Evolution of Shapes Under Some Stationary 2-D Euler Flows

Chumeng Di, Ethan Roy

June & July 2019

Abstract

In this report, we investigate how a vertical line segment, a V shape, and a W shape evolve under some basic stationary 2-D Euler flows.

1 Introduction

The solutions of the two-dimensional (2D) Euler Equations are important for understanding the dynamics of vorticity and for many different applications within fluid dynamics. Right now there is plentiful research on both the stationary [5] and rotating smooth solutions [2] to the 2D Euler Equations. However, they take a more traditional complex approach towards analyzing the evolution of 2-D Euler flows and their evolutions on the 2-D torus $\mathbb{T}^2$. In this presentation we investigate simpler examples of stationary 2-D Euler flows and their evolutions on the 2-D torus $\mathbb{T}^2$. These examples follow previous research on vortex patches [4] and evidence of a singularities [3] within these equations. Our goal for this research is that it is used as a barebones estimate for vorticity in future analysis on this problem. In this report, we investigate some examples of stationary 2-D Euler flows constructed on the two dimensional torus $\mathbb{T}^2$ and how some shapes evolve under them. General Euler equation has the form

$$\omega_t + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u, \tag{1.1}$$

where $(\omega \cdot \nabla)u = 0$ and ω is scalar in 2-D, and $\omega_t = 0$ when it is stationary. Thus the 2-D stationary Euler equation is given by

$$(u \cdot \nabla)\omega = 0. \tag{1.2}$$

By Biot-Savart law, we have

$$u = \nabla^{\perp}(-\Delta)^{-1}\omega, \tag{1.3}$$

where $\nabla^{\perp} = (\partial_2, -\partial_1)$. Combining (1.2) and (1.3) we have

$$(\nabla^{\perp}(-\Delta)^{-1}\omega) \cdot \nabla\omega = 0. \tag{1.4}$$

We denote the stream function ψ by

$$\psi = (-\Delta)^{-1}\omega, \tag{1.5}$$

and thus we have

$$\nabla^{\perp}\psi \cdot \nabla(-\Delta\psi) = 0. \tag{1.6}$$

If $-\Delta\psi = f(\psi)$ [1] for some real-valued smooth function f, then (1.6) holds for those ψ. The eigenfunctions of f, i.e., $f(\psi) = \lambda\psi$, are some simplest examples of stationary 2-D Euler flows.

Stream function of the form

$$\psi = e^{i(\lambda_1 x_1 + \lambda_2 x_2)} \qquad (\lambda_1, \lambda_2 \in \mathbb{R}) \tag{1.7}$$

is an eigenfunction of $f(\psi) = \lambda\psi$ with eigenvalue $\lambda = \lambda_1^2 + \lambda_2^2$. In this report we look at six basic forms of stream functions ψ (for $\lambda_1, \lambda_2 > 0$) as follows:

$$\psi_1 = \cos(\lambda_2 x_2), \tag{1}$$

$$\psi_2 = \sin(\lambda_2 x_2), \tag{2}$$

$$\psi_3 = \cos(\lambda_1 x_1), \tag{3}$$

$$\psi_4 = \sin(\lambda_1 x_1), \tag{4}$$

$$\psi_5 = \cos(\lambda_1 x_1 + \lambda_2 x_2), \tag{5}$$

$$\psi_6 = \cos(\lambda_1 x_1)\sin(\lambda_2 x_2). \tag{6}$$

We investigate how three shapes evolve under these stream functions. For each shape and each stream function ψ, we want to find velocity u given by

$$u = \nabla^{\perp}\psi, \tag{1.8}$$

displacement X given by

$$X = (x_1, x_2) + u \cdot t, \tag{1.9}$$

and accordingly the graph of evolution of the shape under stream flow ψ.

2 Vertical Line Segment

We start by defining a vertical interval $\gamma_0 = \left\{(0, x_2) | -\dfrac{\pi}{\lambda_2} \le x_2 \le \dfrac{\pi}{\lambda_2}\right\}$. On this vertical interval, we analyze the six $2D$ stationary solutions and how they evolve.

