p)^2 c^*_{22} & 0 \\ v c_{31} & v(c_{32} + (v+p)c'_{32}) & c^*_{33} \end{pmatrix},$$

where the starred coefficients are units and the nonstarred coefficients are topologically nilpotent. This is exactly what is given in Table 5 of [31]. The ring $R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}$ is a power series ring over $c_{12}, c_{31}, c_{32}, c'_{32}, c^*_{11} - [\overline{c}^*_{11}], c^*_{22} - [\overline{c}^*_{22}], c^*_{33} - [\overline{c}^*_{33}]$.

Proof

(1) We work with fixed j and set $B = w_j^{-1} A^{(j),\mathrm{univ}} w_j$. The fact that $A^{(j),\mathrm{univ}}$ obeys the degree bounds in Proposition 3.4.3 implies that

- $B_{ii} \in R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}[v]$ and has degree $\leq n - i$;
- for any root $\alpha = \varepsilon_i - \varepsilon_k$ of GL_n, $v^{-\delta_{w_j(\alpha)<0}} B_{ik}$ is in $R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}[v]$ and has degree less than $n - k$.

We claim that these conditions together with the height $\leq$ relations force B to be lower triangular, B_{ii} to be $u_i(v+p)^{n-i}$ with $u_i \in (R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}})^\times$, and B_{ik} to be divisible by $v^{\delta_{w_j(\varepsilon_i - \varepsilon_k)<0}}(v+p)^{n-i}$. This finishes the proof, since we can then uniquely factorize

$$B = D_j (v+p)^{\eta_0} U$$

with $D_j \in T(R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}})$ and U lower triangular unipotent (whose entries obey degree bounds deduced from the bounds for B), and conjugating by w_j yields the desired factorization of $A^{(j),\mathrm{univ}}$. Note that the nondiagonal entries of U are necessarily topologically nilpotent since B is diagonal modulo the maximal ideal.

We now prove the claim by downward induction on the index of the rows and columns. We start by showing the claim for entries in the nth column and nth row of B. The degree bounds imply $B_{in} = 0$ for $i < n$, while $B_{nn} \in R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}$. Furthermore, B_{nn} is a unit since it lifts a unit in the residue field. The claim is empty for all other entries of the nth row of B. Suppose the claim holds for all entries in the last $k-1$ rows and columns. Then the induction hypothesis and the condition that each $k \times k$ minor of $A^{(j),\mathrm{univ}}$ (and hence also each $k \times k$ minor of of B) is divisible by $(v+p)^{k(k-1)/2}$ imply the following.

- Looking at the minor formed by the last k columns, the last $(k-1)$ rows, and the ith row of B for $i \leq n - k + 1$, we get

$$(v+p)^{k(k-1)/2} \mid B_{i(n-k+1)} \prod_{l=0}^{k-2} u_{n-l}(v+p)^l;$$

thus, $(v+p)^{k-1} \mid B_{i(n-k+1)}$. For $i = n-k+1$, since $B_{(n-k+1)(n-k+1)}$ has degree $\leq k-1$, we must have $B_{(n-k+1)(n-k+1)} = u_{(n-k+1)}(v+p)^{k-1}$, and $u_{(n-k+1)}$ is a unit since it lifts a unit in the residue field. On the other hand, for $i < n-k+1$, the degree bounds imply that $B_{i(n-k+1)}$ is of the form v^δ times a polynomial of degree less than $k-1$, for $\delta \in \{0,1\}$. However, if p is

regular in a ring R, then the condition that vP is divisible by $(v+p)^l$ for $P \in R[v]$ is equivalent to P being divisible by $(v+p)^l$. (This can be seen by using the interpretation of this condition in terms of vanishing up to $(l-1)$th-order derivatives of P evaluated at $v=-p$.) Since p is regular in $R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}$, we conclude that $B_{i(n-k+1)}=0$.

- Looking at the minor formed by the last k rows, the last $k-1$ columns, and the ith column of B for $i<n-k+1$, we get

$$(v+p)^{k(k-1)/2} \mid B_{(n-k+1)i} \prod_{l=0}^{k-2} u_{n-l}(v+p)^l;$$

thus, $(v+p)^{k-1} \mid B_{(n-k+1)i}$. We get the claim about divisibility by $v^{\delta_{w_j(\varepsilon_{n-k+1}-\varepsilon_i)<0}}(v+p)^{k-1}$ of $B_{(n-k+1)i}$ by the same reasoning as above.

(2) We observe that, for a polynomial $P \in R[v]$ with given degree, the condition that P is divisible by $(v+p)^l$ is equivalent to solving the first l coefficients of P in terms of the remaining ones.

Thus, for j, there is a quotient $\widetilde{R}_j$ of the ring R_j over which the universal matrix with degree bounds A_j^{univ} has the form in the first part of the proposition (as this is equivalent to asking that each entry either is 0 or is divisible by certain powers of $v+p$ and v), and $\widetilde{R}_j$ is exactly the power series ring in the variables described in the second part of the proposition. Furthermore, as the specialization of the universal matrix A_j^{univ} to $\widetilde{R}_j$ clearly satisfies the determinant and the height conditions defining $R^{\mathrm{expl}}_{w_j(\eta_0)}$, we conclude that there is a factorization $R_j \twoheadrightarrow R^{\mathrm{expl}}_{w_j(\eta_0)} \twoheadrightarrow \widetilde{R}_j$. Part (1) then shows that there is a factorization $\widehat{\bigotimes}_{0\le j<f} R^{\mathrm{expl}}_{w_j(\eta_0)} \twoheadrightarrow \widehat{\bigotimes}_{0\le j<f} \widetilde{R}_j \twoheadrightarrow R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}$. But since $\widehat{\bigotimes}_{0\le j<f} \widetilde{R}_j$ is a power series ring—hence reduced and p-flat—and $R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}$ is the maximal reduced and p-flat quotient of $\widehat{\bigotimes}_{0\le j<f} R^{\mathrm{expl}}_{w_j(\eta_0)}$, the last quotient map is an isomorphism. □

We now recall the monodromy condition on the universal Kisin module $\mathfrak{M}^{\mathrm{univ}}$ over $R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}$, as in [31, Section 5.1] and [28]. We refer to [31] for undefined symbols. On $\mathfrak{M}^{\mathrm{univ}} \otimes \mathcal{O}^{\mathrm{rig}}_{R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}}$, there is a canonical derivation over the differential operator $-\lambda u' \frac{d}{d(u')}$, the *monodromy operator*, which is meromorphic along λ. (In fact, it has poles of order $\le n-2$ due to the finite height conditions we imposed.) The *monodromy condition* is the condition that this operator has no poles. On the closed points of $\operatorname{Spec} R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}[\frac{1}{p}]$, this condition precisely cuts out the (Zariski-closed) locus where the induced Kisin module comes from a potentially crystalline representation (which is necessarily of inertial type τ and Hodge–Tate weight $\le \eta$).

We recall some more deformation problems attached to the current situation, similar to [31, Definition 5.10] (when τ is principal series) and [31, Section 6.2] (for general τ). All data below are understood to be compatible with the given data $\overline{\rho}$, $\overline{\mathfrak{M}}$, $\overline{\beta}$, and so on.

(1) $R_{\overline{\rho}}^{\eta,\tau}$ is the framed potentially crystalline deformation ring of type (η,τ) as in [29]. By [29, Theorem 3.3.4], if this ring is not zero, it has Krull dimension $\dim R_{\overline{\rho}}^{\eta,\tau} = \frac{n(n-1)f}{2} + n^2 + 1$. We denote the deformation problem it represents as $D_{\overline{\rho}}^{\tau,\square}$.

(2) Let $R_{\overline{\mathfrak{M}},\overline{\rho}}^{\tau,\square}$ denote the complete local Noetherian $\mathcal{O}$-algebra which represents the deformation problem

$$D_{\overline{\mathfrak{M}},\overline{\rho}}^{\tau,\square}(A) := \big\{(\mathfrak{M}_A,\rho_A,\delta_A) \,\big|\, \mathfrak{M}_A \in Y^{\mu,\tau}(A), \rho_A \in D_{\overline{\rho}}^{\tau,\square}(A), \delta_A : T_{dd}^*(\mathfrak{M}_A) \cong (\rho_A)|_{G_{K_\infty}}\big\}.$$

Thanks to Proposition 3.2.18 (and our running hypothesis that τ is generic), this deformation problem is representable and, in fact, is representable by $R_{\overline{\rho}}^{\eta,\tau}$ as explained in [31, Section 5.2].

(3) Let $R_{\overline{\mathfrak{M}},\overline{\rho}}^{\tau,\overline{\beta},\square}$ denote the complete local Noetherian $\mathcal{O}$-algebra which represents the deformation problem

$$D_{\overline{\mathfrak{M}},\overline{\rho}}^{\tau,\overline{\beta},\square}(A) = \big\{(\mathfrak{M}_A,\rho_A,\delta_A,\beta_A) \,\big|\, (\mathfrak{M}_A,\rho_A,\delta_A) \in D_{\overline{\mathfrak{M}},\overline{\rho}}^{\tau,\square}(A), \beta_A \text{ a gauge basis for } \mathfrak{M}_A\big\}.$$

(4) Let $R_{\overline{\mathfrak{M}}}^{\tau,\overline{\beta},\square}$ denote the complete local Noetherian $\mathcal{O}$-algebra which represents the deformation problem of triples $(\mathfrak{M}_A,\beta_A,\underline{e}_A)$, where $(\mathfrak{M}_A,\beta_A) \in D_{\overline{\mathfrak{M}}}^{\tau,\overline{\beta}}(A)$ and $\underline{e}_A$ is a basis of $T_{dd}^*(\mathfrak{M}_A)$ lifting the basis on $\overline{\rho}|_{G_{K_\infty}}$ so that $(T_{dd}^*(\mathfrak{M}_A),\underline{e}_A)$ is a framed deformation of $\overline{\rho}|_{G_{K_\infty}}$.

(5) Let $R_{\overline{\mathfrak{M}}}^{\tau,\overline{\beta},\nabla}$ denote the $\mathcal{O}$-flat and reduced quotient of $R_{\overline{\mathfrak{M}}}^{\tau,\overline{\beta}}$ such that $\operatorname{Spec} R_{\overline{\mathfrak{M}}}^{\tau,\overline{\beta},\nabla}[1/p]$ is the locus where the monodromy condition holds on $\operatorname{Spec} R_{\overline{\mathfrak{M}}}^{\tau,\overline{\beta}}[1/p]$. We define $R_{\overline{\mathfrak{M}}}^{\tau,\overline{\beta},\square,\nabla}$ from $R_{\overline{\mathfrak{M}}}^{\tau,\overline{\beta},\square}$ in a similar way.

We recall [31, Diagram (5.9)], which summarizes the relationship between the above deformation problems. The square is Cartesian and "f.s." stands for "formally smooth."

$$\begin{array}{ccccc}
 & & \operatorname{Spf} R^{\tau,\overline{\beta},\square,\nabla}_{\overline{\mathfrak{M}}} & \xrightarrow{f.s.} & \operatorname{Spf} R^{\tau,\overline{\beta},\nabla}_{\overline{\mathfrak{M}}} \\
 & \overset{\xi}{\nearrow} & \downarrow & & \downarrow \\
\operatorname{Spf} R^{\tau,\overline{\beta},\square}_{\overline{\mathfrak{M}},\overline{\rho}} & \hookrightarrow & \operatorname{Spf} R^{\tau,\overline{\beta},\square}_{\overline{\mathfrak{M}}} & \xrightarrow{f.s.} & \operatorname{Spf} R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}} \\
\downarrow f.s. & & & & \\
\operatorname{Spf} R^{\eta,\tau}_{\overline{\rho}} & \xleftarrow{\sim} & \operatorname{Spf} R^{\tau,\square}_{\overline{\mathfrak{M}},\overline{\rho}} & &
\end{array} \tag{3.16}$$

The maps which are formally smooth correspond to forgetting either a framing on the Galois representation or a gauge basis on the Kisin modules. (The fact that adding gauge basis is a formally smooth operation is due to Proposition 3.4.3.) The fact that the horizontal arrow below the dotted arrow is a closed immersion is due to our assumption that τ is (at least) generic, which implies $\mathrm{ad}(\overline{\rho})$ is cyclotomic free (e.g., by looking at the inertial weights, which are easily read off by applying Corollary 3.2.17), and hence the argument of [31, Proposition 5.11] applies. This shows that if the dotted arrow exists, then it must be a closed immersion. We show below that the dotted arrow exists and is furthermore an isomorphism. The fact that it is an isomorphism rather than just a closed immersion is because, in the present situation, the elementary divisors of the matrices of partial Frobenii of $\overline{\mathfrak{M}}$ with respect to $\overline{\beta}$ are *exactly* $(v+p)^{\eta_0}$, and thus, no lift $\mathfrak{M}$ of $\overline{\mathfrak{M}}$ can satisfy a height $\leq \lambda$ relation for $\lambda < \eta$.

PROPOSITION 3.4.10
The natural map $R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}} \to R^{\tau,\overline{\beta},\square}_{\overline{\mathfrak{M}},\overline{\rho}}$ factors through the quotient $R^{\tau,\overline{\beta},\nabla}_{\overline{\mathfrak{M}}}$. The induced map ξ is an isomorphism.

Proof
The proof is completely analogous to the proof of [31, Theorem 5.12]. As both target rings are reduced and p-flat, we only need to check that the factorization exists on closed points of the generic fibers. However, this is just the statement that a Kisin module coming from a potentially crystalline Galois representation satisfies the monodromy condition.

To see that the map ξ is an isomorphism, we note that the only closed points in $\operatorname{Spec} R^{\tau,\overline{\beta},\square,\nabla}_{\overline{\mathfrak{M}}}[\frac{1}{p}]$ that do not come from $\operatorname{Spec} R^{\tau,\overline{\beta},\square}_{\overline{\mathfrak{M}},\overline{\rho}}[\frac{1}{p}]$ are those for which the jth component of the underlying Kisin module has elementary divisors strictly dominated by $(v+p)^{\eta_0}$, for some $0 \leq j < f$. (This corresponds to the condition that the Hodge–Tate weight of the jth embedding of the corresponding Galois representation is less

than η_0.) However, this possibility is ruled out by the form of the universal Kisin module given in Proposition 3.4.8 □

COROLLARY 3.4.11
For $\overline{\rho}, \tau$ as in Theorem 3.4.1, there exists a closed point $x \in \operatorname{Spec} R^{\eta,\tau}_{\overline{\rho}}[\frac{1}{p}]$ such that the corresponding Galois representation ρ_x becomes a direct sum of characters after restriction to a finite-index subgroup. In particular, $R^{\eta,\tau}_{\overline{\rho}} \neq 0$.

