Evolution of Shapes Under Some Stationary 2-D Euler Flows

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Introduction

The solutions of the two-dimensional (2D) Euler Equations are important for understanding the dynamics of vorticity and for many different applications within fluid dynamics. Right now there is plentiful research on both the stationary [5] and rotating smooth solutions [2] to the 2D Euler Equations. However, they take a more traditional complex approach towards analyzing the evolution of 2-D Euler flows and their evolutions on the 2-D torus $\mathbb{T}^2$. In this presentation we investigate simpler examples of stationary 2-D Euler flows and their evolutions on the 2-D torus $\mathbb{T}^2$. These examples follow previous research on vortex patches [4] and evidence of a singularities [3] within these equations. Our goal for this research is that it is used as a barebones estimate for vorticity in future analysis on this problem.

The Problem

We investigate how certain shapes evolve under some stationary 2-D Euler flows on the two dimensional torus $\mathbb{T}^2$. The the 2-D stationary Euler equation is given by [1]

$$(u \cdot \nabla)\omega = 0. \tag{1.1}$$

By Biot-Savart law, we obtain

$$u = \nabla^{\perp}(-\Delta)^{-1}\omega, \tag{1.2}$$

where $\nabla^{\perp} = (\partial_2, -\partial_1)$. We denote the stream function ψ by

$$\psi = (-\Delta)^{-1}\omega, \tag{1.3}$$

and thus we have

$$\nabla^{\perp}\psi \cdot \nabla(-\Delta\psi) = 0. \tag{1.4}$$

If $-\Delta\psi = f(\psi)$ for some real-valued smooth function f, then (1.4) holds for ψ. Stream function of the form below is an eigenfunction of $f(\psi) = \lambda\psi$

$$\psi = e^{i(\lambda_1 x_1 + \lambda_2 x_2)} \qquad (\lambda_1, \lambda_2 \in \mathbb{R}) \tag{1.5}$$

For each shape we investigate how it evolves with six stream functions ψ (for $\lambda_1, \lambda_2 > 0$) as follows:

$$\psi_1 = \cos(\lambda_2 x_2), \psi_2 = \sin(\lambda_2 x_2), \psi_3 = \cos(\lambda_1 x_1),$$

$$\psi_4 = \sin(\lambda_1 x_1), \psi_5 = \cos(\lambda_1 x_1 + \lambda_2 x_2), \psi_6 = \cos(\lambda_1 x_1)\sin(\lambda_2 x_2).$$

In each case we find velocity u and displacement X given by

$$u = \nabla^{\perp}\psi, \qquad X = (x_1, x_2) + u \cdot t,$$

and accordingly generate the graph of evolution of the shape under the flow.

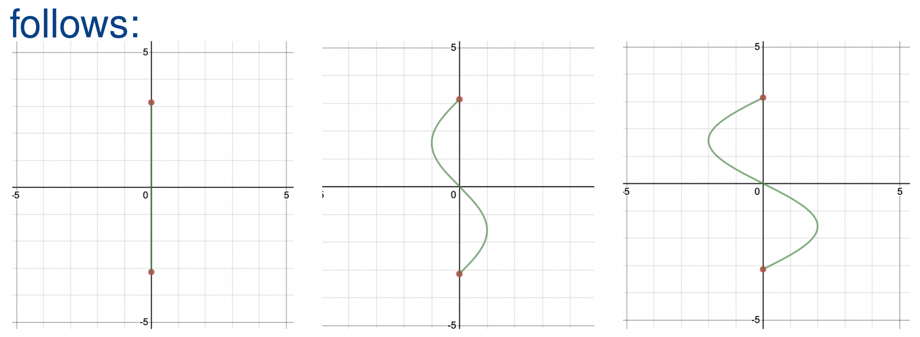
Vertical Line Segment

For $\gamma_0 = \left\{(0, x_2), -\frac{\pi}{\lambda_2} \le x_2 \le \frac{\pi}{\lambda_2}\right\}$ and $\psi = \cos(\lambda_2 x_2)$, we have

$$u(x_1, x_2) = (-\lambda_2 \sin(\lambda_2 x_2), 0) \tag{2.1}$$

$$\begin{aligned} X = (X_1, X_2) &= (x_1 - \lambda_2 \sin(\lambda_2 x_2)t, x_2) \\ &= (-\lambda_2 \sin(\lambda_2 x_2)t, x_2) \end{aligned} \tag{2.2}$$

