KNOTS, 4-dimensions and FRACTALS

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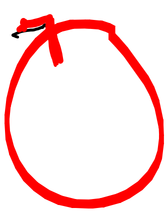
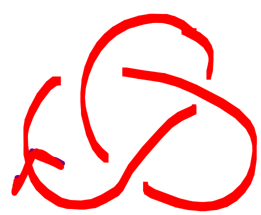
Def: A knot is a smooth embedding

$$f : S^1 \longrightarrow \mathbb{R}^3$$

i.e. take a rope, tie it up and attach the ends



unknot



trefoil

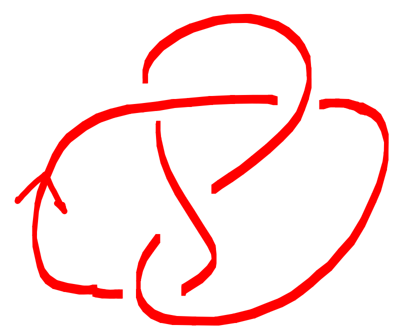


figure-eight

the usual 3-dimensional Equivalence Relation

ISOTOPY

Def: Two knots are isotopic if one can be deformed into the other through embeddings in $\mathbb{R}^3$

Change this to $\mathbb{R}^3 \times [0, \infty)$ to get different 4-DIM equivalence relation

Exterior of Knot K: $E(K) \equiv \mathbb{R}^3 - \text{int}\, N(K)$ is a 3-dimensional manifold with boundary whose fundamental group $\pi_1(E(K))$ is infinite, non-abelian.

Why Knots?

studying knots and links

$\Uparrow$

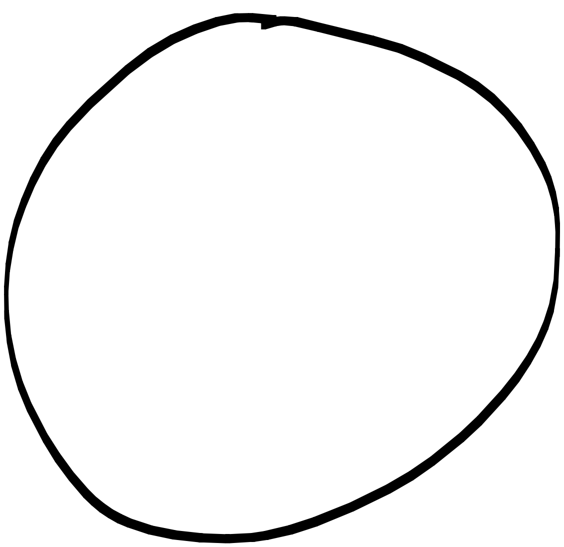
study arbitrary 3-dimensional mfds.