Proof
Recall that we have fixed a gauge basis $\overline{\beta}$ of $\overline{\mathfrak{M}}$ such that, for all j, the matrices of the jth partial Frobenius with respect to $\overline{\beta}$ have the form

$$\overline{A}^{(j)} = \overline{D}_j v^{w_j(\eta_0)} \tag{3.17}$$

with $\overline{D}_j \in T(\mathbb{F})$. Using Proposition 3.4.8, we can produce an $\mathcal{O}$ point of $R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}$ such that the matrices of partial Frobenii are monomial matrices of the form

$$A^{(j)} = D_j (v+p)^{w_j(\eta_0)}$$

by choosing diagonal matrices $D_j \in T(\mathcal{O})$ lifting $\overline{D}_j$. A Kisin module of this form becomes isomorphic to a direct sum of rank 1 Kisin modules after passing to a finite unramified extension $\breve{K}$ of K'. Since the monodromy condition can be checked after base change and always holds for rank 1 Kisin modules (by an easy computation), we deduce that the above Kisin module satisfies the monodromy condition. Lifting this point along the formally smooth maps in the diagram (3.16) yields a closed point $x \in \operatorname{Spec} R^{\eta,\tau}_{\overline{\rho}}[\frac{1}{p}]$. As the underlying Kisin module of x decomposes into a direct sum of rank 1 Kisin modules over $\breve{K}$, ρ_x becomes a direct sum of characters over $\breve{K}$.

Alternatively, we can also directly produce ρ_x as a direct sum of inductions of (potentially crystalline) characters for unramified extensions of K and then check that it comes from a Kisin module with the above form and, hence, has type (η, τ). □

PROPOSITION 3.4.12
We have that $R^{\tau,\overline{\beta},\nabla}_{\overline{\mathfrak{M}}}/\varpi$ is a quotient of a power series ring over $\mathbb{F}$ in $\frac{n(n+1)f}{2}$ variables.

Proof
We work over $R = R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}$ with universal Kisin module $\mathfrak{M} = \mathfrak{M}^{\mathrm{univ}}$ and universal gauge basis β. This determines the matrices $A^{(j)}$ as before. It will suffice to analyze the monodromy condition on $\mathfrak{M}$ viewed as a Kisin module over K' with descent data corresponding to the base changed type τ' (which is a principal series type), and we

will do so by closely following the computations in [31, Section 5.1] (especially [31, Theorem 5.6]). As in [31], we have the matrices $C^{(j)}$ which are determined by the $A^{(j)}$'s and our chosen presentation of τ, and we have a ring $\mathcal{O}_R^{\mathrm{rig}}$. We note that the variable u in [31] corresponds to our variable u', the variable v there is the same as our $v=(u')^{e'}$, and $E(u)$ there is $(u')^{e'}+p=v+p$. Exactly as in [31, Lemma 5.2] we have a formula for the jth component $N_\infty^{(j)}$ of the monodromy operator:

$$N_\infty^{(j)}=N_1^{(j)}+\sum_{i=1}^{\infty}\Big(\prod_{k=0}^{i-1}\varphi^k(C^{(j-k-1)})\Big)\varphi^i(N_1^{(j-i)})\Big(\prod_{k=i-1}^{0}\varphi^k(C^{(j-k-1),*})\Big),$$

where $C^{(j),*}:=(v+p)(C^{(j)})^{-1}$,

$$N_1^{(j)}=\lambda u'\frac{d}{du'}(C^{(j-1)})(C^{(j-1)})^{-1},$$

and the convergence happens inside $\lambda^{2-n}\operatorname{Mat}_n(\mathcal{O}_R^{\mathrm{rig}})$.

As in [31, Theorem 5.6] we can write

$$p^{n-1}\lambda^{n-2}N_\infty^{(j)}=(p\lambda)^{n-1}u'\frac{d}{du'}(C^{(j-1)})(C^{(j-1)})^{-1}+\sum_{i=1}^{\infty}X_i^{(j)},$$

where

$$X_i^{(j)}:=\frac{\varphi^{i+1}(\lambda)^{n-1}}{p^{i(n-2)}}\Big(\prod_{k=0}^{i-1}\varphi^k(C^{(j-k-1)})\Big)\varphi^i\Big(u'\frac{d}{du'}C^{(j-i-1)}\Big)$$
$$\times\Big(\prod_{k=i}^{0}\varphi^k\big((v+p)^{n-2}C^{(j-k-1),*}\big)\Big).$$

If $z\in\mathbb{Z}^n$, we use the shorthand $\operatorname{Diag}(z)$ to denote the diagonal matrix with entries $z_1,z_2,\dots,z_n$. Also for $M,N\in M_n(R[\![v]\!])$, $[M,N]:=MN-NM$. By "removing the descent data" as in [31, Theorem 5.6] (see (3.6) for notation), we obtain

$$p^{n-1}\operatorname{Ad}\big((s'_{\mathrm{or},j})^{-1}(u')^{-\mathbf{a}'^{(j)}_{(s,\mu)}}\big)(\lambda^{n-2}N_\infty^{(j)})$$
$$=-\varphi(\lambda)^{n-1}P_N(A^{(j-1)})+\sum_{i=1}^{\infty}\varphi^{i+1}(\lambda)^{n-1}Z_i^{(j)},$$

where (see [31, Lemma 5.4])

$$P_N(A^{(j-1)})\overset{\mathrm{def}}{=}\Big(-e'v\frac{d}{dv}A^{(j-1)}-\big[\operatorname{Diag}\big((s'_{\mathrm{or},j})^{-1}(\mathbf{a}'^{(j)}_{(s,\mu)})\big),A^{(j-1)}\big]\Big)$$
$$\times(v+p)^{n-1}(A^{(j-1)})^{-1}$$

and

$$Z_i^{(j)} = \mathrm{Ad}\big((s'_{\mathrm{or},j})^{-1}(u')^{-\mathbf{a}'^{(j)}_{(s,\mu)}}\big)\Big(\frac{1}{\varphi^{i+1}(\lambda)^{n-1}}X_i^{(j)}\Big).$$

Now exactly as in the last part of the proof of [31, Theorem 5.6], using that τ' is $(2n-1)$-generic, we get $Z_i^{(j)} \in \frac{v^{(2n-2)p^{i-1}}}{p^{i(n-2)}}M_n(R[\![v]\!])$ for $i>1$ and $Z_1^{(j)} \in \frac{v^{2n-1}}{p^{n-2}}M_n(R[\![v]\!])$. This fact together with a simple computation with derivatives shows that $M^{(j)} := \frac{1}{\varphi(\lambda)^{n-1}}\sum_{i=1}^{\infty}\varphi^{i+1}(\lambda)^{n-1}Z_i^{(j)}$ satisfies $(\frac{d}{dv})^t M^{(j)}|_{v=-p} \in p^{2n-1-(n-2)-t}M_n(R)$ and $(\frac{d}{dv})^t v^{-1}M^{(j)}|_{v=-p} \in p^{2n-1-(n-2)-t-1}M_n(R)$ for $0 \le t \le n-3$.

As in the proof of [31, Proposition 5.3], the monodromy condition is equivalent to $\lambda^{n-2}N_\infty^{(j)}$ vanishing to order $n-2$ at $u' = (-p)^{\frac{1}{e'}}$. Thus the upshot of the above discussion is that the monodromy condition is equivalent to $-P_N(A^{(j-1)}) + M^{(j)}$ vanishing to order $n-2$ at $v=-p$ in $M_n(\mathcal{O}_R^{\mathrm{rig}})$. Note that this condition is preserved under multiplication by any power of v and can be expressed as the vanishing of all derivatives $(\frac{d}{dv})^t$ at $v=-p$ for $0 \le t \le n-3$.

We now recall Proposition 3.4.8, which gives the decomposition

$$A^{(j)} = D_j(v+p)^{w_j(\eta_0)}U^{(j)},$$

and $R = R^{\tau,\overline{\beta}}_{\overline{\mathfrak{M}}}$ is the formal power series ring over $\mathcal{O}$ in the variables $X_\alpha^{(j),l}$ (where $0 \le j < f$, $\alpha<0$ is a negative root of GL_n, and $0 \le l < -\langle \eta_0, \alpha^\vee\rangle$) and the variables $c_{ii}^{(j)}$ (where $1 \le i \le n$, $0 \le j < f$), which (up to a translation in the case of $c_{ii}^{(j)}$) give the coefficients of the entries of $U^{(j)}$ and D_j, respectively.

Substituting the above expression, we get

$$\begin{aligned} P_N(A^{(j-1)}) = &-(v+p)^{n-1}\Big(e'v\frac{d}{dv}(v+p)^{w_{j-1}(\eta_0)}\Big)(v+p)^{-w_{j-1}(\eta_0)} \\ &-(v+p)^{n-1}\,\mathrm{Ad}\big(D_{j-1}(v+p)^{w_{j-1}(\eta_0)}\big)\Big(\Big(e'v\frac{d}{dv}U^{(j-1)} \\ &+\big[\mathrm{Diag}\big((s'_{\mathrm{or},j})^{-1}(\mathbf{a}'^{(j)}_{(s,\mu)})\big), U^{(j-1)}\big]\Big)(U^{(j-1)})^{-1}\Big). \end{aligned}$$

To understand the second term, recall that also from Proposition 3.4.8, for each negative root $\alpha = \varepsilon_i - \varepsilon_k$ of GL_n, the αth entry of $w_{j-1}^{-1}U^{(j-1)}w_{j-1}$ is given by

$$v^{\delta_{w_j(\alpha)<0}}f_\alpha^{(j-1)}(v) = v^{\delta_{w_{j-1}(\alpha)<0}}\sum_{l=0}^{-\langle\eta_0,\alpha^\vee\rangle-1}X_\alpha^{(j-1),l}v^l.$$

Thus, the αth entry of $\mathrm{Ad}((D_{j-1}w_{j-1})^{-1})(P_N(A^{(j-1)}))$ is of the form

$$\begin{aligned}
&-(v+p)^{n-1+\langle\eta_0,\alpha^\vee\rangle}\Big(e'v\frac{d}{dv}+\big\langle(s'_{\mathrm{or},j})^{-1}(\mathbf{a}'^{(j)}_{(s,\mu)}),w_{j-1}\alpha^\vee\big\rangle\Big)\\
&\qquad\times(v^{\delta_{w_{j-1}(\alpha)<0}}f_\alpha^{(j-1)})+\cdots)\\
&=-(v+p)^{n-1+\langle\eta_0,\alpha^\vee\rangle}\Big(\sum_{l=0}^{-\langle\eta_0,\alpha^\vee\rangle-1}\big(e'(l+\delta_{w_{j-1}(\alpha)<0})\\
&\qquad+\big\langle(s'_{\mathrm{or},j})^{-1}(\mathbf{a}'^{(j)}_{(s,\mu)}),w_{j-1}\alpha^\vee\big\rangle\big)X_\alpha^{(j-1),l}v^{l+\delta_{w_{j-1}(\alpha)<0}}+\cdots\Big),
\end{aligned}$$

where the ellipsis in the first expression is an R-linear combination of terms of the form

$$e'\frac{d}{dv}(v^{\delta_{w_{j-1}(\alpha_0)<0}}f_{\alpha_0}^{(j-1)})\prod_{i\neq0}v^{\delta_{w_{j-1}(\alpha_i)<0}}f_{\alpha_i}^{(j-1)}$$

and

$$\prod_i v^{\delta_{w_{j-1}(\alpha_i)<0}}f_{\alpha_i}^{(j-1)},$$

where the α_i's are negative roots of GL_n with $\alpha=\sum_i\alpha_i$ and the sum has at least two terms. In particular, the remaining terms are polynomials divisible by $v^{\delta_{w_{j-1}(\alpha)<0}}$ and whose coefficients only involve $X_{\alpha'}^{(j-1)}$ for roots α' strictly larger than α. Thus, the monodromy condition on the αth entry of $v^{-\delta_{w_{j-1}(\alpha)<0}}\mathrm{Ad}(w_{j-1}^{-1}(D_{j-1})^{-1})\times(-P_N(A^{(j-1)})+M^{(j)})$ has the form

$$\begin{aligned}
&\Big(\frac{d}{dv}\Big)^t\Big((v+p)^{n-1+\langle\eta_0,\alpha^\vee\rangle}\Big(\sum_{l=0}^{-\langle\eta_0,\alpha^\vee\rangle-1}\big(e'(l+\delta_{w_{j-1}(\alpha)<0})\\
&\qquad+\big\langle(s'_{\mathrm{or},j})^{-1}(\mathbf{a}'^{(j)}_{(s,\mu)}),w_{j-1}\alpha^\vee\big\rangle\big)X_\alpha^{(j-1),l}v^l\Big)\Big)\Big|_{v=-p}\\
&=O\big((X_{\alpha'}^{(j-1),l})_{0>\alpha'>\alpha,l}\big)+O(p^{2n-1-(n-2)-t-1})
\end{aligned}$$

for $0\le t\le n-3$, where the right-hand side only involves variables indexed by strictly larger roots and a term in R that is divisible by $p^{2n-1-(n-2)-t-1}$ and hence is divisible by p.