With our displacement equation we graph the evolution of γ_0 (taking $\lambda_2 = 1$) as follows:



t = 0 t = 1 t = 2

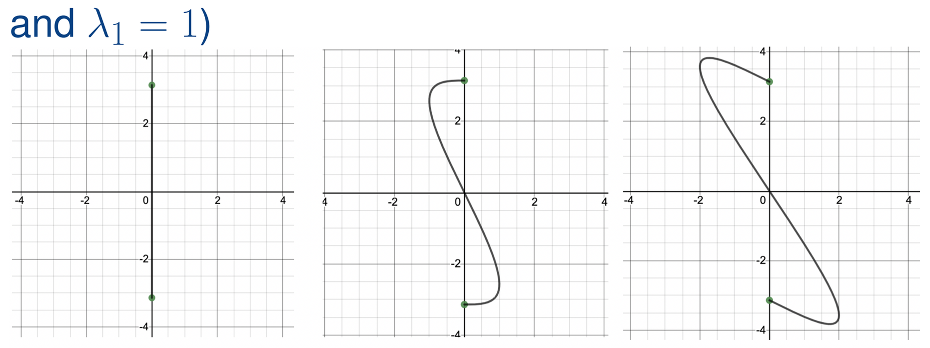
Vertical Line Segment Cont.

For the same γ_0 and taking $\psi = \cos(\lambda_1 x_1 + \lambda_2 x_2)$, we can find our u and X to be

$$u(x_1, x_2) = (-\lambda_2 \sin(\lambda_1 x_1 + \lambda_2 x_2), \lambda_1 \sin(\lambda_1 x_1 + \lambda_2 x_2)) \tag{2.3}$$

$$\begin{aligned} X = (X_1, X_2) &= (x_1 - \lambda_2 \sin(\lambda_1 x_1 + \lambda_2 x_2)t, x_2 + \lambda_1 \sin(\lambda_1 x_1 + \lambda_2 x_2)t) \\ &= (-\lambda_2 \sin(\lambda_2 x_2)t, x_2 + \lambda_1 \sin(\lambda_2 x_2)t) \end{aligned} \tag{2.4}$$

Using the displacement equation we can graph the evolution of γ_0 (taking $\lambda_2 = 1$ and $\lambda_1 = 1$)



t = 0 t = 1 t = 2

Note that ψ_2 is a shifted of ψ_1 ; ψ_4 is a ship of ψ_3, both of which not deformed in this case. Finally ψ_6 takes on the same evolution as ψ_2, which is due to $x_1 = 0$ for this shape.

V shape

Define f on $[-k\frac{\pi}{\lambda_i}, k\frac{\pi}{\lambda_i}]$ as

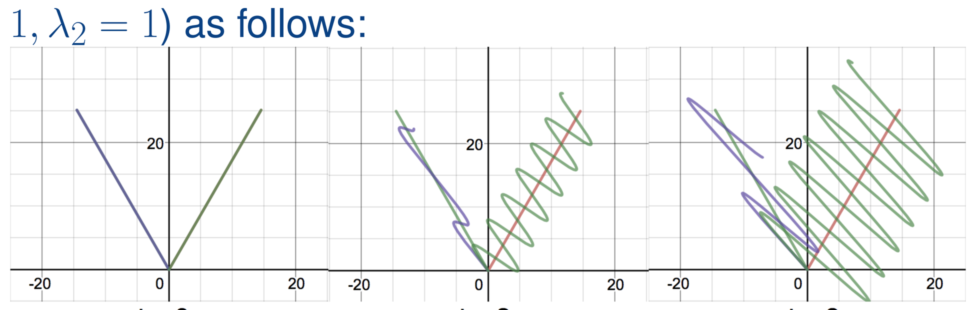
$$f(x) = \begin{cases} -\sqrt{3}x, & x \in [-k\frac{\pi}{\lambda_i}, 0] \\ \sqrt{3}x, & x \in (0, k\frac{\pi}{\lambda_i}] \end{cases} \qquad (k \in \mathbb{R}^+, i = 1, 2) \tag{3.1}$$

Let $\gamma_0 = \{(x_1, x_2) \mid x_2 = f(x_1), x_1 \in [-k\frac{\pi}{\lambda_i}, k\frac{\pi}{\lambda_i}]\}$. Let us consider how this V-shape curve γ_0 evolves with some of the stationary Euler flows.