In 3-dimensions a knot is equivalent to the trivial knot

$\Updownarrow$

it bounds a disk embedded in $\mathbb{R}^3$

trivial knot = unknot



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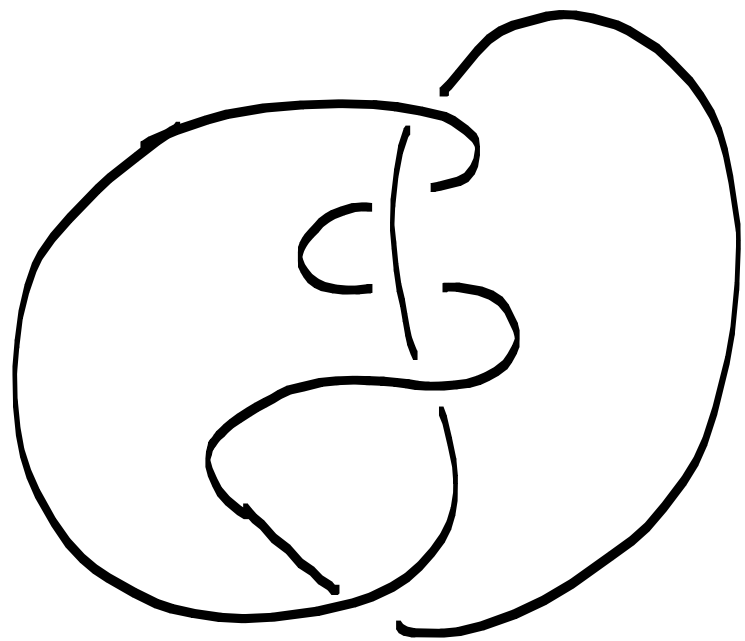
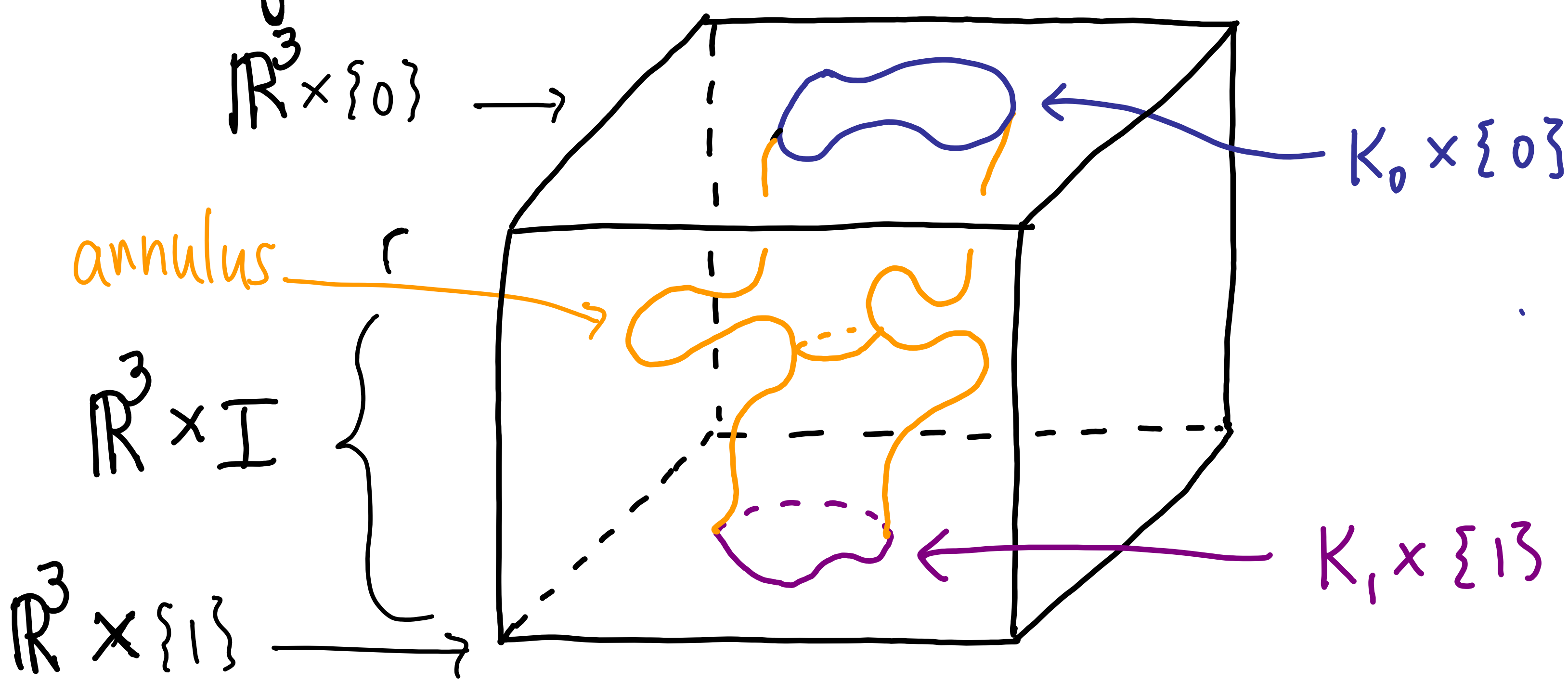


A 4-dim'l equivalence relation on Knots

Def: Knots K_0 and K_1 are **Concordant** if $K_0 \times \{0\}$ and $K_1 \times \{1\}$ cobound a smoothly embedded annulus in $\mathbb{R}^3 \times [0,1]$