Thus, the above system of equations holds in the quotient $R^{\tau,\overline{\beta},\nabla}_{\overline{\mathfrak{M}}}/\varpi$ of R, where it implies (noting $e'=-1$ in $\mathbb{F}$)

$$\Big(\frac{d}{dv}\Big)^t\Big(\sum_{l=0}^{-\langle\eta_0,\alpha^\vee\rangle-1}\big(-l-\delta_{w_{j-1}(\alpha)<0}+\big\langle(s'_{\mathrm{or},j})^{-1}(\mathbf{a}'^{(j)}_{(s,\mu)}),w_{j-1}\alpha^\vee\big\rangle\big)X_\alpha^{(j-1),l}v^l\Big)\Big|_{v=0}=O\big((X_{\alpha'}^{(j-1),l})_{0>\alpha'>\alpha,l}\big)$$

for $0\le t\le(n-3)-(n-1+\langle\eta_0,\alpha^\vee\rangle)=-\langle\eta_0,\alpha^\vee\rangle-2$. Since τ is generic and $p>n$, all the coefficients $-l-\delta_{w_{j-1}^{-1}(\alpha)<0}+\langle(s'_{\mathrm{or},j})^{-1}(\mathbf{a}'^{(j)}_{(s,\mu)}),w_{j-1}\alpha^\vee\rangle$ as well as the constants introduced by taking derivatives are nonzero in $\mathbb{F}$, and hence this system of equations solves $X_\alpha^{(j-1),l}$ for $l<-\langle\eta_0,\alpha^\vee\rangle-1$ in terms of variables indexed by strictly larger roots. It follows that $R^{\tau,\overline{\beta},\nabla}_{\overline{\mathfrak{M}}}/\varpi$ is topologically generated by the top degree coefficients of $f_\alpha^{(j)}$ and the $c_{ii}^{(j)}-[\overline{c}_{ii}^{(j)}]$ for $0\le j<f$, $1\le i\le n$, and negative roots α of GL_n. Hence it is topologically generated by $n(n-1)f/2+nf=n(n+1)f/2$ elements. □

Proof of Theorem 3.4.1

We already know $R^{\eta,\tau}_{\overline{\rho}}\neq 0$ by Corollary 3.4.11. We look at diagram (3.16). By Proposition 3.4.3, $\mathrm{Spf}\,R^{\tau,\overline{\beta},\square}_{\overline{\mathfrak{M}},\overline{\rho}}\to\mathrm{Spf}\,R^{\eta,\tau}_{\overline{\rho}}$ is a torsor for $(\widehat{\mathbb{G}}_m)^{nf}$; hence $d\overset{\mathrm{def}}{=}\dim R^{\tau,\overline{\beta},\square}_{\overline{\mathfrak{M}},\overline{\rho}}=\dim R^{\eta,\tau}_{\overline{\rho}}+nf=n(n-1)f/2+n^2+nf+1=n(n+1)f/2+n^2+1$. On the other hand, $\mathrm{Spf}\,R^{\tau,\overline{\beta},\square,\nabla}_{\overline{\mathfrak{M}}}\to\mathrm{Spf}\,R^{\tau,\overline{\beta},\nabla}_{\overline{\mathfrak{M}}}$ is formally smooth of relative dimension n^2; hence Proposition 3.4.12 shows that there is a surjection $\mathcal{O}[\![x_1,\dots,x_{d-1}]\!]\twoheadrightarrow R^{\tau,\overline{\beta},\square,\nabla}_{\overline{\mathfrak{M}}}$. From Proposition 3.4.10, we obtain a surjection $\mathcal{O}[\![x_1,\dots,x_{d-1}]\!]\twoheadrightarrow R^{\tau,\overline{\beta},\square}_{\overline{\mathfrak{M}},\overline{\rho}}$. Since the quotient ring has dimension $d=\dim\mathcal{O}[\![x_1,\dots,x_{d-1}]\!]$, the kernel of this surjection must be trivial; hence the surjection is an isomorphism. It follows that $R^{\eta,\tau}_{\overline{\rho}}$ is formally smooth over $\mathcal{O}$. □

COROLLARY 3.4.13

With $\overline{\rho},\tau$ as in Theorem 3.4.1, any potentially crystalline lift of $\overline{\rho}$ of type (η,τ) is potentially diagonalizable.

Proof

This follows from Theorem 3.4.1 and Corollary 3.4.11. □

4. Main results

In this section, we deduce our main results using Section 3. In Section 4.2, we deduce weight elimination in an axiomatic context and in the context of definite unitary groups. In Section 4.3, we use weight elimination and the change of weight techniques

of [3] to deduce the modularity of obvious weights. In Section 4.4, we use the above results to classify congruences between RACSDC (regular algebraic, conjugate self-dual, cuspidal) GL_n-automorphic representations of trivial weight and generic tame type τ in residually tame cases and solve the lifting problem for residually tame Galois representations to potentially crystalline representations of type (η, τ). We introduce combinatorial results on Serre weights and affine Weyl groups as needed. The key theme in these combinatorial results is the close relationship between certain reduced factorizations of admissible elements and Jantzen's description of the Jordan–Hölder factors of types.

Recall the (nonstandard) definition of the dot action in Definition 2.1.6. We write $\underline{W}_a^+ \subset \widetilde{\underline{W}}^+$ for the subsets of $\underline{W}_a \subset \widetilde{\underline{W}}$, respectively, which map $\underline{C}_0$ to a dominant alcove under this dot action.

Let $\underline{\Omega} \subset \widetilde{\underline{W}}$ be the stabilizer of $\underline{C}_0$. Then we have the decomposition $\widetilde{\underline{W}} = \underline{W}_a \rtimes \underline{\Omega}$. We extend the Coxeter length function ℓ on $\underline{W}_a$ to $\widetilde{\underline{W}}$ by setting $\ell(\widetilde{w}\delta) = \ell(\widetilde{w})$ if $\widetilde{w} \in \underline{W}_a$ and $\delta \in \underline{\Omega}$. Recall that one can calculate lengths from minimal galleries (see [19, Section 2]). We will use galleries in a fixed direction (see [19, Definition 5.2]), which are necessarily minimal by [19, Lemma 5.3].

Recall the upper arrow ($\uparrow$) ordering on p-alcoves (see [27, Section II.6.5]), and extend it to $\widetilde{\underline{W}}$ by writing $\widetilde{w}_1 \uparrow \widetilde{w}_2$ if $\widetilde{w}_1 \cdot \underline{C}_0 \uparrow \widetilde{w}_2 \cdot \underline{C}_0$ and $\underline{W}_a \widetilde{w}_1 = \underline{W}_a \widetilde{w}_2$ for $\widetilde{w}_1$ and $\widetilde{w}_2 \in \widetilde{\underline{W}}$ (elements of different right $\underline{W}_a$-cosets are incomparable). We also use $\uparrow$ to denote the ordering on $X^*(\underline{T})$ defined in [27, Section II.6.4]. Recall from Section 2.1 the Bruhat ordering $\leq$ on $\underline{W}_a$ defined by the *dominant* base alcove. As with the upper arrow ordering, we extend this to a partial ordering on $\widetilde{\underline{W}}$ by setting $\widetilde{w}_1\delta \leq \widetilde{w}_2\delta$ if $\widetilde{w}_1$ and $\widetilde{w}_2 \in \underline{W}_a$, $\widetilde{w}_1 \leq \widetilde{w}_2$, and $\delta \in \underline{\Omega}$ (and elements of different right $\underline{W}_a$-cosets are incomparable).

Let $\widetilde{w}_h = w_0 t_{-\eta} \in \widetilde{\underline{W}}$. Note that $\widetilde{w}_h \cdot \underline{C}_0$ is the highest p-restricted alcove and $\widetilde{w}_h \cdot \lambda = \mathcal{R}(\lambda)$ from Definition 2.2.11.

4.1. Combinatorics of weights and types

In this section, we deduce the key combinatorial results, especially Corollary 4.1.12. We will use the following theorem of Wang (see [39, Theorem 4.3]) without comment.

THEOREM 4.1.1
If $\widetilde{w}_1$ and $\widetilde{w}_2 \in \widetilde{\underline{W}}^+$, then $\widetilde{w}_1 \leq \widetilde{w}_2$ if and only if $\widetilde{w}_1 \uparrow \widetilde{w}_2$.

We also use the following proposition without comment.

PROPOSITION 4.1.2
If $\widetilde{w}_1$ and $\widetilde{w}_2 \in \underline{\widetilde{W}}$, then $\widetilde{w}_1 \uparrow \widetilde{w}_2$ if and only if $\widetilde{w}_h\widetilde{w}_2 \uparrow \widetilde{w}_h\widetilde{w}_1$ if and only if $\widetilde{w}_h^{-1}\widetilde{w}_2 \uparrow \widetilde{w}_h^{-1}\widetilde{w}_1$.

Proof
From the definition of the up ordering, it is clear that w_0 reverses and translation preserves the ordering. The proposition now follows from the definition of $\widetilde{w}_h$. □

PROPOSITION 4.1.3
Suppose that μ is $2n$-deep in $\underline{C}_0$, and suppose that λ is a p-restricted weight. Then $F(\lambda) \in \mathrm{JH}(\overline{R}_s(\mu+\eta))$ if and only if there exists $\widetilde{w} = wt_\nu \in \underline{\widetilde{W}}^+$ such that

$$\widetilde{w}\cdot(\mu - s\pi\nu) \uparrow \widetilde{w}_h\cdot\lambda \qquad \text{and} \qquad \widetilde{w}\cdot\underline{C}_0 \uparrow \widetilde{w}_h\cdot\underline{C}_0. \tag{4.1}$$

Proof
We use [15, Proposition 10.1.8]. (Note that, by the depth assumption, the proof of [15, Proposition 10.1.2] based on [26, Satz 4.3] applies.) The proposition cited shows that $F(\lambda) \in \mathrm{JH}(\overline{R}_s(\mu+\eta))$ if and only if there is ν such that

$$\sigma t_\nu\cdot(\mu - s\pi\nu) \uparrow \widetilde{w}_h\cdot\lambda \quad \text{for all } \sigma \in W(\underline{G}). \tag{4.2}$$

It suffices to show that the existence of ν satisfying (4.2) and $\widetilde{w} \in \underline{\widetilde{W}}^+$ satisfying (4.1) are equivalent.

We begin with the "backward" implication, for which the following remark is useful.

Remark 4.1.4
If $\widetilde{w} = wt_\nu$ satisfies $\widetilde{w}\cdot\underline{C}_0 \uparrow \widetilde{w}_h\cdot\underline{C}_0$, then ν lies in the convex hull of the Weyl orbit of η, and hence $\max_{\alpha^\vee}\{|\langle\nu,\alpha^\vee\rangle|\} \le n-1$.

Suppose that $\widetilde{w} = wt_\nu$ satisfies (4.1). With the depth assumption on μ, the above remark implies that $\mu - s\pi\nu$ is in $\underline{C}_0$ so that $wt_\nu\cdot(\mu - s\pi\nu)$ is the unique dominant element of the $W(\underline{G})$ (dot) orbit of $\mu - s\pi\nu + p\nu$. This implies that ν satisfies (4.2).

For the "forward" implication, suppose that ν satisfies (4.2), and take $\widetilde{w}$ to be the unique element $wt_\nu \in \underline{\widetilde{W}}^+$ with $w \in W(\underline{G})$. Then $\widetilde{w}\cdot(\mu - s\pi\nu) \uparrow \widetilde{w}_h\cdot\lambda$ by assumption, and it suffices to show that $\widetilde{w}\cdot\underline{C}_0 \uparrow \widetilde{w}_h\cdot\underline{C}_0$. We claim that ν satisfying (4.2) must automatically satisfy $|\langle\nu,\alpha^\vee\rangle| \le n-1$ for all $\alpha^\vee \in \underline{R}^\vee$. Admitting this claim for the moment, we again have that $\widetilde{w}\cdot(\mu - s\pi\nu)$ is in alcove $\widetilde{w}\cdot\underline{C}_0$, so that $\widetilde{w}\cdot(\mu - s\pi\nu) \uparrow \widetilde{w}_h\cdot\lambda$ implies that $\widetilde{w}\cdot\underline{C}_0 \uparrow \widetilde{w}_h\cdot\underline{C}_0$.

Going back to our claim, (4.2) implies that $\mu + p\nu - s\pi\nu + \eta$ is in the convex hull of the Weyl orbit of $\widetilde{w}_h\cdot\lambda + \eta$. The same argument as in the proof of Lemma 2.1.5

shows that

$$\max_{\alpha^\vee}\{|\langle \mu + p\nu - s\pi\nu + \eta, \alpha^\vee\rangle|\} \leq \max_{\alpha^\vee}\{|\langle \widetilde{w}_h \cdot \lambda + \eta, \alpha^\vee\rangle|\} \leq p(n-1).$$

The same argument as in the proof of Corollary 2.2.13, using that μ is $2n$-deep in $\underline{C}_0$, shows that if $M = \max_{\alpha^\vee}\{|\langle \nu, \alpha^\vee\rangle|\}$, then $(p-1)M < p(n-1)+p-2n = (p-2)n$, and thus $M \leq n-1$ as desired. □

We will often fix

(P1) a generic semisimple Galois representation $\overline{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F})$;

(P2) a pair $(s_{\overline{\rho}}, \mu_{\overline{\rho}})$ such that $\overline{\rho}|_{I_K} \cong \overline{\tau}(s_{\overline{\rho}}, \mu_{\overline{\rho}} + \eta)$ with $\mu_{\overline{\rho}}$ in $\underline{C}_0$; and

(P3) a lowest alcove presentation $(s, \mu - \eta)$ of a tame inertial type $\tau \cong \tau(s, \mu) : I_K \to \mathrm{GL}_n(\mathcal{O})$ such that $\mu_{\overline{\rho}} - \mu \in \underline{\Lambda}_R$. Such a presentation is called compatible with $(s_{\overline{\rho}}, \mu_{\overline{\rho}})$.

Note that, under **(P1)** and **(P2)**, $\mu_{\overline{\rho}}$ is always $(3n-1)$-deep in $\underline{C}_0$ by Proposition 2.2.15.

LEMMA 4.1.5

Suppose that $\tau(s, \mu) \cong \tau(s', \mu')$ is 1-generic, suppose that $\mu - \eta$ and $\mu' - \eta$ are both in $\underline{C}_0$, and suppose that $\mu - \mu' \in \underline{\Lambda}_R$. Then we have $(s, \mu) = (s', \mu')$.

Proof

By Proposition 2.2.15, since $(s', \mu' - \eta)$ and $(s, \mu - \eta)$ are two lowest alcove presentations of $\tau(s, \mu)$, we have $(s, \mu) = {}^{(\nu,\sigma)}(s', \mu')$ with $t_\nu \sigma \in \underline{\Omega}$. Since $\mu - \mu' \in \underline{\Lambda}_R$, we also must have that $(p - \pi)\nu|_{\underline{Z}} = 0$ or, equivalently, that $\nu|_{\underline{Z}} = 0$. Combining these facts, we have that $t_\nu \sigma$ is the identity. □

Definition 4.1.6

Fix **(P1)**–**(P3)** as above. Then, we define $w = s^{-1} s_{\overline{\rho}}$, $\nu = s^{-1}(\mu_{\overline{\rho}} + \eta - \mu)$, and $\widetilde{w}^*(\overline{\rho}, \tau) \stackrel{\text{def}}{=} t_\nu w$.

Remark 4.1.7

Definition 4.1.6 a priori depends on the choice of $(s_{\overline{\rho}}, \mu_{\overline{\rho}})$ and the compatible presentation $(s, \mu - \eta)$ of the type τ. By Lemma 4.1.5, if τ is 1-generic, there is at most one compatible presentation, so that Definition 4.1.6 depends only on the choice of $(s_{\overline{\rho}}, \mu_{\overline{\rho}})$. Furthermore, if a compatible presentation $(s, \mu - \eta)$ exists for one choice of $(s_{\overline{\rho}}, \mu_{\overline{\rho}})$, then a compatible presentation exists for all other choices, and changing the choice of $(s_{\overline{\rho}}, \mu_{\overline{\rho}})$ conjugates $\widetilde{w}^*(\overline{\rho}, \tau)$ by an element of $\underline{\Omega}$.