2.1 $\psi = \cos(\lambda_2 x_2)$

If we start with the solution

$$\psi_1 = \cos(\lambda_2 x_2) \tag{2.1.1}$$

then our velocity vector filed u becomes

$$u(x_1, x_2) = (-\lambda_2 \sin(\lambda_2 x_2), 0) \tag{2.1.2}$$

consequently

$$X = (X_1, X_2) = (x_1 - \lambda_2 \sin(\lambda_2 x_2)t, x_2) \tag{2.1.3}$$

And as a result of being on the vertical line segment $x_1 = 0$, thus

$$X = (-\lambda_2 \sin(\lambda_2 x_2)t, x_2) \tag{2.1.4}$$

and using our displacement equation we can graph the evolution of γ_0 while also taking $\lambda_2 = 1$

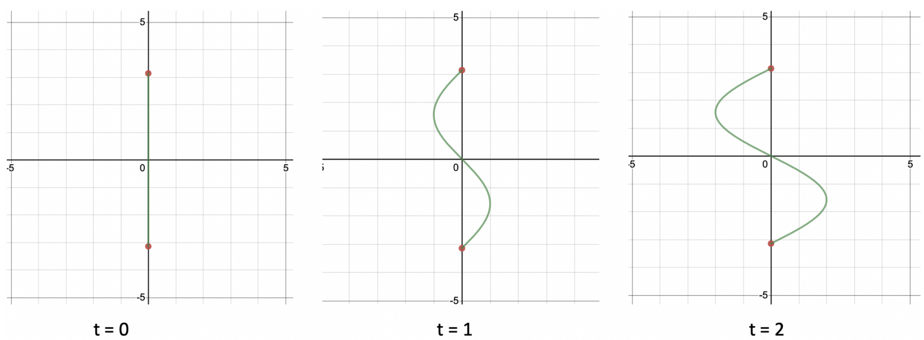

Figure 1: Evolution of vertical line segment under ψ_1

2.2 $\psi = \sin(\lambda_2 x_2)$

Next, if we analyze

$$\psi = \sin(\lambda_2 x_2) \tag{2.2.1}$$

then our u and X are

$$u(x_1, x_2) = (\lambda_2 \cos(\lambda_2 x_2), 0) \tag{2.2.2}$$

$$X = (X_1, X_2) = (x_1 + \lambda_2 \cos(\lambda_2 x_2)t, x_2) \tag{2.2.3}$$

If we substitute $x_1 = 0$ into our equation then

$$X = (\lambda_2 \cos(\lambda_2 x_2)t, x_2) \tag{2.2.4}$$

and using our displacement equation we can graph the evolution of γ_0 while also taking $\lambda_2 = 1$.

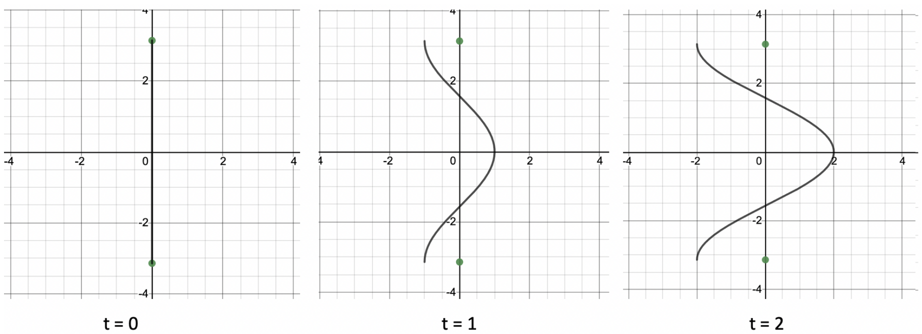

Figure 2: Evolution of vertical line segment under ψ_2

2.3 $\psi = \cos(\lambda_1 x_1)$

After analyzing both versions of ψ involving cosine and sine of λ_2, we now start with

$$\psi = \cos(\lambda_1 x_1). \tag{2.3.1}$$

This results in

$$u(x_1, x_2) = (0, \lambda_1 \sin(\lambda_1 x_1)), \tag{2.3.2}$$

$$X = (X_1, X_2) = (x_1, x_2 + \lambda_1 t \sin(\lambda_1 x_1)). \tag{2.3.3}$$

Since, we are on the vertical line segment $x_1 = 0$ and as a result

$$X = (0, x_2). \tag{2.3.4}$$

Due to our $x_1 = 0$ the graph for $\psi = \cos(\lambda_1 x_1)$ does not change from the vertical line