Knot is "trivial" under concordance

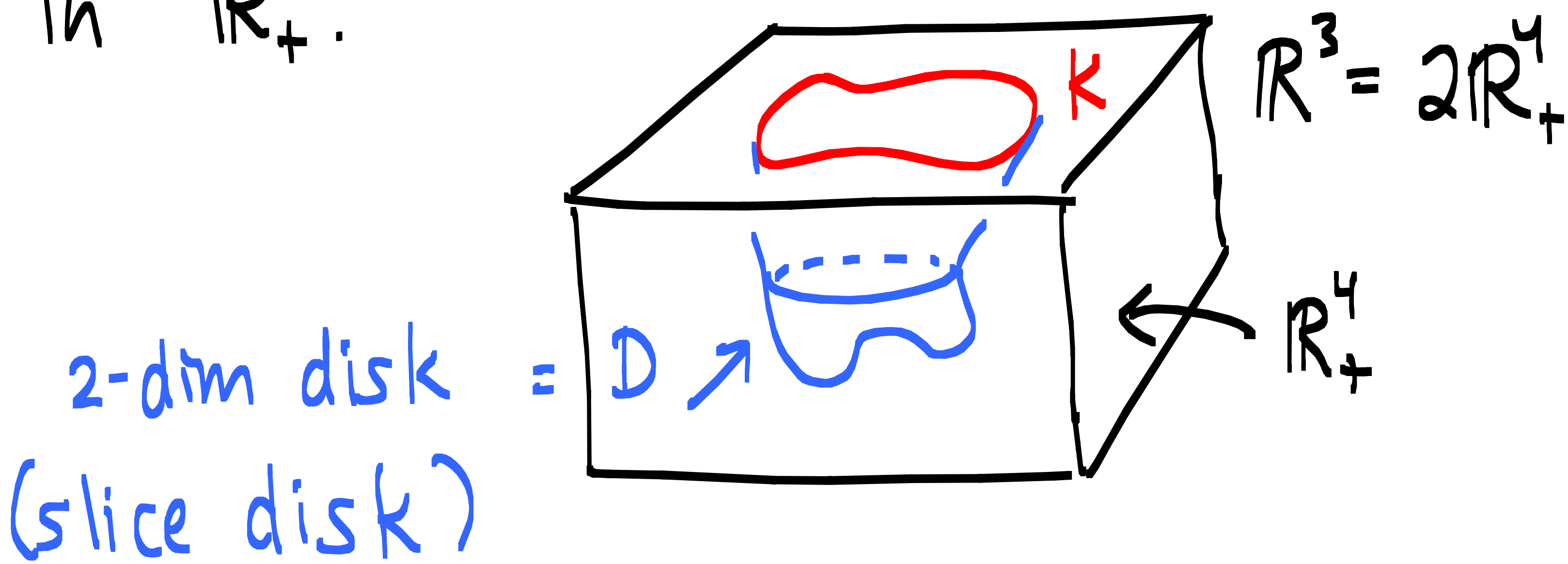
$\Updownarrow$

concordant to unknot

$\Updownarrow$

is a "slice knot"

Def: A knot $K \subset \mathbb{R}^3$ is slice if it is the boundary of a smoothly embedded 2-disk in $\mathbb{R}^4_+$.

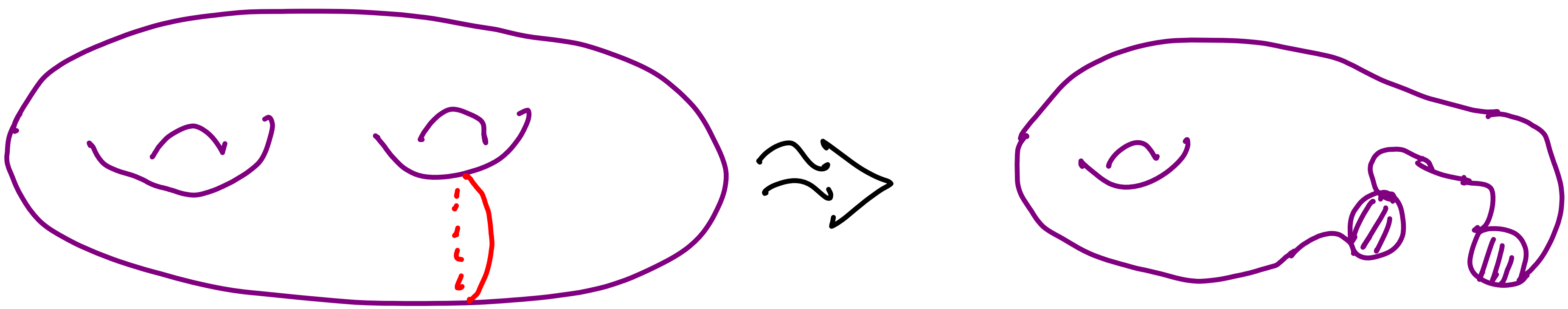


Why do we care?

$$\frac{\text{Knots}}{\text{isotopy}} \longleftrightarrow \text{classification of 3-mfds}$$