PROPOSITION 4.1.8
Let τ be a generic type with lowest alcove presentation $(s, \mu - \eta)$. Assume that there exists $\overline{\mathfrak{M}} \in Y^{\eta,\tau}(\mathbb{F})$ as in Definition 3.2.19 such that $T^*_{\mathrm{dd}}(\overline{\mathfrak{M}}) \cong \overline{\rho}|_{G_{K_\infty}}$, where $\overline{\rho}$ is moreover semisimple. Then there is a pair $(s_{\overline{\rho}}, \mu_{\overline{\rho}})$ as in (**P2**) *such that* (**P3**) *holds and $\widetilde{w}(\overline{\rho}, \tau)^* = \widetilde{w}^*(\overline{\rho}, \tau)$.*

Proof
By Theorem 3.2.26 combined with Corollary 3.2.17, we have

$$\overline{\rho}|_{I_K} \cong \overline{\tau}(w, \nu + \eta),$$

where $\widetilde{w}(\overline{\rho}, \tau) s^* t_{\mu^* - \eta^*} = w^* t_{\nu^*}$. Thus, $\widetilde{w}(\overline{\rho}, \tau)^* = s^{-1} t_{\nu - \mu + \eta} w$. Since $\widetilde{w}(\overline{\rho}, \tau)^* \in \mathrm{Adm}(\eta)$, $\nu - \mu \in \underline{\Lambda}_R$. Moreover, $\nu - \mu + \eta$ is in the convex hull of $\underline{W}\eta$ by Remark 4.1.4, so that ν is 1-deep in $\underline{C}_0$, since $\mu - \eta$ is n-deep in $\underline{C}_0$. Then, by letting $\mu_{\overline{\rho}} = \nu$ and $s_{\overline{\rho}} = w$, (**P2**) and (**P3**) are satisfied. Comparing, we see that $\widetilde{w}(\overline{\rho}, \tau)^*$ agrees with Definition 4.1.6. □

LEMMA 4.1.9
Suppose that $\widetilde{w}_1, \widetilde{w}_2 \in \underline{\widetilde{W}}^+$. Then $\ell(\widetilde{w}_2^{-1} w_0 \widetilde{w}_1) = \ell(\widetilde{w}_2^{-1}) + \ell(w_0) + \ell(\widetilde{w}_1)$.

Proof
The length $\ell(\widetilde{w})$ is the length of a minimal gallery between $\widetilde{w}_2^{-1} w_0 \widetilde{w}_1 \cdot \underline{C}_0$ and $\underline{C}_0$, which is the length of a minimal gallery between $w_0 \widetilde{w}_1 \cdot \underline{C}_0$ and $\widetilde{w}_2 \cdot \underline{C}_0$. Such a gallery can be taken to start with a gallery from $w_0 \widetilde{w}_1 \cdot \underline{C}_0$ to $w_0 \cdot \underline{C}_0$, then to $\underline{C}_0$, and then to $\widetilde{w}_2 \cdot \underline{C}_0$, all in the dominant direction. This decomposition of a minimal gallery in the dominant direction gives the desired equality. □

LEMMA 4.1.10
Fix (**P1**)–(**P2**) *as above. Suppose that $\lambda \in X_1(\underline{T})$ is $3n$-deep in its alcove and such that, for all $s \in W(\underline{G})$, $\tau(s, \widetilde{w}_h \cdot \lambda + \eta)$ is a tame inertial type which admits a compatible presentation as in* (**P3**) *and $\widetilde{w}^*(\overline{\rho}, \tau(s, \widetilde{w}_h \cdot \lambda + \eta)) \in \mathrm{Adm}(\eta)$. Then $F(\lambda) \in W^?(\overline{\rho})$.*

Proof
Let $s \in W(\underline{G})$, and let $\tau = \tau(s, \widetilde{w}_h \cdot \lambda + \eta)$. If $\widetilde{w}_h \cdot \lambda \in \widetilde{\sigma} \cdot \underline{C}_0$ for $\widetilde{\sigma} = \sigma t_\nu$, then τ has a lowest alcove presentation $(\sigma^{-1} s \pi(\sigma), \widetilde{\sigma}^{-1} \widetilde{w}_h \cdot \lambda + \sigma^{-1} s \pi(\sigma \nu))$ with $\widetilde{\sigma}^{-1} \widetilde{w}_h \cdot \lambda + \sigma^{-1} s \pi(\sigma \nu)$ $2n$-deep in $\underline{C}_0$ by Proposition 2.2.4(1). By assumption, $\tau \cong \tau(s', \lambda')$ has a lowest alcove presentation $(s', \lambda' - \eta)$ as in (**P3**). Combining these with Proposition 2.2.4(2), we have that $s' = w_2^{-1} s \pi(w_2)$ and $\lambda' = \widetilde{w}_2^{-1} \widetilde{w}_h \cdot \lambda + s' \pi \nu_2 + \eta$ for some $\widetilde{w}_2 = w_2 t_{\nu_2} \in \underline{\widetilde{W}}^+$. If we let $\lambda^{(0)}$ be $\widetilde{w}_2^{-1} \widetilde{w}_h \cdot \lambda$, then $\lambda^{(0)} + s' \pi \nu_2$ is in $\underline{C}_0$

since λ (and, thus, $\lambda^{(0)}$) is $3n$-deep in its alcove and $\max_{\alpha^\vee}\{|\langle \nu_2, \alpha^\vee\rangle|\} \le n-1$ by Remark 4.1.4.

Let $\widetilde{w} = \widetilde{w}^*(\overline{\rho}, \tau)$. Then by assumption, we have that $\mu_{\overline{\rho}} - \lambda^{(0)} - \pi\nu_2 - \eta \in \underline{\Lambda}_R$. Note that this condition and the condition that $\widetilde{w}_h \cdot \lambda \in \widetilde{w}_2 \cdot \underline{C}_0$ uniquely determines $\widetilde{w}_2$; hence, $\widetilde{w}_2$ does not change when s changes. Thus, as s runs over $W(\underline{G})$, s' runs over all of $W(\underline{G})$.

By definition, we have

$$\widetilde{w} = s'^{-1} t_{\mu_{\overline{\rho}} - \lambda^{(0)} - s'\pi\nu_2} s_{\overline{\rho}} = s'^{-1} t_{\delta - s'\pi\nu_2} s_{\overline{\rho}} = t_{-\pi\nu_2} s'^{-1} t_\delta s_{\overline{\rho}} \in \mathrm{Adm}(\eta),$$

where $\delta = \mu_{\overline{\rho}} - \lambda^{(0)}$. Let $\sigma \in W(\underline{G})$ (unrelated to the use of σ in the first paragraph) be the unique element such that $\sigma t_{s_{\overline{\rho}}^{-1}\delta} \in \underline{\widetilde{W}}^+$. Note that the π-action on $\underline{\widetilde{W}}$ preserves $\underline{\widetilde{W}}^+$. We now take $s' = s_{\overline{\rho}}\sigma^{-1} w_0 \pi(w_2)$, so that $\widetilde{w} = \pi(t_{-\nu_2} w_2^{-1}) w_0 \sigma t_{s_{\overline{\rho}}^{-1}\delta} = \pi(\widetilde{w}_2^{-1}) w_0 \sigma t_{s_{\overline{\rho}}^{-1}\delta}$. Since $\sigma t_{s_{\overline{\rho}}^{-1}\delta} \in \underline{\widetilde{W}}^+$, we have that $w_0\pi(\widetilde{w}_2)\widetilde{w} \in \underline{\widetilde{W}}^+$. Thus, $w_0\pi(w_2)\widetilde{w} = t_{-\pi(w_0 w_2 \nu_2)} w_0 \pi(\widetilde{w}_2)\widetilde{w}$ is in $\underline{\widetilde{W}}^+$, since $-\pi(w_0 w_2 \nu_2)$ is a dominant weight. Note that there is a gallery in the dominant direction from $w_0\pi(w_2)(\underline{C}_0)$ to $w_0\pi(w_2)\widetilde{w}(\underline{C}_0)$ passing through $\underline{C}_0$ and hence a gallery in the $(w_0\pi(w_2))^{-1}$-direction from $\underline{C}_0$ to $\widetilde{w}(\underline{C}_0)$. By [20, Corollary 4.4], $\widetilde{w} \le t_{(w_0\pi(w_2))^{-1}\eta}$. (The reference [20] uses the Bruhat order defined with respect to the *antidominant* base alcove. However, by applying w_0-conjugation which interchanges the two Bruhat orders, the cited corollary holds for the Bruhat order defined with respect to the *dominant* base alcove.)

Note that

$$\begin{aligned} t_{(w_0\pi(w_2))^{-1}\eta} &= \big(w_0\pi(w_2)\big)^{-1} t_\eta \big(w_0\pi(w_2)\big) = \big(\widetilde{w}_h^{-1}\pi(\widetilde{w}_2)\big)^{-1} \widetilde{w}_h^{-1} w_0 \big(\widetilde{w}_h^{-1}\pi(\widetilde{w}_2)\big) \\ &= \pi(\widetilde{w}_2)^{-1} w_0 \big(\widetilde{w}_h^{-1}\pi(\widetilde{w}_2)\big). \end{aligned}$$

Since $\ell(\widetilde{w}) = \ell(\pi(\widetilde{w}_2)^{-1}) + \ell(w_0) + \ell(\sigma t_{s_{\overline{\rho}}^{-1}\delta})$ by Lemma 4.1.9, we conclude that $\sigma t_{s_{\overline{\rho}}^{-1}\delta} \le \widetilde{w}_h^{-1}\pi(\widetilde{w}_2)$ by standard facts about Coxeter groups. Since both sides of this inequality are elements of $\underline{\widetilde{W}}^+$, we conclude that $\sigma t_{s_{\overline{\rho}}^{-1}\delta} \uparrow \widetilde{w}_h^{-1}\pi(\widetilde{w}_2)$. In other words, we have

$$\pi^{-1}(\sigma) t_{\pi^{-1}(s_{\overline{\rho}}^{-1}\delta)} \cdot (\mu_{\overline{\rho}} - \delta) \uparrow \lambda.$$

By Proposition 4.1.3, $F(\widetilde{w}_h^{-1}\lambda) \in \mathrm{JH}(\overline{R}_{s_{\overline{\rho}}}(\mu_{\overline{\rho}} + \eta))$ or, equivalently, $F(\lambda) \in W^?(\overline{\rho})$. □

Remark 4.1.11

Regarding the hypotheses in Lemma 4.1.10, if λ is n-deep in a p-restricted alcove,

then by Proposition 2.2.16, $\tau = \tau(s, \widetilde{w}_h \cdot \lambda + \eta)$ is 1-generic. By Proposition 2.2.15, all lowest alcove presentations $(s', \mu' - \eta)$ of τ are of the form

$$(s', \mu') =^{(\nu,\sigma)} (\widetilde{w}_h \cdot \lambda + \eta)$$

with $(t_\nu \sigma \widetilde{w}_h) \cdot \lambda \in \underline{C}_0$. The condition that there is one compatible lowest alcove presentation as in (**P3**) above is equivalent to $(\widetilde{w}_h \cdot \lambda + \eta - \mu_{\overline{\rho}})|_{\underline{Z}} \in (p - \pi) X^*(\underline{Z})$. This condition is the condition that the central character of $F(\lambda)$ agrees with the central character of any element or, equivalently, all elements of $W^?(\overline{\rho})$.

COROLLARY 4.1.12

Suppose that $\overline{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F})$ is a $5n$-generic Galois representation. If λ is in $X_1(\underline{T})$ and $(2n-1)$-deep in its alcove and $F(\lambda)$ is not in $W^?(\overline{\rho}^{\mathrm{ss}})$, then there exists an n-generic tame inertial type τ such that $F(\lambda) \in \mathrm{JH}(\overline{\sigma}(\tau))$ and $\overline{\rho}$ does not have a potentially crystalline lift of type (η, τ).

Proof

Suppose that λ is not $3n$-deep. By Proposition 2.2.16 and its proof, the tame type $\tau \stackrel{\mathrm{def}}{=} \tau(1, \widetilde{w}_h \cdot \lambda + \eta)$ is n-generic, but not $4n$-generic (by comparing $(1, \widetilde{w}_h \cdot \lambda + \eta)$ with a lowest alcove presentation). By Corollary 2.3.5 and [24, Lemma 2.5], $F(\lambda) \in \mathrm{JH}(\overline{\sigma}(\tau(1, \widetilde{w}_h \cdot \lambda + \eta)))$. Then $\overline{\rho}$ is not the reduction of a potentially crystalline representation of type (η, τ) by Proposition 3.3.2.

Now suppose that λ is $3n$-deep in its alcove. Suppose that $\overline{\rho}$ is the reduction of a potentially crystalline representation of type (η, τ) for all τ such that $F(\lambda) \in \mathrm{JH}(\overline{\sigma}(\tau))$. We claim that τ can be taken to be $\tau(s, \widetilde{w}_h \cdot \lambda + \eta)$ for any $s \in \underline{W}$. Suppose that $\widetilde{w} = w t_\nu$, and suppose that $\widetilde{w}_h \cdot \lambda \in \widetilde{w} \cdot \underline{C}_0$. Let μ be $\widetilde{w}^{-1} \widetilde{w}_h \cdot \lambda + w^{-1} s \pi(w\nu)$. Then $\tau(w^{-1} s \pi(w), \mu + \eta) \cong \tau(s, \widetilde{w}_h \cdot \lambda + \eta)$ by Proposition 2.2.4, and μ is $2n+1$-deep in $\underline{C}_0$ by Remark 4.1.4. Then $\overline{\sigma}(\tau(s, \widetilde{w}_h \cdot \lambda + \eta)) = \overline{R}_{w^{-1} s \pi(w)}(\mu + \eta)$ contains $F(\lambda)$ as a Jordan–Hölder factor by Proposition 4.1.3 ($\widetilde{w} \cdot (\mu - w^{-1} s \pi(w\nu)) = \widetilde{w}_h \cdot \lambda$). Thus, for all $s \in W(\underline{G})$, $\overline{\rho}$ is the reduction of a potentially crystalline representation of type $(\eta, \tau(s, \widetilde{w}_h \cdot \lambda + \eta))$.

By Theorem 3.2.1 and Proposition 4.1.8, for each $s \in W(\underline{G})$, there is a pair $(s_{\overline{\rho}}, \mu_{\overline{\rho}})$ for $\overline{\rho}^{\mathrm{ss}}$ as in (**P2**) such that $\widetilde{w}^*(\overline{\rho}^{\mathrm{ss}}, \tau(s, \widetilde{w}_h \cdot \lambda + \eta)) \in \mathrm{Adm}(\eta)$ (which implies (**P3**) holds). By Remark 4.1.7, these conditions hold for any choice of $(s_{\overline{\rho}}, \mu_{\overline{\rho}})$ satisfying (**P2**), since $\mathrm{Adm}(\eta)$ is stable under $\underline{\Omega}$-conjugation. Thus we can use the same pair $(s_{\overline{\rho}}, \mu_{\overline{\rho}})$ for all choices of s. Then by Lemma 4.1.10, we see that $F(\lambda) \in W^?(\overline{\rho}^{\mathrm{ss}})$. □

Remark 4.1.13

In the proof of Corollary 4.1.12, we used Proposition 4.1.3 to show that $F(\lambda) \in \mathrm{JH}(\overline{\sigma}(\tau(s, \widetilde{w}_h \cdot \lambda + \eta)))$. This weight is the reflection of an obvious weight or a

diagonal prediction in [22, Definition 12.8], so the above membership is likely true with weaker genericity hypotheses on the Deligne–Lusztig representation. Corollary 4.1.12 also probably holds with weaker genericity hypotheses.