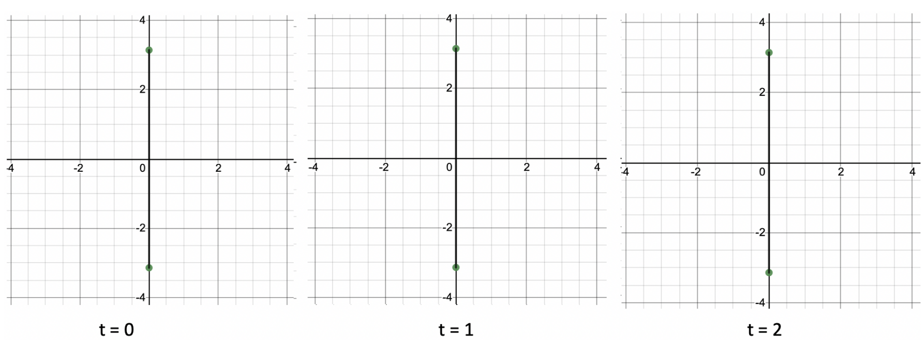

Figure 3: Evolution of vertical line segment under ψ_3

2.4 $\psi = \sin(\lambda_1 x_1)$

Furthermore, if we analyze

$$\psi = \sin(\lambda_1 x_1) \tag{2.4.1}$$

This results in

$$u(x_1, x_2) = (0, -\lambda_1 \cos(\lambda_1 x_1)) \tag{2.4.2}$$

$$X = (X_1, X_2) = (x_1, x_2 - \lambda_1 t \cos(\lambda_1 x_1)) \tag{2.4.3}$$

Since, we are on the vertical line segment $x_1 = 0$ and as a result

$$X = (0, x_2 - \lambda_1) \tag{2.4.4}$$

due to our $x_1 = 0$ the graph for $\psi = \sin(\lambda_1 x_1)$ does not change from the vertical line $x_2 - \lambda_1$ where $\lambda_1 = 1$

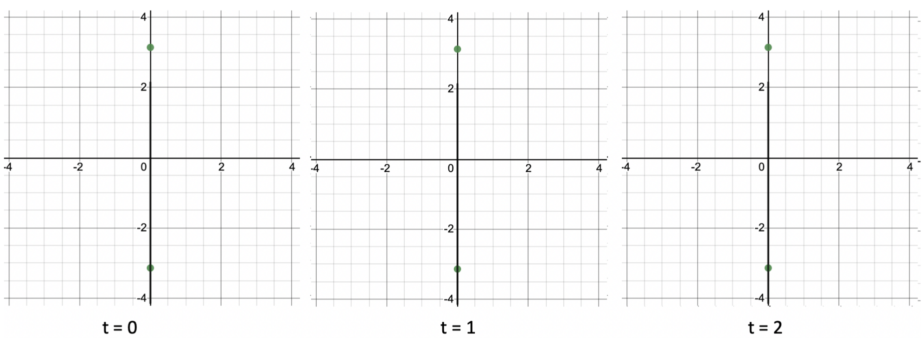

Figure 4: Evolution of vertical line segment under ψ_4

2.5 $\psi = \cos(\lambda_1 x_1 + \lambda_2 x_2)$

Now, if we combine our λ_1 and λ_2 into one ψ equation, we can analyze the solution

$$\psi = \cos(\lambda_1 x_1 + \lambda_2 x_2). \tag{2.5.1}$$

Then we can find

$$u(x_1, x_2) = (-\lambda_2 \sin(\lambda_1 x_1 + \lambda_2 x_2), \lambda_1 \sin(\lambda_1 x_1 + \lambda_2 x_2)) \tag{2.5.2}$$

$$X = (X_1, X_2) = (x_1 - \lambda_2 \sin(\lambda_1 x_1 + \lambda_2 x_2)t, x_2 + \lambda_1 \sin(\lambda_1 x_1 + \lambda_2 x_2)t). \tag{2.5.3}$$

Since, $x_1 = 0$ on the vertical line segment, our displacement equation X becomes

$$X = (-\lambda_2 \sin(\lambda_2 x_2)t, x_2 + \lambda_1 \sin(\lambda_2 x_2)t) \tag{2.5.4}$$

and using our displacement equation we can graph the evolution of γ_0 while also taking $\lambda_2 = 1$ and $\lambda_1 = 1$