We start with $\psi = \cos(\lambda_1 \mathbf{x_1} + \lambda_2 \mathbf{x_2})$. By equation (2.4) we find our displacement equation to be

$$X(t) = (X_1(t), X_2(t)) = (x_1 - \lambda_2 \sin(\lambda_1 x_1 + \lambda_2 x_2)t, x_2 + \lambda_1 \sin(\lambda_1 x_1 + \lambda_2 x_2)t), \tag{3.2}$$

and accordingly we can graph the evolution of γ_0 (by taking $k = 8, i = 2, \lambda_1 = 1, \lambda_2 = 1$) as follows:



t = 0 t = 3 t = 8

Next if we analyze $\psi = \cos(\lambda_1 \mathbf{x_1})\sin(\lambda_2 \mathbf{x_2})$ then we can find our u and X to be

$$u(x_1, x_2) = (\lambda_2 \cos(\lambda_1 x_1)\cos(\lambda_2 x_2), \lambda_1 \sin(\lambda_1 x_1)\sin(\lambda_2 x_2)) \tag{3.3}$$

$$X = (X_1, X_2) = (x_1 + t\lambda_2 \cos(\lambda_1 x_1)\cos(\lambda_2 x_2), x_2 + t\lambda_1 \sin(\lambda_1 x_1)\sin(\lambda_2 x_2)) \tag{3.4}$$

and accordingly we can graph the evolution of γ_0 (by taking $k = 8, i = 2, \lambda_1 = 1, \lambda_2 = 1$) as follows:

t = 0 t = 2 t = 5

W shape

Define f_1, f_2, f_3, and f_4 on $[-\frac{\sqrt{3}\pi}{4\lambda_2}, 0]$ as follows:

$$f_1(x) = -\frac{1}{\sqrt{3}}x, \qquad f_2(x) = \frac{1}{\sqrt{3}}x + \frac{\pi}{2\lambda_2}$$

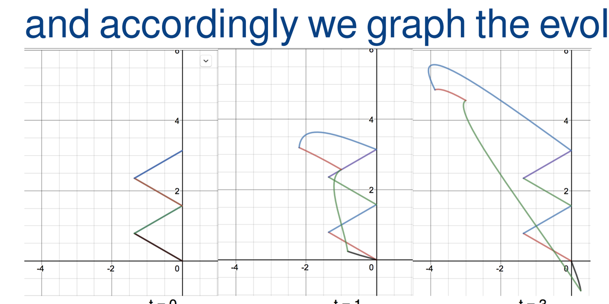
$$f_3(x) = -\frac{1}{\sqrt{3}}x + \frac{\pi}{2\lambda_2}, \qquad f_4(x) = \frac{1}{\sqrt{3}}x + \frac{\pi}{\lambda_2}.$$

Let γ_0 be the shape generated by these functions and let us see how it evolves with some stationary Euler flows.

We first consider how γ_0 evolves with $\psi = \cos(\lambda_1 x_1 + \lambda_2 x_2)$. We have the displacement equation

$$X(t) = (X_1(t), X_2(t)) = (x_1 - \lambda_2 \sin(\lambda_1 x_1 + \lambda_2 x_2)t, x_2 + \lambda_1 \sin(\lambda_1 x_1 + \lambda_2 x_2)t), \tag{4.1}$$

and accordingly we graph the evolution of γ_0 (by taking $\lambda_1 = 1, \lambda_2 = 1$) as follows:

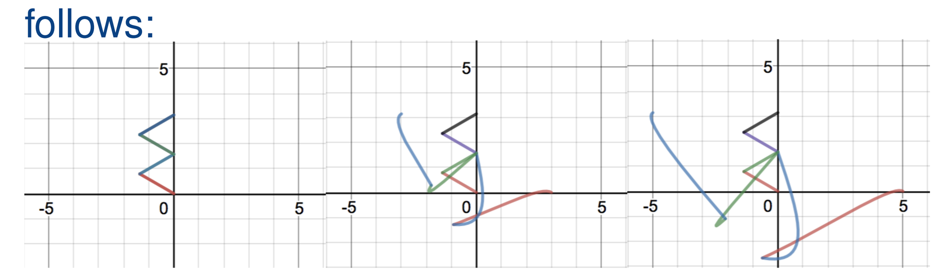


t = 0 t = 1 t = 3

We then consider the evolution of γ_0 with $\psi = \cos(\lambda_1 x_1)\sin(\lambda_2 x_2)$. we have the displacement equation

$$X(t) = (X_1(t), X_2(t)) = (x_1 + \lambda_2 \cos(\lambda_1 x_1)\cos(\lambda_2 x_2)t, x_2 + \lambda_1 \sin(\lambda_1 x_1)\sin(\lambda_2 x_2)t), \tag{4.2}$$

and accordingly we can graph the evolution of γ_0 (by taking $\lambda_1 = 1, \lambda_2 = 1$) as follows:



t = 0 t = 3 t = 5

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