4.2. Weight elimination

In this section, we deduce our main weight elimination result (Corollary 4.2.7). We begin with an axiomatic setup for our method and then proceed to the case of modular forms for definite unitary groups.

4.2.1. Axiomatic setup

We begin with an axiomatic setup for modular Serre weights. This is related to the axiomatic setup of [11, Section 4.2]. Let $\overline{\rho}: G_K \to \mathrm{GL}_n(\mathbb{F})$ be a Galois representation. (We no longer assume (**P1**)–(**P3**).) We write $\mathbb{F}[\mathrm{GL}_n(k)]$-mod for the category of finite $\mathbb{F}[\mathrm{GL}_n(k)]$-modules and $\mathrm{Vect}_{/\mathbb{F}}$ for the category of $\mathbb{F}$-modules (i.e., vector spaces).

Definition 4.2.1

We say that a functor $S : \mathbb{F}[\mathrm{GL}_n(k)]\text{-mod} \to \mathrm{Vect}_{/\mathbb{F}}$ is an *arithmetic cohomology functor for* $\overline{\rho}$ if

- $S \neq 0$;
- whenever $S(V) \neq 0$ for a Serre weight V and $V \in \mathrm{JH}(\overline{\sigma}(\tau))$ for a regular tame inertial type τ, $\overline{\rho}$ has a potentially crystalline lift of type (η, τ); and
- whenever $S(V) \neq 0$ for a Serre weight V and $V \in \mathrm{JH}(\overline{R}_1(\mu))$, $\overline{\rho}$ has a potentially semistable lift of type $(\eta, \tau(1, \mu))$.

We now fix an arithmetic cohomology functor S for $\overline{\rho}$.

Definition 4.2.2

We say that $\overline{\rho}$ is *modular of weight* V if $S(V) \neq 0$. Let $W(\overline{\rho})$ be the set of isomorphism classes of Serre weights for which $\overline{\rho}$ is modular. Let $W_{\mathrm{elim}}(\overline{\rho})$ be the subset of $W(\overline{\rho})$ consisting of isomorphism classes of Serre weights represented by $F(\lambda)$, where λ is $(2n-1)$-deep in its alcove.

THEOREM 4.2.3

If $\overline{\rho}$ is $5n$-generic, then $W_{\mathrm{elim}}(\overline{\rho}) \subset W^?(\overline{\rho}^{\mathrm{ss}})$.

Proof

If λ is $(2n-1)$-deep in its alcove and $F(\lambda) \notin W^?(\overline{\rho}^{\mathrm{ss}})$, then by Corollary 4.1.12 there is a regular (n-generic even) tame inertial type τ such that $F(\lambda) \in \mathrm{JH}(\overline{\sigma}(\tau))$

and $\overline{\rho}$ does not have a potentially crystalline lift of type (η, τ). We conclude that $F(\lambda) \notin W(\overline{\rho})$. □

Theorem 8 in [12] shows that, for sufficiently generic $\overline{\rho}$, $W(\overline{\rho})$ does not contain weights near the boundary of alcoves. Combining this with Theorem 4.2.3, we obtain an upper bound for $W(\overline{\rho})$ rather than $W_{\mathrm{elim}}(\overline{\rho})$.

COROLLARY 4.2.4
If $\overline{\rho}$ is $(6n-2)$-generic, then $W(\overline{\rho}) \subset W^?(\overline{\rho}^{\mathrm{ss}})$.

Proof
Suppose that $\lambda \in X_1(\underline{T})$ is 0-deep but not $(2n-1)$-deep in its alcove. Let $\lambda^{(0)}$ be the unique weight in $\underline{C}_0$ linked to λ. Then $\lambda^{(0)}$ is also not $(2n-1)$-deep in its alcove so that $\lambda^{(0)} + \eta$ is not $2n$-generic in the sense of [12, Definition 2]. We conclude that λ is not $(4n-2)$-generic in the sense of [12, Definition 1]. Note that even if λ is not 0-deep in its alcove, this last conclusion still holds. Now $\overline{\rho}$ is $(6n-2)$-generic in the sense of [12, Definition 2] by Remark 2.2.8. Then the proof of [12, Theorem 8] shows that $\overline{\rho}$ does not have a potentially semistable lift of type $(\eta, \tau(1, \lambda))$, while $F(\lambda) \in \mathrm{JH}(\overline{R}_1(\lambda))$ by [24, Lemma 2.5]. We conclude that $F(\lambda) \notin W(\overline{\rho})$. □

4.2.2. Algebraic modular forms for unitary groups
We closely follow the setup of [11, Section 7.1] (see also [25, Section 4.1]). Let $F/\mathbb{Q}$ be a CM field, and let F^+ be its maximal totally real subfield. Assume that $F^+ \neq \mathbb{Q}$, and assume that all places of F^+ dividing p split in F and are unramified. We write c for the generator of $\mathrm{Gal}(F/F^+)$. For $u \nmid \infty$ (resp., $v \nmid \infty$) a place of F (resp., F^+) we denote by k_u (resp., k_v) the residue field of F_u (resp., F_v^+).

We let $G_{/F^+}$ be a reductive group, which is an outer form of GL_n which splits over F. We assume that $G(F_v^+) \cong U_n(\mathbb{R})$ for all $v|\infty$. By the argument of [9, Section 3.3], G admits a model $\mathcal{G}$ over $\mathcal{O}_{F^+}$ such that $\mathcal{G} \times \mathcal{O}_{F_v^+}$ is reductive for all places v of F^+ that split in F. For any such place v of F^+ and $u|v$ of F we get an isomorphism $\iota_u : G(F_v^+) \xrightarrow{\sim} \mathrm{GL}_n(F_u)$ which restricts moreover to an isomorphism $\iota_u : \mathcal{G}(\mathcal{O}_{F_v^+}) \xrightarrow{\sim} \mathrm{GL}_n(\mathcal{O}_{F_u})$. Let Σ be a finite set of places in F^+. If $U^\Sigma \leq G(\mathbb{A}_{F^+}^{\infty,\Sigma})$ is a compact open subgroup, then the space $S(U^\Sigma)$ of infinite-level algebraic automorphic forms on G is defined to be the set of continuous functions $f : G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty)/U^\Sigma \to \mathbb{F}$, where $\mathbb{F}$ is given the discrete topology.

We recall that the level $U \leq G(\mathbb{A}_{F^+}^\infty)$ is said to be *sufficiently small* if, for all $t \in G(\mathbb{A}_{F^+}^\infty)$, the finite group $t^{-1}G(F^+)t \cap U$ is of order prime to p. We say that U^Σ is *sufficiently small* if

$$U = U^{\Sigma} \prod_{v \in \Sigma} \mathcal{G}(\mathcal{O}_{F_v^+})$$

is sufficiently small. For a finite place v of F^+ we say that U is *unramified* at v if v splits in F and one has a decomposition $U = \mathcal{G}(\mathcal{O}_{F_v^+})U^v$ for some compact open subgroup $U^v \leq G(\mathbb{A}_{F^+}^{\infty,v})$.

Let $\mathcal{P}_U$ denote the set consisting of finite places u of F such that $v \stackrel{\text{def}}{=} u|_{F^+}$ is split in F, $v \nmid p$, and U is unramified at v. If $\mathcal{P} \subseteq \mathcal{P}_U$ is a subset of finite complement that is closed under complex conjugation and disjoint from Σ, we write $\mathbb{T}^{\mathcal{P}} = \mathcal{O}[T_u^{(i)} : u \in \mathcal{P}, i \in \{0, 1, \ldots, n\}]$ for the abstract Hecke algebra on $\mathcal{P}$, where the Hecke operator $T_u^{(i)}$ acts on the space $S(U^{\Sigma})$ as the usual double coset operator

$$\iota_u^{-1}\left[\mathrm{GL}_n(\mathcal{O}_{F_u})\begin{pmatrix}\varpi_u \mathrm{Id}_i & \\ & \mathrm{Id}_{n-i}\end{pmatrix}\mathrm{GL}_n(\mathcal{O}_{F_u})\right],$$

where ϖ_u denotes a uniformizer of F_u. If $\overline{r} : G_F \to \mathrm{GL}_n(\mathbb{F})$ is a continuous, absolutely irreducible Galois representation, we further write $\mathfrak{m}_{\overline{r}}$ for the maximal ideal of $\mathbb{T}^{\mathcal{P}}$ with residue field $\mathbb{F}$ defined by the formula

$$\det\bigl(1 - \overline{r}^{\vee}(\mathrm{Frob}_u)X\bigr) = \sum_{j=0}^{n} (-1)^j \mathbf{N}_{F_u/\mathbb{Q}_p}(u)^{\binom{j}{2}} (T_u^{(j)} \bmod \mathfrak{m}_{\overline{r}}) X^j \quad \forall u \in \mathcal{P}.$$

Definition 4.2.5
We say that $\overline{r}$ is *modular* of (prime-to-Σ) level U^{Σ} if $S(U^{\Sigma})_{\mathfrak{m}_{\overline{r}}} \neq 0$. We say $\overline{r}$ is modular if $\overline{r}$ is modular of some level U^{Σ}.

Assume that $\overline{r}$ is modular of level U^{Σ}, and assume that Σ contains all places dividing p and all places divisible by places in F where $\overline{r}$ is ramified. Fix places $u \mid v \mid p$ of F and F^+. Then we define the functor $S : \mathbb{F}[\mathrm{GL}_n(k_u)]\text{-mod} \to \mathrm{Vect}_{/\mathbb{F}}$ by $S(V) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_{F_u})}(V^{\vee}, S(U^{\Sigma})_{\mathfrak{m}_{\overline{r}}})$, where $(\cdot)^{\vee}$ denotes the contragradient representation and $\mathrm{GL}_n(\mathcal{O}_{F_u})$ acts on V by inflation and on $S(U^{\Sigma})_{\mathfrak{m}_{\overline{r}}}$ via ι_u^{-1}.

PROPOSITION 4.2.6
If U^{Σ} is sufficiently small, then S is an arithmetic cohomology functor for $\overline{r}|_{G_{F_u}}$.

Proof
This essentially follows from the proof of Proposition 7.4.4 of [11]. Note that the modularity of $\overline{r}$ of level U^{Σ} implies that the functor S is nonzero. Suppose that $S(V) \neq 0$ and $V \in \mathrm{JH}(\overline{\sigma}(\tau))$. Let $v' \in \Sigma$ be a place not dividing p (if one exists). Since $S(V)$ is a smooth representation of $G(F_{v'}^+)$, there exists a compact open subgroup $K_{v'}$ of

$G(\mathcal{O}_{F^+_{v'}})$ with nonzero invariants. Inductively choosing $K_{v'}$ and replacing $S(U^\Sigma)_{\mathfrak{m}_{\overline{r}}}$ with

$$S\Big(U^\Sigma \prod_{v'\in\Sigma, v'\nmid p} K_{v'}\Big)_{\mathfrak{m}_{\overline{r}}},$$

we assume without loss of generality that Σ is exactly the places of F^+ dividing p, and we write $U^\Sigma = U^p$. For each place $v' \mid p$ of F^+, choose a place $\widetilde{v}' \mid v'$ of F such that $\widetilde{v}'|_{F^+} = v'$ and $\widetilde{v} = u$. For an $\mathcal{O}[\![\mathrm{GL}_n(\mathcal{O}_{F,\widetilde{v}'})]\!]$-representation $V_{\widetilde{v}'}$, let $V_{v'}$ be the corresponding $\mathcal{O}[\![\mathcal{G}(\mathcal{O}_{F^+,v'})]\!]$-representation via $\iota_{\widetilde{v}}$. There exist irreducible $\mathbb{F}[\![\mathrm{GL}_n(\mathcal{O}_{F,\widetilde{v}'})]\!]$-representations $V_{\widetilde{v}'}$ for every $v' \mid p$ such that $V_{\widetilde{v}} = V$ and $\mathrm{Hom}_{\mathcal{G}(\mathcal{O}_{F^+,p})}(\bigotimes_{v'|p} V_{v'}^\vee, S(U^p)_{\mathfrak{m}_{\overline{r}}}) \neq 0$. Choose an $\mathcal{O}[\![\mathcal{G}(\mathcal{O}_{F^+,p})]\!]$-representation $W = \bigotimes_{v'|p} W_{v'}$ such that $W_{\widetilde{v}}$ is an $\mathcal{O}$-lattice in $\sigma(\tau)$, $W_{\widetilde{v}'}$ is an $\mathcal{O}$-lattice in an algebraic representation over E for $v' \neq v$ (say, a suitable Weyl module), and $V_{\widetilde{v}'} \in \mathrm{JH}(\overline{W}_{\widetilde{v}'})$ for all $v' \mid p$ (where the bar denotes mod ϖ reduction). For U^p sufficiently small, $S(U^p)_{\mathfrak{m}_{\overline{r}}}$ is an injective $\mathbb{F}[\![\mathcal{G}(\mathcal{O}_{F^+,p})]\!]$-module; hence $\mathrm{Hom}_{\mathbb{F}[\![\mathcal{G}(\mathcal{O}_{F^+,p})]\!]}(-, S(U^p)_{\mathfrak{m}_{\overline{r}}})$ is exact. Thus, $\mathrm{Hom}_{\mathcal{G}(\mathcal{O}_{F^+,p})}(\overline{W}^\vee, S(U^p)_{\mathfrak{m}_{\overline{r}}}) \neq 0$. Then the proof of [11, Proposition 7.4.4 Ã1] holds without modification and one then constructs π as in the proof of [11, Proposition 7.4.4 Ã1]. Then $r_\pi^\vee|_{G_{F_u}}$ provides the required lift by [11, Theorem 7.2.1], Theorem 2.3.1, and Proposition 2.3.4. □

Assume that $\overline{r}$ is modular. Then $\overline{r}$ is modular of some sufficiently small level U^Σ. Let $W(\overline{r})$ be the set of isomorphism classes of irreducible $\mathcal{G}(\mathcal{O}_{F^+,p})$-representations V over $\mathbb{F}$ such that

$$\mathrm{Hom}_{\mathcal{G}(\mathcal{O}_{F^+,p})}\big(V^\vee, S(U^\Sigma)_{\mathfrak{m}_{\overline{r}}}\big) \neq 0.$$

For each place $v \mid p$ of F^+ choose a place $\widetilde{v} \mid v$ of F. Let $W^?_{\widetilde{v},\mathrm{ss}}(\overline{r})$ be $W^?((\overline{r}|_{G_{F_{\widetilde{v}}}})^{\mathrm{ss}})$, and let $W^?_{v,\mathrm{ss}}(\overline{r})$ be the set of isomorphism classes of $\mathcal{G}(\mathcal{O}_{F^+_v})$-representations over $\mathbb{F}$ corresponding to $W^?_{\widetilde{v},\mathrm{ss}}(\overline{r})$ via $\iota_{\widetilde{v}}$. Note that the definition of $W^?_{v,\mathrm{ss}}(\overline{r})$ does not depend on the choice of place $\widetilde{v}$. Let $W^?_{\mathrm{ss}}(\overline{r})$ be $\bigotimes_{v|p} W^?_{v,\mathrm{ss}}(\overline{r})$.