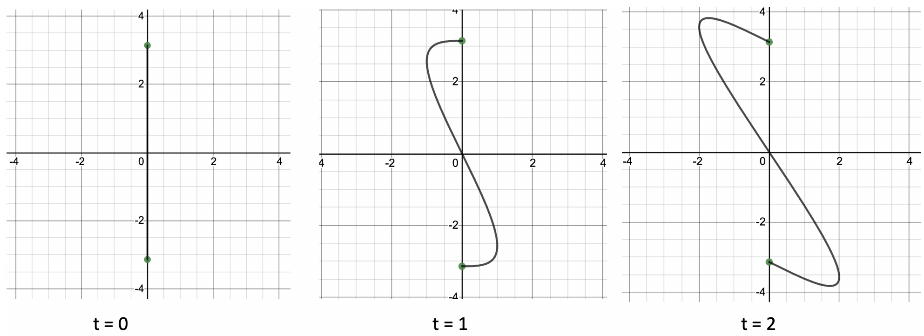

Figure 5: Evolution of vertical line segment under ψ_5

2.6 $\psi = \cos(\lambda_1 x_1)\sin(\lambda_2 x_2)$

Finally, if we analyze our last combination of λ_1 and λ_2, we see a

$$\psi = \cos(\lambda_1 x_1)\sin(\lambda_2 x_2). \tag{2.6.1}$$

Then we can find our u and displacement equation to be

$$u(x_1, x_2) = (\lambda_2 \cos(\lambda_1 x_1)\cos(\lambda_2 x_2), \lambda_1 sin(\lambda_1 x_1)\sin(\lambda_2 x_2)) \tag{2.6.2}$$

$$X = (X_1, X_2) = (x_1 + t\lambda_2 \cos(\lambda_1 x_1)\cos(\lambda_2 x_2), x_2 + t\lambda_1 \sin(\lambda_1 x_1)\sin(\lambda_2 x_2)) \tag{2.6.3}$$

Consequently, by including that $x_1 = 0$ our displacement equation becomes

$$X = (t\lambda_2 \cos(\lambda_2 x_2), x_2) \tag{2.6.4}$$

and using our displacement equation, we can graph the evolution of γ_0 while also taking $\lambda_2 = 1$.

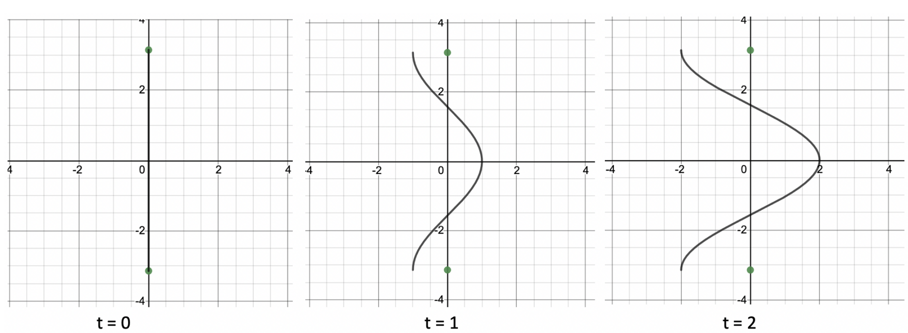

Figure 6: Evolution of vertical line segment under ψ_6

3 V shape

Define f on $[-k\frac{\pi}{\lambda_i}, k\frac{\pi}{\lambda_i}]$ as

$$f(x) = \begin{cases} -\sqrt{3}x, & x \in [-k\frac{\pi}{\lambda_i}, 0] \\ \sqrt{3}x, & x \in (0, k\frac{\pi}{\lambda_i}] \end{cases} \qquad (k \in \mathbb{R}^+, i = 1, 2) \tag{7}$$

Let $\gamma_0 = \{(x_1, x_2) \mid x_2 = f(x_1), x_1 \in [-k\frac{\pi}{\lambda_i}, k\frac{\pi}{\lambda_i}]\}$. Let us consider how this V-shape curve γ_0 evolves with the six stationary Euler flows.

3.1 $\psi = \cos(\lambda_2 x_2)$

We consider how γ_0 evolves with stream function

$$\psi = \cos(\lambda_2 x_2). \tag{3.1.1}$$

By equation (2.1), we have the displacement equation

$$X(t) = (X_1(t), X_2(t)) = (x_1 - \lambda_2 \sin(\lambda_2 x_2)t, x_2), \tag{3.1.2}$$

and accordingly we can graph the evolution of γ_0 (by taking $k = 8, i = 2, \lambda_2 = 1$) as follows:

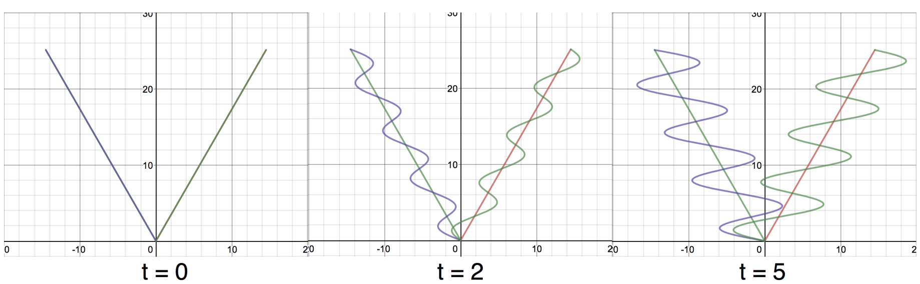

Figure 7: Evolution of V shape under ψ_1

We further investigate how the magnitude of angle evolve as the shape evolves. Let the angle be θ and we parametrize two legs of θ using x_2 as

$$\mathbf{r_1}(x_2) = (-\frac{x_2}{\sqrt{3}} - \lambda_2 \sin(\lambda_2 x_2)t, x_2) \tag{3.1.3}$$

$$\mathbf{r_2}(x_2) = (\frac{x_2}{\sqrt{3}} - \lambda_2 \sin(\lambda_2 x_2)t, x_2) \tag{3.1.4}$$

Thereby we compute

$$\lim_{x_2 \to 0} \mathbf{r_1}'(x_2) = (-\frac{1}{\sqrt{3}} - \lambda_2^2 t, 1) \tag{3.1.5}$$

$$\lim_{x_2 \to 0} \mathbf{r_2}'(x_2) = (\frac{1}{\sqrt{3}} - \lambda_2^2 t, 1) \tag{3.1.6}$$

Thus we have

$$\cos(\theta) = \frac{(-\frac{1}{\sqrt{3}} - \lambda_2^2 t, 1) \cdot (\frac{1}{\sqrt{3}} - \lambda_2^2 t, 1)}{\sqrt{((-\frac{1}{\sqrt{3}} - \lambda_2^2 t)^2 + 1)((\frac{1}{\sqrt{3}} - \lambda_2^2 t)^2 + 1)}}, \tag{3.1.7}$$

and we graph $\theta(t)$ as follows:

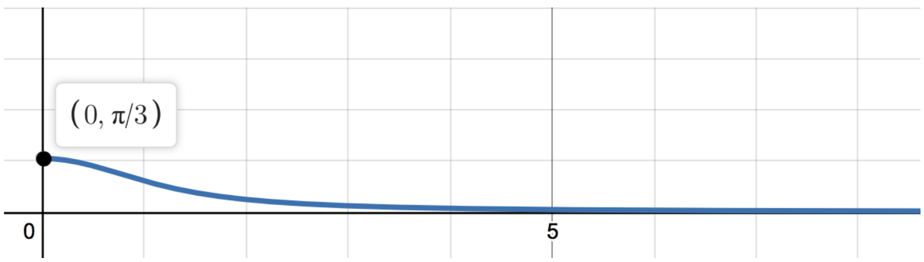

Figure 8: θ change with time

3.2 $\psi = \sin(\lambda_2 x_2)$

We consider how γ_0 evolves with stream function

$$\psi = \sin(\lambda_2 x_2). \tag{3.2.1}$$

By equation (2.2), we have the displacement equation

$$X(t) = (X_1(t), X_2(t)) = (x_1 + \lambda_2 \cos(\lambda_2 x_2) t, x_2), \tag{3.2.2}$$

and accordingly we can graph the evolution of γ_0 as follows:

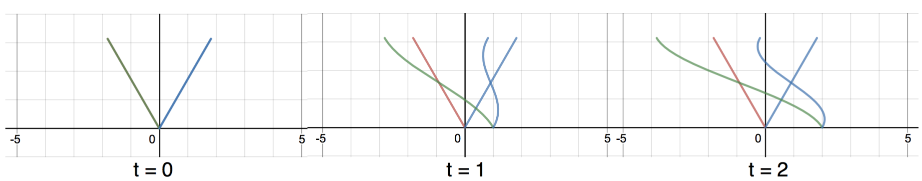

Figure 9: Evolution of V shape under ψ_2

3.3 $\psi = \cos(\lambda_1 x_1)$

We consider how γ_0 evolves with stream function

$$\psi = \cos(\lambda_1 x_1). \tag{3.3.1}$$

By equation (2.3), we have the displacement equation

$$X(t) = (X_1(t), X_2(t)) = (x_1, x_2 + \lambda_1 \sin(\lambda_1 x_1)t), \tag{3.3.2}$$

and accordingly we can graph the evolution of γ_0 (by taking $k = 8, i = 1, \lambda_1 = 1$) as follows:

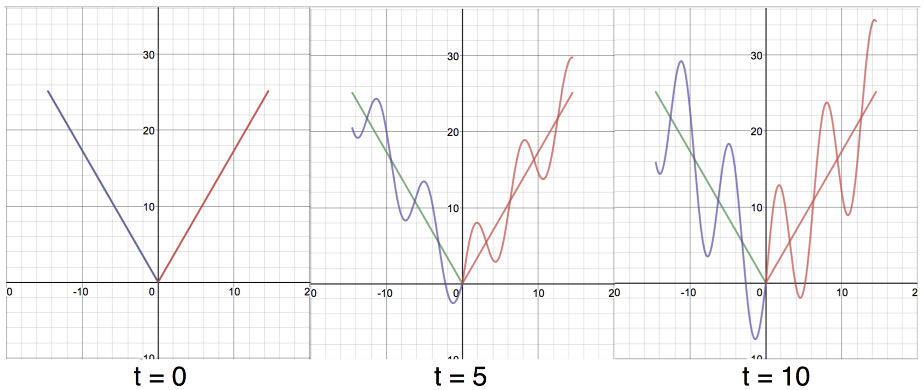

Figure 10: Evolution of V shape under ψ_3

3.4 $\psi = \sin(\lambda_1 x_1)$

We consider how γ_0 evolves with stream function

$$\psi = \sin(\lambda_1 x_1). \tag{3.4.1}$$

By equation (2.4), we have the displacement equation

$$X(t) = (X_1(t), X_2(t)) = (x_1, x_2 - \lambda_1 \cos(\lambda_1 x_1)t), \tag{3.4.2}$$

and accordingly we can graph the evolution of γ_0 (by taking $k = 8, i = 1, \lambda_1 = 1$) as follows:

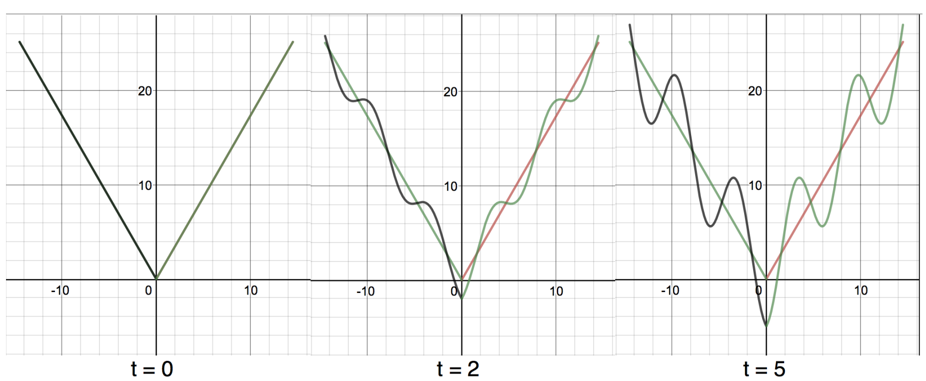

Figure 11: Evolution of V shape under ψ_4

3.5 $\psi = \cos(\lambda_1 \mathbf{x_1} + \lambda_2 \mathbf{x_2})$

We consider how γ_0 evolves with stream function

$$\psi = \cos(\lambda_1 x_1 + \lambda_2 x_2). \tag{3.5.1}$$

By equation (2.5), we have the displacement equation

$$X(t) = (X_1(t), X_2(t)) = (x_1 - \lambda_2 \sin(\lambda_1 x_1 + \lambda_2 x_2)t, x_2 + \lambda_1 \sin(\lambda_1 x_1 + \lambda_2 x_2)t), \tag{3.5.2}$$

and accordingly we can graph the evolution of γ_0 (by taking $k = 8, i = 2, \lambda_1 = 1, \lambda_2 = 1$) as follows:

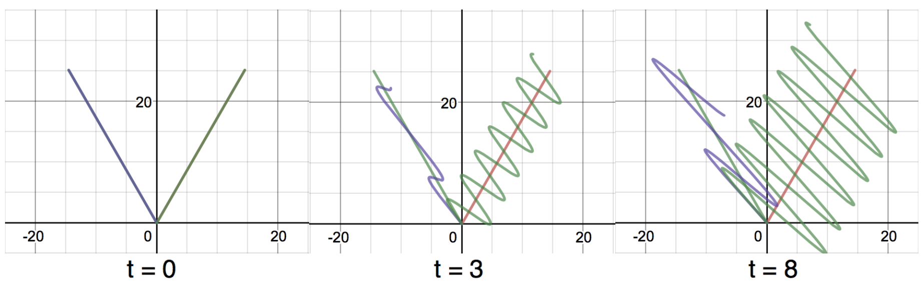

Figure 12: Evolution of V shape under ψ_5

3.6 $\psi = \cos(\lambda_1 \mathbf{x_1}) \sin(\lambda_2 \mathbf{x_2})$

We consider how γ_0 evolves with stream function

$$\psi = \cos(\lambda_1 x_1) \sin(\lambda_2 x_2). \tag{3.6.1}$$

By equation (2.6), we have the displacement equation

$$X(t) = (X_1(t), X_2(t)) = (x_1 + \lambda_2 \cos(\lambda_1 x_1) \cos(\lambda_2 x_2)t, x_2 + \lambda_1 \sin(\lambda_1 x_1) \sin(\lambda_2 x_2)t), \tag{3.6.2}$$

and accordingly we can graph the evolution of γ_0 (by taking $k = 8, i = 2, \lambda_1 = 1, \lambda_2 = 1$) as follows:

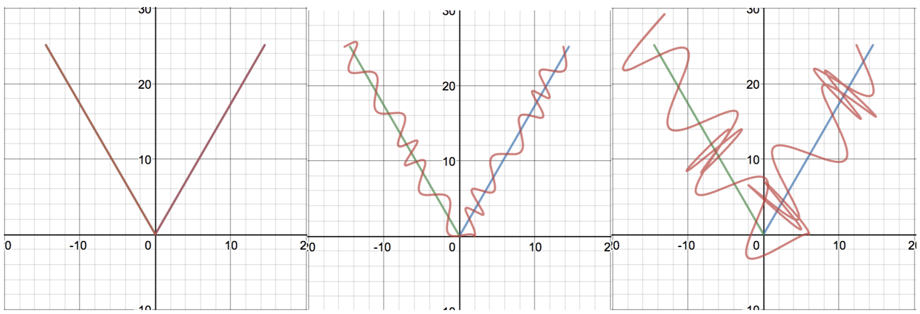

Figure 13: Evolution of V shape under ψ_6

In this graph we see an unexpected touching of trajectories after time t. For $A, B \in \gamma_0$ with $A \neq B$, we have

$$X(A) = X_0(A) + (\nabla^{\perp}\psi)(A)t$$

$$X(B) = X_0(B) + (\nabla^{\perp}\psi)(B)t.$$

We set $X(A) = X(B)$, which gives us

$$\psi(A) \neq \psi(B) \tag{3.6.3}$$

Since $\omega = -\Delta\psi = \lambda\psi$, (3.6.3) gives us

$$\omega(A) \neq \omega(B), \tag{3.6.4}$$

which implies an unexpected singularity.

4 W shape

Define f_1, f_2, f_3, and f_4 on $[-\frac{\sqrt{3}\pi}{4\lambda_2}, 0]$ as follows:

$$f_1(x) = -\frac{1}{\sqrt{3}}x \tag{4.0.1}$$

$$f_2(x) = \frac{1}{\sqrt{3}}x + \frac{\pi}{2\lambda_2} \tag{4.0.2}$$

$$f_3(x) = -\frac{1}{\sqrt{3}}x + \frac{\pi}{2\lambda_2} \tag{4.0.3}$$

$$f_4(x) = \frac{1}{\sqrt{3}}x + \frac{\pi}{\lambda_2}. \tag{4.0.4}$$

Let γ_0 be the shape generated by these four functions and let us consider how it evolves with some stationary Euler flows.

4.1 $\psi = \cos(\lambda_1 \mathbf{x_1} + \lambda_2 \mathbf{x_2})$

We consider how γ_0 evolves with stream function

$$\psi = \cos(\lambda_1 x_1 + \lambda_2 x_2). \tag{4.1.1}$$

By equation (2.5), we have the displacement equation

$$X(t) = (X_1(t), X_2(t)) = (x_1 - \lambda_2 \sin(\lambda_1 x_1 + \lambda_2 x_2)t, x_2 + \lambda_1 \sin(\lambda_1 x_1 + \lambda_2 x_2)t), \tag{4.1.2}$$

and accordingly we can graph the evolution of γ_0 (by taking $\lambda_1 = 1, \lambda_2 = 1$) as follows:

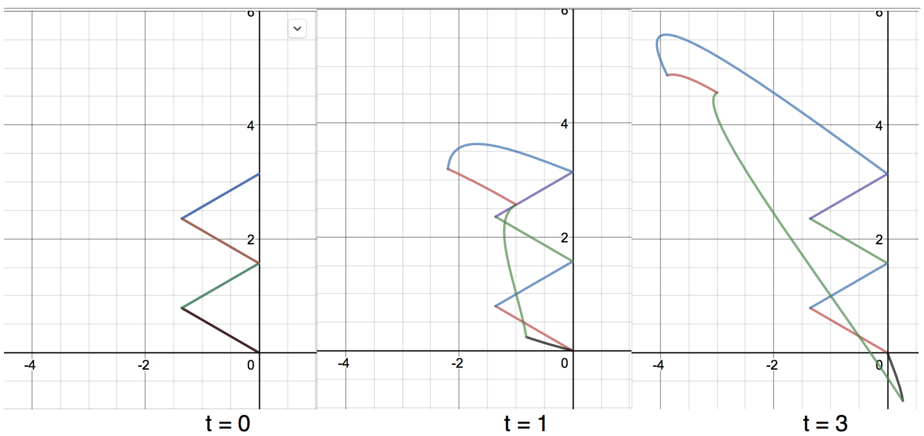

Figure 14: Evolution of W shape under ψ_5

4.2 $\psi = \cos(\lambda_1 \mathbf{x_1}) \sin(\lambda_2 \mathbf{x_2})$

We consider how γ_0 evolves with stream function

$$\psi = \cos(\lambda_1 x_1) \sin(\lambda_2 x_2). \tag{4.2.1}$$

By equation (2.6), we have the displacement equation

$$X(t) = (X_1(t), X_2(t)) = (x_1 + \lambda_2 \cos(\lambda_1 x_1) \cos(\lambda_2 x_2)t, x_2 + \lambda_1 \sin(\lambda_1 x_1) \sin(\lambda_2 x_2)t), \tag{4.2.2}$$

and accordingly we can graph the evolution of γ_0 (by taking $\lambda_1 = 1, \lambda_2 = 1$) as follows:

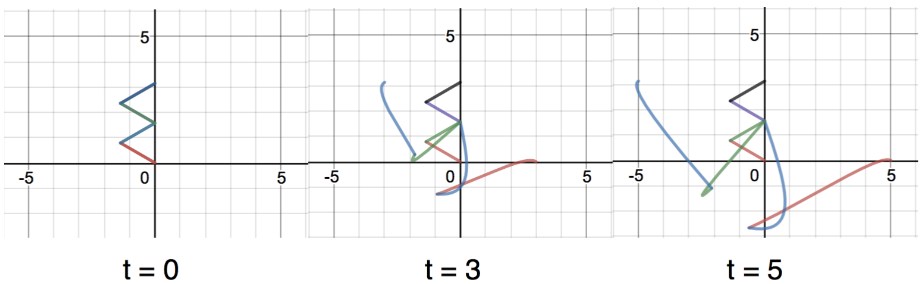

Figure 15: Evolution of W shape under ψ_6

Acknowledgments

We would like to thank Dr. Betul Orcan for her help on this project. Not only did she introduce and teach the topic to the two of us, she has also consistently checked in on our progress as well as help us with any problems we encountered. This project would not exist without her, and we are incredibly grateful for her advice and guidance.

References

[1] L. Alexander Kiselev, Jean-Michel Roquejoffre. *Notes on Partial Differential Equations.*

[2] A. Castro, D. Córdoba, and J. Gómez-Serrano. Uniformly rotating smooth solutions for the incompressible 2d euler equations. *Archive for Rational Mechanics and Analysis*, 231(2):719–785, Feb 2019.

[3] A. Cordoba, Diego and Fontelos, Marco A and Mancho. Evidence of singularities for a family of contour dynamics equations. *Proceedings of the National Academy of Sciences of the United States of America*, 102(17):5949–5952, 04 2005.

[4] T. M. Elgindi and I.-J. Jeong. On Singular Vortex Patches, I: Well-posedness Issues. *arXiv e-prints*, page arXiv:1903.00833, Mar 2019.

[5] A. Tur and V. Yanovsky. Point vortices with a rational necklace: New exact stationary solutions of the two-dimensional euler equation. *Physics of Fluids*, 16(8):2877–2885, 2004.