COROLLARY 4.2.7
Suppose that $(\overline{r}|_{G_{F_{\widetilde{v}}}})^{\mathrm{ss}}$ *is* $(6n-2)$*-generic for all places* $v \mid p$ *of* F^+. *We have* $W(\overline{r}) \subset W^?_{\mathrm{ss}}(\overline{r})$.

Proof
Suppose that $\mathrm{Hom}_{\mathcal{G}(\mathcal{O}_{F^+,p})}(\bigotimes_{v|p} F(\lambda_v)^\vee, S(U^p)_{\mathfrak{m}_{\overline{r}}}) \neq 0$. For each place $v \mid p$ of F^+, let $S_{\widetilde{v}}$ be $\mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_{\widetilde{v}})}((\cdot)^\vee, S(U^p)_{\mathfrak{m}_{\overline{r}}})$. By Proposition 4.2.6, $S_{\widetilde{v}}$ is an arithmetic cohomology functor for $\overline{r}|_{G_{F_{\widetilde{v}}}}$, from which we define $W(\overline{r}|_{G_{F_{\widetilde{v}}}})$ as in Definition 4.2.2. If $F(\lambda_{\widetilde{v}})$ corresponds to $F(\lambda_v)$ via $\iota_{\widetilde{v}}$, then $S_{\widetilde{v}}(F(\lambda_{\widetilde{v}}))$ is nonzero, and so

$F(\lambda_{\widetilde{v}}) \in W(\overline{r}|_{G_{F_{\widetilde{v}}}})$. By Corollary 4.2.4, $F(\lambda_{\widetilde{v}}) \in W^?_{\mathrm{ss}}(\overline{r}|_{G_{F_{\widetilde{v}}}})$. Thus, $\bigotimes_{v|p} F(\lambda_v) \in W^?_{\mathrm{ss}}(\overline{r})$. □

4.3. Modularity of "obvious" Serre weights

In this section, we deduce the modularity of obvious Serre weights for sufficiently generic semisimple local Galois representations under mild hypotheses. We show that for each obvious weight there is a type containing it as a Jordan–Hölder factor so that no other Jordan–Hölder factor is modular (Corollary 4.2.7 and Proposition 4.3.6). Fortunately, these are precisely the types to which we can apply Corollary 3.4.13 and the results of [3] to deduce the modularity of obvious Serre weights. That these types isolate weights can be seen as a consequence of Theorem 3.4.1 and the Breuil–Mézard philosophy (see Remark 4.3.7). We first summarize the results we need from [3] and refer the reader to [3] for any undefined notation and terminology.

THEOREM 4.3.1

Let $p > 2$, and let F be a CM field with maximal totally real subfield F^+ such that $\zeta_p \notin F$. Assume that F/F^+ is split at all places dividing p. Suppose that $\overline{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is an irreducible representation with the following additional properties.

(1) *$\overline{r}$ is potentially diagonalizably automorphic, that is, there is an RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that*
   - *$\overline{r} \cong \overline{r}_{p,\iota}(\Pi)$; and*
   - *for each place $u \mid p$ of F, $r_{p,\iota}(\Pi)|_{G_{F_u}}$ is potentially diagonalizable.*

(2) *The image of $\overline{r}(G_{F(\zeta_p)})$ is adequate.*

Let Σ be a finite set of places of F^+ containing all places dividing p and all the places of F^+ divisible by places at which $\overline{r}$ ramifies. For each place $v \in \Sigma$, choose a place $\widetilde{v} \mid v$ of F, and choose a lift $\rho_{\widetilde{v}} : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(\overline{\mathbb{Z}}_p)$. Suppose that, for $v \mid p$, $\rho_{\widetilde{v}}$ is potentially crystalline and potentially diagonalizable with distinct Hodge–Tate weights for every embedding $F_{\widetilde{v}} \hookrightarrow \overline{\mathbb{Q}}_p$. Then there is an RACSDC automorphic representation π such that

- *$\overline{r} \cong \overline{r}_{p,\iota}(\pi)$;*
- *π_u is unramified at all places u of F that do not divide a place in Σ; and*
- *$r_{p,\iota(\pi)}|_{G_{F_{\widetilde{v}}}} \sim \rho_{\widetilde{v}}$ for all places $v \in \Sigma$.*

Proof

This is [2, Theorem 3.1.3], except for two differences:

(1) for places $v \mid p$ of F^+, $\overline{\rho}_{\widetilde{v}}$ is allowed to be potentially crystalline rather than crystalline; and

(2) Σ may contain places which do not split in F.

However, the proof of [2, Theorem 3.1.3] still applies with two corresponding modifications:

(1) π_u is not necessarily unramified if u is a place of F dividing p; and
(2) we replace the use of [2, Theorem A.4.1] with [4, Theorem 5.2.1]. □

Define (see Definition 2.2.11 and Section 2.3)

$$W^?(\overline{\rho},\tau) := W^?(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\tau)).$$

We can characterize $W^?(\overline{\rho},\tau)$ in terms of the element $\widetilde{w}^*(\overline{\rho},\tau)$ from Definition 4.1.6.

PROPOSITION 4.3.2
Fix **(P1)**–**(P3)** *from Section 4.1 with $\mu-\eta$ $2n$-deep in $\underline{C}_0$, and let $\widetilde{w} = \widetilde{w}^*(\overline{\rho},\tau) = t_\nu w$. Then for λ a dominant p-restricted character, $F(\lambda)$ is in $W^?(\overline{\rho},\tau)$ if and only if there exist $\widetilde{w}_\lambda, \widetilde{w}_1, \widetilde{w}_2 \in \widetilde{\underline{W}}^+$ with $\widetilde{w}_\lambda \cdot \underline{C}_0$ p-restricted and $w' \in W(\underline{G})$ such that*

- $\pi^{-1}(\widetilde{w}) = \widetilde{w}_2^{-1} w' \widetilde{w}_1$;
- $\widetilde{w}_1 \uparrow \widetilde{w}_\lambda \uparrow \widetilde{w}_h^{-1}\widetilde{w}_2$; *and*
- $\lambda = \widetilde{w}_\lambda \cdot (\mu - s\pi\nu_2 - \eta)$, *where* $\widetilde{w}_2 = w_2 t_{\nu_2}$.

Proof
Recall from Definition 2.2.11 that $F(\lambda) \in W^?(\overline{\rho})$ if and only if $F(\widetilde{w}_h^{-1}\cdot\lambda) \in \mathrm{JH}(\overline{R}_{sw}(\mu+s\nu))$ since $\widetilde{w}_h\cdot\lambda = \mathcal{R}(\lambda)$. (Recall that $\mu+s\nu = \mu_{\overline{\rho}}+\eta$ so that $\nu \equiv \eta \pmod{\underline{\Lambda}_R}$ and $sw = s_{\overline{\rho}}$.) By Proposition 4.1.3, $F(\lambda) \in \mathrm{JH}(\overline{\sigma}(\tau))$ (resp., $F(\widetilde{w}_h^{-1}\cdot\lambda) \in \mathrm{JH}(\overline{R}_{sw}(\mu+s\nu))$) if and only if there exists a $\widetilde{w}_2 \in \widetilde{\underline{W}}^+$ (resp., $\widetilde{w}_1 \in \widetilde{\underline{W}}^+$) such that $\widetilde{w}_2\cdot(\mu-s\pi\nu_2-\eta) \uparrow \widetilde{w}_h\cdot\lambda$ (resp., $\widetilde{w}_1\cdot(\mu+s\nu-sw\pi\nu_1-\eta)\uparrow\lambda$), where $\widetilde{w}_2 = w_2 t_{\nu_2}$ (resp., $\widetilde{w}_1 = w_1 t_{\nu_1}$) is such that $\widetilde{w}_2\cdot\underline{C}_0 \uparrow \widetilde{w}_h\cdot\underline{C}_0$ (and $\widetilde{w}_1\cdot\underline{C}_0 \uparrow \widetilde{w}_h\cdot\underline{C}_0$). In summary, $F(\lambda)\in W^?(\overline{\rho},\tau)$ if and only if there exist $\widetilde{w}_1$ and $\widetilde{w}_2 \in \widetilde{\underline{W}}^+$ with $\widetilde{w}_i\cdot\underline{C}_0 \uparrow \widetilde{w}_h\cdot\underline{C}_0$ for $i=1,2$ such that

$$\widetilde{w}_1\cdot(\mu+s\nu-sw\pi\nu_1-\eta) \uparrow \lambda \uparrow \widetilde{w}_h^{-1}\widetilde{w}_2\cdot(\mu-s\pi\nu_2-\eta).$$

By our assumption, $\mu_{\overline{\rho}} = \mu+s\nu-\eta$ and $\mu-\eta$ are both $2n$-deep in $\underline{C}_0$, so Remark 4.1.4 implies that both $\mu+s\nu-sw\pi\nu_1-\eta$ and $\mu-s\pi\nu_2-\eta$ are in $\underline{C}_0$. Thus, the above condition is equivalent to the existence of $\widetilde{w}_\lambda, \widetilde{w}_1, \widetilde{w}_2 \in \widetilde{\underline{W}}^+$ with $\widetilde{w}_\lambda\cdot\underline{C}_0$ p-restricted such that

- $\widetilde{w}_1\cdot(\mu+s\nu-sw\nu_1-\eta)$ is linked to $\widetilde{w}_h^{-1}\widetilde{w}_2\cdot(\mu-s\pi\nu_2-\eta)$;
- $\widetilde{w}_1 \uparrow \widetilde{w}_\lambda \uparrow \widetilde{w}_h^{-1}\widetilde{w}_2$; and
- $\lambda = \widetilde{w}_\lambda\cdot(\mu-s\pi\nu_2-\eta)$.

We claim that $\widetilde{w}_1\cdot(\mu+s\nu-sw\pi\nu_1-\eta)$ and $\widetilde{w}_h^{-1}\widetilde{w}_2\cdot(\mu-s\pi\nu_2-\eta)$ are linked if and only if $\widetilde{w}_2\pi^{-1}(\widetilde{w})\widetilde{w}_1^{-1} \in W(\underline{G})$.

We first show that $\widetilde{w}_1 \cdot (\mu + s\nu - sw\pi\nu_1 - \eta)$ and $\widetilde{w}_h^{-1}\widetilde{w}_2 \cdot (\mu - s\pi\nu_2 - \eta)$ are linked if and only if $\nu + \pi\nu_2 = w\pi\nu_1$. If $\nu + \pi\nu_2 = w\pi\nu_1$, then one sees directly that $\widetilde{w}_1^{-1}\widetilde{w}_h^{-1}\widetilde{w}_2$ is in $\underline{W}_a$, from which we see that $\widetilde{w}_1 \cdot (\mu + s\nu - sw\pi\nu_1 - \eta)$ and $\widetilde{w}_h^{-1}\widetilde{w}_2 \cdot (\mu - s\pi\nu_2 - \eta)$ are linked. Now suppose that $\widetilde{w}_1 \cdot (\mu + s\nu - sw\pi\nu_1 - \eta)$ and $\widetilde{w}_h^{-1}\widetilde{w}_2 \cdot (\mu - s\pi\nu_2 - \eta)$ are linked. Then the restriction of the difference of $\widetilde{w}_1 \cdot (\mu + s\nu - sw\pi\nu_1 - \eta)$ and $\widetilde{w}_h^{-1}\widetilde{w}_2 \cdot (\mu - s\pi\nu_2 - \eta)$ to the center $\underline{Z}$ of $\underline{G}$ must be trivial. Equivalently, the restriction of $(p-\pi)(\nu - \nu_1 + \nu_2)$ and therefore $\nu - \nu_1 + \nu_2$ to $\underline{Z}$ must be trivial. By noting that $\nu \equiv \eta \mod \underline{\Lambda}_R$, this implies that $\widetilde{w}^{-1}\widetilde{w}_h^{-1}\widetilde{w}_2$ is in $\underline{W}_a$. We conclude that $\mu + s\nu - sw\pi\nu_1 - \eta$ and $\mu - s\pi\nu_2 - \eta$ are linked. Since $\mu + s\nu - sw\pi\nu_1 - \eta$ and $\mu - s\pi\nu_2 - \eta$ are both in $\underline{C}_0$, they must therefore be equal. This equality implies the equality $\nu + \pi\nu_2 = w\pi\nu_1$.

Finally, $\nu + \pi\nu_2 = w\pi\nu_1$ if and only if

$$\begin{aligned}\widetilde{w}_2\pi^{-1}(\widetilde{w})\widetilde{w}_1^{-1} &= w_2 t_{\nu_2}\pi^{-1}(t_\nu w)t_{-\nu_1}w_1^{-1}\\ &= w_2\pi^{-1}(t_{\pi\nu_2+\nu-w\pi\nu_1}w)w_1^{-1} \in W(\underline{G}).\end{aligned}$$

□

Remark 4.3.3
Note that $F(\lambda)$ is an obvious weight (Definition 2.2.14) if and only if $\widetilde{w}_\lambda = \widetilde{w}_1$.

LEMMA 4.3.4
Let $\widetilde{w} \in \underline{\widetilde{W}}$, and let $w \in W(\underline{G})$ and $\widetilde{w}^+ \in \underline{\widetilde{W}}^+$ be the unique elements such that $\widetilde{w} = w\widetilde{w}^+$. Then $\widetilde{w}^+ \le \widetilde{w}$.

Proof
The length $\ell(w\widetilde{w}^+)$ is the length of a minimal gallery from $\underline{C}_0$ to $w\widetilde{w}^+ \cdot \underline{C}_0$, which is the length of a minimal gallery from $w^{-1} \cdot \underline{C}_0$ to $\widetilde{w}^+ \cdot \underline{C}_0$. A minimal gallery can be taken through $\underline{C}_0$ in the dominant direction. Hence $\ell(w\widetilde{w}^+) = \ell(w) + \ell(\widetilde{w}^+)$, and therefore $\widetilde{w}^+ \le \widetilde{w}$. □

PROPOSITION 4.3.5
If $\widetilde{w}_2 \uparrow \widetilde{w}_1$ and $\widetilde{w}_2 \in \underline{\widetilde{W}}^+$, then $\widetilde{w}_2 \le \widetilde{w}_1$.

Proof
Let $w \in W$ be the unique element such that $w\widetilde{w}_1 \in \underline{\widetilde{W}}^+$. Then $\widetilde{w}_1 \uparrow w\widetilde{w}_1$ by [27, Section II 6.5 (5)]. Then $\widetilde{w}_2 \le w\widetilde{w}_1$ by Theorem 4.1.1. Since $w\widetilde{w}_1 \le \widetilde{w}_1$ by Lemma 4.3.4, $\widetilde{w}_2 \le \widetilde{w}_1$. □

PROPOSITION 4.3.6
Let $\overline{\rho}$, τ, and $\widetilde{w}$ be as in Proposition 4.3.2, and suppose that $\widetilde{w} = t_{\pi s^{-1}(\eta)}$. Then

$W^?(\overline{\rho},\tau)=\{F(\lambda)\}$, where $F(\lambda)\in W_{\mathrm{obv}}(\overline{\rho})$ is the obvious weight corresponding to s (see Definition 2.2.14).

Proof
Suppose that $\pi^{-1}(\widetilde{w})=t_{s^{-1}(\eta)}=\widetilde{w}_2^{-1}w'\widetilde{w}_1$, where $w'\in W(\underline{G})$, $\widetilde{w}_1,\widetilde{w}_2\in\underline{\widetilde{W}}^+$, and $\widetilde{w}_1\uparrow\widetilde{w}_\lambda\uparrow\widetilde{w}_h^{-1}\widetilde{w}_2$ for some $\widetilde{w}_\lambda$ with $\widetilde{w}_\lambda\cdot\underline{C}_0$ p-restricted. We have

$$\begin{aligned}\ell(t_{s^{-1}(\eta)})&\le\ell(\widetilde{w}_2^{-1})+\ell(w')+\ell(\widetilde{w}_1)\le\ell\big((\widetilde{w}_h\widetilde{w}_\lambda)^{-1}\big)+\ell(w_0)+\ell(\widetilde{w}_\lambda)\\&=\ell\big((\widetilde{w}_h\widetilde{w}_\lambda)^{-1}w_0\widetilde{w}_\lambda\big)=\ell(t_{w_\lambda^{-1}(\eta)}),\end{aligned}$$

where $w_\lambda\in W(\underline{G})$ is the projection of $\widetilde{w}_\lambda$. The first inequality is obvious, while the first equality follows from Lemma 4.1.9. For the second inequality, since $w'\le w_0$, it suffices to show that $\widetilde{w}_2\le\widetilde{w}_h\widetilde{w}_\lambda$ and $\widetilde{w}_1\le\widetilde{w}_\lambda$. Since $\widetilde{w}_\lambda\uparrow\widetilde{w}_h^{-1}\widetilde{w}_2$, we have that $\widetilde{w}_2\uparrow\widetilde{w}_h\widetilde{w}_\lambda$. Since $\widetilde{w}_1$ and $\widetilde{w}_2\in\underline{\widetilde{W}}^+$, we have that $\widetilde{w}_2\le\widetilde{w}_h\widetilde{w}_\lambda$ and $\widetilde{w}_1\le\widetilde{w}_\lambda$ by Proposition 4.3.5.

Since $\ell(t_{s^{-1}(\eta)})=\ell(t_{w_\lambda^{-1}(\eta)})$, we have $w'=w_0$ and $\widetilde{w}_2=\widetilde{w}_h\widetilde{w}_\lambda=\widetilde{w}_h\widetilde{w}_1$. This implies that $s=w_\lambda$. Now suppose that $F(\lambda)\in W^?(\overline{\rho},\tau)$. We now use notation from Proposition 4.3.2, particularly from (**P2**) and Definition 4.1.6. Then by Proposition 4.3.2, $\lambda=\widetilde{w}_\lambda\cdot(\mu_{\overline{\rho}}-s_{\overline{\rho}}\pi\nu_\lambda)$, where we write $\widetilde{w}_\lambda=w_\lambda t_{\nu_\lambda}$ and $\overline{\rho}|_{I_K}=\tau(s_{\overline{\rho}},\mu_{\overline{\rho}}+\eta)$. This is exactly the obvious weight corresponding to $s=w_\lambda$ (Definition 2.2.14). □

Remark 4.3.7
One could show using Theorem 3.4.1, Corollary 4.2.7, and Kisin's approach to the Breuil–Mézard conjecture that, with the hypotheses of Proposition 4.3.6, $\#W^?(\overline{\rho},\tau)\le 1$. This leads to an alternate proof of Proposition 4.3.6, which we eschew in favor of our more direct approach.

In the setting of Proposition 4.3.6, if $\widetilde{w}^*(\overline{\rho},\tau)=t_{\pi w^{-1}\eta}$ for some $w\in W(\underline{G})$, we say that τ is the *obvious type* for the obvious weight of $\overline{\rho}$ corresponding to w. (Note that this notion depends on the choice of $(s_{\overline{\rho}},\mu_{\overline{\rho}})$.) Such a type τ always exists and is uniquely determined by the corresponding obvious weight.

We use the setup and notation of Section 4.2.2. For each place $v\mid p$ of F^+ choose a place $\widetilde{v}\mid v$ of F. Let $\overline{r}:G_F\to\mathrm{GL}_n(\mathbb{F})$ be a modular Galois representation such that, for each place $v\mid p$ of F^+, $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is semisimple. Let $W_{\mathrm{obv},\widetilde{v}}(\overline{r})\overset{\mathrm{def}}{=}W_{\mathrm{obv}}(\overline{r}|_{G_{F_{\widetilde{v}}}})$. Let $W_{\mathrm{obv},v}(\overline{r})$ be the set of isomorphism classes of $\mathcal{G}(\mathcal{O}_{F_v^+})$-representations over $\mathbb{F}$ corresponding to $W_{\mathrm{obv},\widetilde{v}}(\overline{r})$. Note that the definition of $W_{\mathrm{obv},v}(\overline{r})$ does not depend on the choice of place $\widetilde{v}$. Let $W_{\mathrm{obv}}(\overline{r})$ be $\bigotimes_{v\mid p}W_{\mathrm{obv},v}(\overline{r})$.

THEOREM 4.3.8
Suppose that $\zeta_p \notin F$, and suppose that $\overline{r} : G_F \to \mathrm{GL}_n(\mathbb{F})$ is a modular Galois representation such that $\overline{r}(G_{F(\zeta_p)})$ is adequate. Assume that, for all $\widetilde{v} \mid p$, $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is semisimple and $(6n-2)$-generic. Then the following are equivalent:

(1) $W_{\mathrm{obv}}(\overline{r}) \cap W(\overline{r}) \neq \emptyset$;
(2) *$\overline{r}$ is potentially diagonalizably automorphic (see Theorem 4.3.1(1)); and*
(3) $W_{\mathrm{obv}}(\overline{r}) \subset W(\overline{r})$.

Proof
Clearly, (3) implies (1). We next show that (1) implies (2). For each place $v \mid p$ of F^+, choose a place $\widetilde{v} \mid v$ of F. Suppose that $\bigotimes_{v|p} F(\lambda_v) \in W_{\mathrm{obv}}(\overline{r}) \cap W(\overline{r})$. Then

$$\mathrm{Hom}_{\mathcal{G}(\mathcal{O}_{F^+,p})}\Big(\bigotimes_{v|p} F(\lambda_v)^\vee, S(U^p)_{\mathfrak{m}_{\overline{r}}}\Big) \neq 0$$

for some sufficiently small compact open subgroup $U^p \leq G(\mathbb{A}_{F^+}^{\infty,p})$. (We can replace Σ with the set of places dividing p as in the proof of Proposition 4.2.6.) Say $F(\lambda_v)$ corresponds to $F(\lambda_{\widetilde{v}}) \in W_{\mathrm{obv}}(\overline{r}|_{G_{F_{\widetilde{v}}}})$ via $\iota_{\widetilde{v}}$, and say that $F(\lambda_{\widetilde{v}}) \in W_{\mathrm{obv}}(\overline{r}|_{G_{F_{\widetilde{v}}}})$ (resp., $\tau_{\widetilde{v}}$) is the obvious weight (resp., obvious type) for $\overline{r}|_{G_{F_{\widetilde{v}}}}$ corresponding to w_v (after choosing a lowest alcove presentation of $\overline{r}|_{I_{F_{\widetilde{v}}}}$). One checks directly from the definition of obvious type that $\tau_{\widetilde{v}}$ is $2n$-generic, and hence any lowest alcove presentation of $\tau_{\widetilde{v}}$ satisfies the hypothesis of Theorem 3.4.1, by Proposition 2.2.15. Let $\sigma(\tau_v)$ be the $\mathcal{G}(\mathcal{O}_{F_v^+})$-representation corresponding to $\sigma(\tau_{\widetilde{v}})$ via $\iota_{\widetilde{v}}$. Note that $S(U^p)_{\mathfrak{m}_{\overline{r}}}$ is an injective $\mathbb{F}[\![\mathcal{G}(\mathcal{O}_{F^+,p})]\!]$-module as U^p is sufficiently small; thus $\mathrm{Hom}_{\mathbb{F}[\![\mathcal{G}(\mathcal{O}_{F^+,p})]\!]}(\cdot, S(U^p)_{\mathfrak{m}_{\overline{r}}})$ is exact. Since

$$\mathrm{Hom}_{\mathcal{G}(\mathcal{O}_{F^+,p})}\Big(\bigotimes_{v|p} F(\lambda_v)^\vee, S(U^p)_{\mathfrak{m}_{\overline{r}}}\Big) \neq 0,$$

we have that

$$\mathrm{Hom}_{\mathcal{G}(\mathcal{O}_{F^+,p})}\Big(\bigotimes_{v|p} \overline{\sigma}(\tau_v)^\vee, S(U^p)_{\mathfrak{m}_{\overline{r}}}\Big) \neq 0,$$

where $\overline{\sigma}(\tau_v)$ is the reduction of some $\mathcal{O}$-lattice for each $v \mid p$. A nonzero element of

$$\mathrm{Hom}_{\mathcal{G}(\mathcal{O}_{F^+,p})}\Big(\bigotimes_{v|p} \overline{\sigma}(\tau_v)^\vee, S(U^p)_{\mathfrak{m}_{\overline{r}}}\Big) \neq 0$$

gives an automorphic lift $r_{p,\iota}(\Pi)$ of $\overline{r}$ whose restriction at $\widetilde{v}$ is potentially crystalline of type $(\eta, \tau_{\widetilde{v}})$ by [11, Theorem 7.2.1], Theorem 2.3.1, and Proposition 2.3.4. Thus $r_{p,\iota}(\Pi)|_{G_{F_u}}$ is potentially diagonalizable for each place $u \mid p$ of F by Corollary 3.4.13.

Finally, we show that (2) implies (3). Assuming (2), we see that $\overline{r}$ satisfies the enumerated hypotheses of Theorem 4.3.1. Suppose now that $\bigotimes_{v|p} F(\lambda_v) \in W_{\mathrm{obv}}(\overline{r})$ is arbitrary, and let $F(\lambda_{\widetilde{v}})$, $\tau_{\widetilde{v}}$, $\sigma(\tau_{\widetilde{v}})$, and $\sigma(\tau_v)$ be as in the last paragraph. For each place $v \mid p$ of F^+, let $\rho_{\widetilde{v}}$ be a potentially diagonalizable, potentially crystalline lift of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ of type $(\eta, \tau_{\widetilde{v}})$ (say, the lift from Corollary 3.4.11). By Theorem 4.3.1, there is an automorphic lift $r_{p,\iota}(\pi)$ of $\overline{r}$ whose restriction at $\widetilde{v}$ is potentially crystalline of type $(\eta, \tau_{\widetilde{v}})$, which is unramified outside the places where $\overline{r}$ is ramified. Thus, $\mathrm{Hom}_{\mathcal{G}(\mathcal{O}_{F^+,p})}(\bigotimes_{v|p} \overline{\sigma}(\tau_v)^\vee, S(U^\Sigma)_{\mathfrak{m}_{\overline{r}}}) \neq 0$ for any Σ containing all places dividing p and all places divisible by places in F where $\overline{r}$ is ramified and any sufficiently small U^Σ. By Corollary 4.2.7 and Proposition 4.3.6, we conclude that $\mathrm{Hom}_{\mathcal{G}(\mathcal{O}_{F^+,p})}(\bigotimes_{v|p} F(\lambda_v)^\vee, S(U^\Sigma)_{\mathfrak{m}_{\overline{r}}}) \neq 0$. Thus, $\bigotimes_{v|p} F(\lambda_v) \in W(\overline{r})$. □

Remark 4.3.9

In [2], it is shown that if $\overline{r}$ is modular of a Fontaine–Laffaille weight, $\bigotimes_{v|p} F(\lambda_v) \in W_{\mathrm{obv}}(\overline{r})$, and p splits completely in F, then

$$\mathrm{Hom}_{\mathcal{G}(\mathcal{O}_{F^+,p})}\Big(\bigotimes_{v|p} \overline{W}(\lambda_v)^\vee, S(U^\Sigma)_{\mathfrak{m}_{\overline{r}}}\Big) \neq 0,$$

which is strictly weaker than Theorem 4.3.8. In Section 6 of [14], it is shown that if $\overline{r}$ is assumed to be modular and ordinary at p, then $\overline{r}$ is modular of all ordinary obvious weights. (These are all obvious weights if p splits completely but is strictly smaller otherwise.) In Section 6 of [14], $\overline{r}$ is no longer assumed to be semisimple above p.

4.4. Type-changing congruences and a local lifting problem

In this section, we give a classification (Theorem 4.4.3) of congruences between RACSDC GL_n-automorphic representations of trivial weight and generic tame type whose associated Galois representations are residually tamely ramified at p. We also solve the corresponding local Galois lifting problem. Throughout this section, we are in the setting of **(P1)**–**(P3)** from Section 4.1; that is, we fix

(1) a generic semisimple Galois representation $\overline{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F})$;
(2) a pair $(s_{\overline{\rho}}, \mu_{\overline{\rho}})$ such that $\overline{\rho}|_{I_K} \cong \overline{\tau}(s_{\overline{\rho}}, \mu_{\overline{\rho}} + \eta)$ with $\mu_{\overline{\rho}}$ in $\underline{C}_0$; and
(3) a lowest alcove presentation $(s, \mu - \eta)$ of a tame inertial type $\tau \cong \tau(s, \mu) : I_K \to \mathrm{GL}_n(\mathcal{O})$ such that $\mu_{\overline{\rho}} - \mu \in \underline{\Lambda}_R$.

PROPOSITION 4.4.1

Let $\overline{\rho}$ and τ be as above with $\mu - \eta$ $2n$-deep in $\underline{C}_0$. The set $W^?(\overline{\rho}, \tau)$ is nonempty if and only if the set $W_{\mathrm{obv}}(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\tau))$ is nonempty.

Proof

The "if" part of the claim is clear. Suppose that $W^?(\overline{\rho}, \tau)$ is nonempty. Let $\widetilde{w} =$

$\widetilde{w}^*(\overline{\rho},\tau)$. By Proposition 4.3.2, $\pi^{-1}(\widetilde{w}) = \widetilde{w}_2^{-1} w' \widetilde{w}_1$, where $w' \in W$, $\widetilde{w}_1, \widetilde{w}_2 \in \underline{\widetilde{W}}^+$, and $\widetilde{w}_1 \uparrow \widetilde{w}_h^{-1}\widetilde{w}_2$. Let ω be a weight (unique up to weights whose restrictions to the derived group are trivial) such that $t_{-\omega}\widetilde{w}_1 \cdot (\underline{C}_0)$ is p-restricted. Note that ω is dominant since the set of dominant alcoves is exactly the set of dominant translates of the restricted ones. Then $\pi^{-1}(\widetilde{w}) = (t_{-w'\omega}\widetilde{w}_2)^{-1} w' (t_{-\omega}\widetilde{w}_1)$. Let $t_{-w'\omega}\widetilde{w}_2 = w^{-1}\widetilde{w}_3$, where $w \in W$ and $\widetilde{w}_3 \in \underline{\widetilde{W}}^+$. It suffices to show that $t_{-\omega}\widetilde{w}_1 \uparrow \widetilde{w}_h^{-1}\widetilde{w}_3$, since then by Proposition 4.3.2, taking $\widetilde{w}_\lambda = t_{-\omega}\widetilde{w}_1$ we see that $W^?(\overline{\rho},\tau)$ contains the obvious weight corresponding to the permutation part of $t_{-\omega}\widetilde{w}_1$ via the bijection in the proof of Corollary 2.2.13 (see also Remark 4.3.3).

Using that $\widetilde{w}_1 \uparrow \widetilde{w}_h^{-1}\widetilde{w}_2$, it suffices to show that $\widetilde{w}_h^{-1}\widetilde{w}_2 \uparrow \widetilde{w}_h^{-1} t_{w_0\omega}\widetilde{w}_3$ or, equivalently, that $t_{w_0\omega}\widetilde{w}_3 \uparrow \widetilde{w}_2$. Now $t_{w_0\omega}\widetilde{w}_3 = t_{w_0\omega - ww'\omega} w \widetilde{w}_2$ by definition. Note that $w_0\omega - ww'\omega$ is a sum of negative roots since ω is dominant. Then $t_{w_0\omega - ww'\omega} w\widetilde{w}_2 \uparrow w\widetilde{w}_2 \uparrow \widetilde{w}_2$ by [27, Section II 6.5 (3), (5)]. □

PROPOSITION 4.4.2

Let $\overline{\rho}$ and τ be as in Proposition 4.4.1. Then $\widetilde{w}^*(\overline{\rho},\tau) \in \mathrm{Adm}(\eta)$ if and only if $W^?(\overline{\rho},\tau)$ is nonempty.

Proof

Suppose that $W^?(\overline{\rho},\tau)$ is nonempty, and let $\widetilde{w} = \widetilde{w}^*(\overline{\rho},\tau)$. By Proposition 4.3.2, $\pi^{-1}(\widetilde{w}) = \widetilde{w}_2^{-1} w' \widetilde{w}_1$, where $w' \in W$, $\widetilde{w}_1, \widetilde{w}_2 \in \underline{\widetilde{W}}^+$, and $\widetilde{w}_1 \uparrow \widetilde{w}_h^{-1}\widetilde{w}_2$. By an argument analogous to the proof of Proposition 4.4.1 applied to $\pi^{-1}(\widetilde{w}^{-1}) = \widetilde{w}_1^{-1} w'^{-1} \widetilde{w}_2$ (i.e., we replace $\widetilde{w}$, w', $\widetilde{w}_1$, and $\widetilde{w}_2$ with $\widetilde{w}^{-1}$, w'^{-1}, $\widetilde{w}_2$, and $\widetilde{w}_1$, respectively), we can assume without loss of generality that $\widetilde{w}_2$ is p-restricted. It suffices to show that $\pi^{-1}(\widetilde{w}) \le t_{w_2^{-1}w_0\eta}$. Note that

$$t_{w_2^{-1}w_0\eta} = \widetilde{w}_2^{-1}\widetilde{w}_h t_\eta \widetilde{w}_h^{-1}\widetilde{w}_2 = \widetilde{w}_2^{-1} w_0 (\widetilde{w}_h^{-1}\widetilde{w}_2)$$

and that $\ell(\widetilde{w}_2^{-1} w_0 (\widetilde{w}_h^{-1}\widetilde{w}_2)) = \ell(\widetilde{w}_2^{-1}) + \ell(w_0) + \ell(\widetilde{w}_h^{-1}\widetilde{w}_2)$ by Lemma 4.1.9. Then since $w' \le w_0$ and $\widetilde{w}_1 \le \widetilde{w}_h^{-1}\widetilde{w}_2$ by Proposition 4.3.5 (since $\widetilde{w}_1 \uparrow \widetilde{w}_h^{-1}\widetilde{w}_2$), we have that $\pi^{-1}(\widetilde{w}) \le t_{w_2^{-1}w_0\eta}$.

Conversely, suppose that $\widetilde{w} \in \mathrm{Adm}(\eta)$. Then there exists $w_2 \in W$ such that $\pi^{-1}(\widetilde{w}) \le t_{w_2^{-1}w_0\eta}$. Let $\widetilde{w}_2 \in \underline{\widetilde{W}}^+$ be such that $\widetilde{w}_2$ has projection $w_2 \in W$ and $\widetilde{w}_2 \cdot (\underline{C}_0)$ is p-restricted. (Such elements differ by weights whose restrictions to the derived group are trivial.) Since $t_{w_2^{-1}w_0\eta} = \widetilde{w}_2^{-1} w_0 (\widetilde{w}_h^{-1}\widetilde{w}_2)$ and $\ell(\widetilde{w}_2^{-1} w_0 (\widetilde{w}_h^{-1}\widetilde{w}_2)) = \ell(\widetilde{w}_2^{-1}) + \ell(w_0) + \ell(\widetilde{w}_h^{-1}\widetilde{w}_2)$ as in the last paragraph, $\pi^{-1}(\widetilde{w}) = (\widetilde{w}'_2)^{-1} w' \widetilde{w}'_1$, where $\widetilde{w}'_2 \le \widetilde{w}_2$, $\widetilde{w}'_1 \le \widetilde{w}_h^{-1}\widetilde{w}_2$, and $w' \le w_0$. In particular, $w' \in W(\underline{G})$. If w'_1 and $w'_2 \in W(\underline{G})$ and $\widetilde{w}_1^+$ and $\widetilde{w}_2^+ \in \underline{\widetilde{W}}^+$ are the unique elements such that $\widetilde{w}'_2 = w'_2\widetilde{w}_2^+$ and $\widetilde{w}'_1 = w'_1\widetilde{w}_1^+$, then $\widetilde{w}_2^+ \le \widetilde{w}'_2 \le \widetilde{w}_2$ and $\widetilde{w}_1^+ \le \widetilde{w}'_1 \le \widetilde{w}_h^{-1}\widetilde{w}_2$ by Lemma 4.3.4. Thus $\widetilde{w}_2^+ \uparrow \widetilde{w}_2$ and $\widetilde{w}_1^+ \uparrow \widetilde{w}_h^{-1}\widetilde{w}_2$. Letting

$w'' = (w_2')^{-1} w' w_1'$, we get that $\pi^{-1}(\widetilde{w}) = (\widetilde{w}_2^+)^{-1} w'' \widetilde{w}_1^+$. Since $\widetilde{w}_1^+ \uparrow \widetilde{w}_h^{-1}\widetilde{w}_2$ and $\widetilde{w}_2^+ \uparrow \widetilde{w}_2$ or, equivalently, $\widetilde{w}_h^{-1}\widetilde{w}_2 \uparrow \widetilde{w}_h^{-1}\widetilde{w}_2^+$, we have that $\widetilde{w}_1^+ \uparrow \widetilde{w}_h^{-1}\widetilde{w}_2^+$. By the proof of Proposition 4.4.1 applied to $\pi^{-1}(\widetilde{w}) = (\widetilde{w}_2^+)^{-1} w'' \widetilde{w}_1^+$ (i.e., we replace w', $\widetilde{w}_1$, and $\widetilde{w}_2$ with w'', $\widetilde{w}_1^+$, and $\widetilde{w}_2^+$, respectively), modifying the factorization if necessary, we can assume without loss of generality that $\widetilde{w}_1^+ \cdot \underline{C}_0$ is p-restricted. By Proposition 4.3.2 taking $\widetilde{w}_\lambda$ to be $\widetilde{w}_1^+$, we see that $W^?(\overline{\rho}, \tau)$ is nonempty. □

THEOREM 4.4.3

Let $\overline{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F})$ be a $(6n-2)$-generic semisimple Galois representation, and let τ be $2n$-generic tame inertial type. Let F be a CM field such that $\zeta_p \notin F$, and let $\overline{r} : G_F \to \mathrm{GL}_n(\mathbb{F})$ be a Galois representation as in Section 4.2.2 satisfying the following hypotheses.

- *$\overline{r}$ is potentially diagonalizably automorphic; that is, there is an RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that*
  - *$\overline{r} \cong \overline{r}_{p,\iota}(\Pi)$; and*
  - *for each place $u \mid p$ of F, $r_{p,\iota}(\Pi)|_{G_{F_u}}$ is potentially diagonalizable.*
- *The image of $\overline{r}(G_{F(\zeta_p)})$ is adequate.*
- *$\overline{r}$ is generic and semisimple at all places dividing p and $\overline{r}|_{G_{F_{\widetilde{v}}}} \cong \overline{\rho}$ for a place $\widetilde{v} \mid p$ of F.*

Then the following are equivalent:

(1) *there is an RACSDC representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $\overline{r} \cong \overline{r}_{p,\iota}(\Pi)$ and the restriction of $r_{p,\iota}(\Pi)$ at $\widetilde{v}$ is potentially crystalline of type (η, τ);*

(2) *$\overline{\rho}$ has a potentially crystalline lift of type (η, τ);*

(3) *$\widetilde{w}^*(\overline{\rho}, \tau) \in \mathrm{Adm}(\eta)$;*

(4) *$W^?(\overline{\rho}, \tau) \neq \emptyset$; and*

(5) *$W_{\mathrm{obv}}(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\tau)) \neq \emptyset$.*

Remark 4.4.4

The assumption that τ is $2n$-generic can be relaxed to n-generic. Suppose that τ is n-generic, but not $2n$-generic. Then (4) can be checked to be false, since $\mathrm{JH}(\overline{\sigma}(\tau))$ will contain only weights which are not $3n$-deep in their alcoves by [23, Theorem 5.2]. (Note that, by the linkage principle, the depth of the Jordan–Hölder factors appearing in [23, Theorem 5.2] coincides with that of $\mu - w\epsilon'_{w_0\tau}$ and $\max_{\alpha^\vee} |\langle \epsilon'_{w_0\tau}, \alpha^\vee \rangle| \leq n-1$ for all $\tau \in W$ by Remark 4.1.4.) This implies that (5) is false as well. If (3) holds, then a direct computation shows that τ is $4n$-generic, using that $\max_{\alpha^\vee} |\langle \widetilde{w}(0), \alpha^\vee \rangle| \leq n-1$ for any $\widetilde{w} \in \mathrm{Adm}(\eta)$. So (3) is false. Then (2) is false by Theorem 3.2.1, which immediately implies that (1) is false.

Remark 4.4.5

For $\overline{\rho}$ as in the theorem, there always exists a representation $\overline{r}$ as in the theorem. Indeed, since $\overline{\rho}$ is Fontaine–Laffaille, [10, Conjecture A.3] holds for $\overline{\rho}$. (Alternatively, one can use Corollary 3.4.13.) Let $\overline{r} : G_F \to \mathrm{GL}_n(\mathbb{F})$ be a suitable globalization of $\overline{\rho}$ as constructed by [10, Corollary A.7]. By [10, Corollary A.7 and Lemma A.5], $\overline{r}$ satisfies the required hypotheses. Thus, by removing sentences containing $\overline{r}$, the above theorem can be interpreted as giving existence criteria for potentially crystalline lifts of type (η, τ) for generic semisimple $\overline{\rho}$ and tame generic types τ.

Proof

First, (1) immediately implies (2). Then (2) implies (3) by Theorem 3.2.1 and Proposition 4.1.8. The equivalence of (3), (4), and (5) follows from Propositions 4.4.1 and 4.4.2. It remains to show that (5) implies (1).

Assume that $W_{\mathrm{obv}}(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\tau)) \neq \emptyset$. By Theorem 4.3.8, $W_{\mathrm{obv}}(\overline{r}) \subset W(\overline{r})$ in the notation of Section 4.2.2. Following the notation of the proof of Theorem 4.3.8, let $\bigotimes_{v'|p} F(\lambda_{v'}) \in W_{\mathrm{obv}}(\overline{r})$ be such that $F(\lambda_{\widetilde{v}}) \in W_{\mathrm{obv}}(\overline{\rho}) \cap \mathrm{JH}(\overline{\sigma}(\tau))$. Then, as in the proof of Theorem 4.3.8, we obtain the required automorphic representation Π. □

Acknowledgments. We would like to thank Matthew Emerton, Thomas Haines, Florian Herzig, and Stefano Morra for many helpful conversations, and Florian Herzig for detailed comments on an earlier draft of this paper. We thank the referees for very detailed and helpful feedback which greatly improved the paper. We would also like to thank the Institut Henri Poincaré for their hospitality during part of this project.

Le's work was supported by the National Science Foundation under agreement numbers DMS-1128155 and DMS-1703182 and an American Mathematical Society–Simons travel grant. Le Hung's work was supported in part by the National Science Foundation under grant number DMS-1802037.

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Le
Department of Mathematics, University of Toronto, Toronto, Ontario, Canada;
le@math.toronto.edu

Le Hung
Mathematics Department, Northwestern University, Evanston, Illinois, USA;
lhvietbao@googlemail.com

Levin
Department of Mathematics, University of Arizona, Tucson, Arizona, USA;
bwlevin@math.arizona.edu