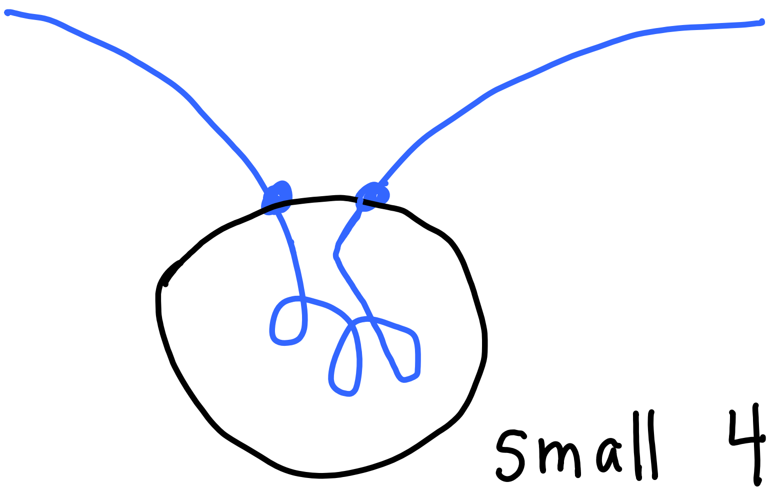
$$\frac{\text{Knots}}{\text{concordance}} \longleftrightarrow \text{classification of 4-manifolds}$$

Classification of 2-manifolds: Find embedded 1-manifold, cut open and induct



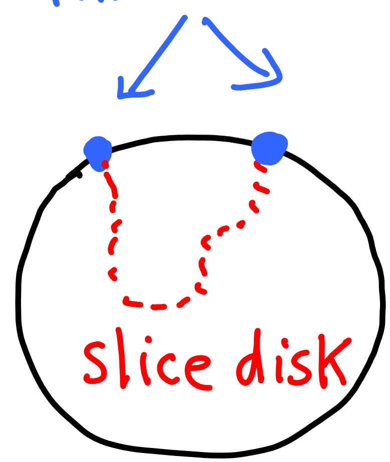
Classification of 4-manifolds: Find embedded 2-spheres, cut open, induct.

1960 Milnor, Fox, Kervaire: isolated singularities of 2-spheres in 4-manifolds



small 4-ball

KNOT OR LINK

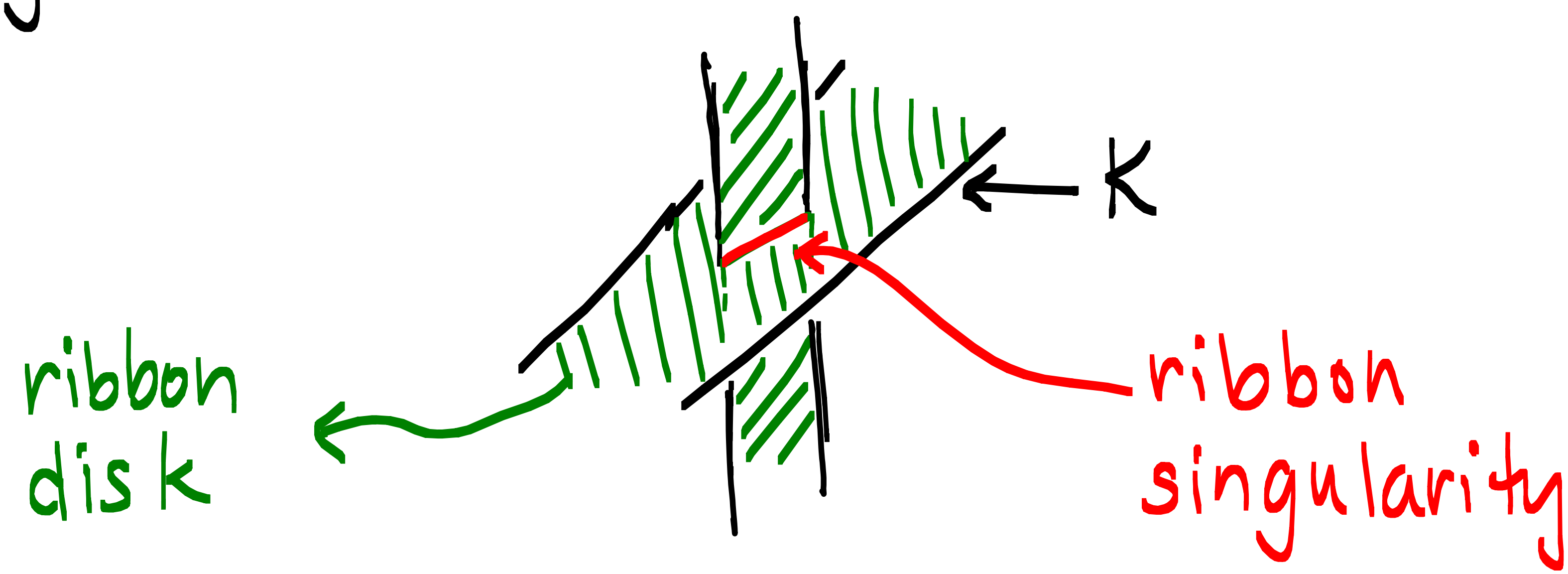


$$\partial B^4 = S^3$$

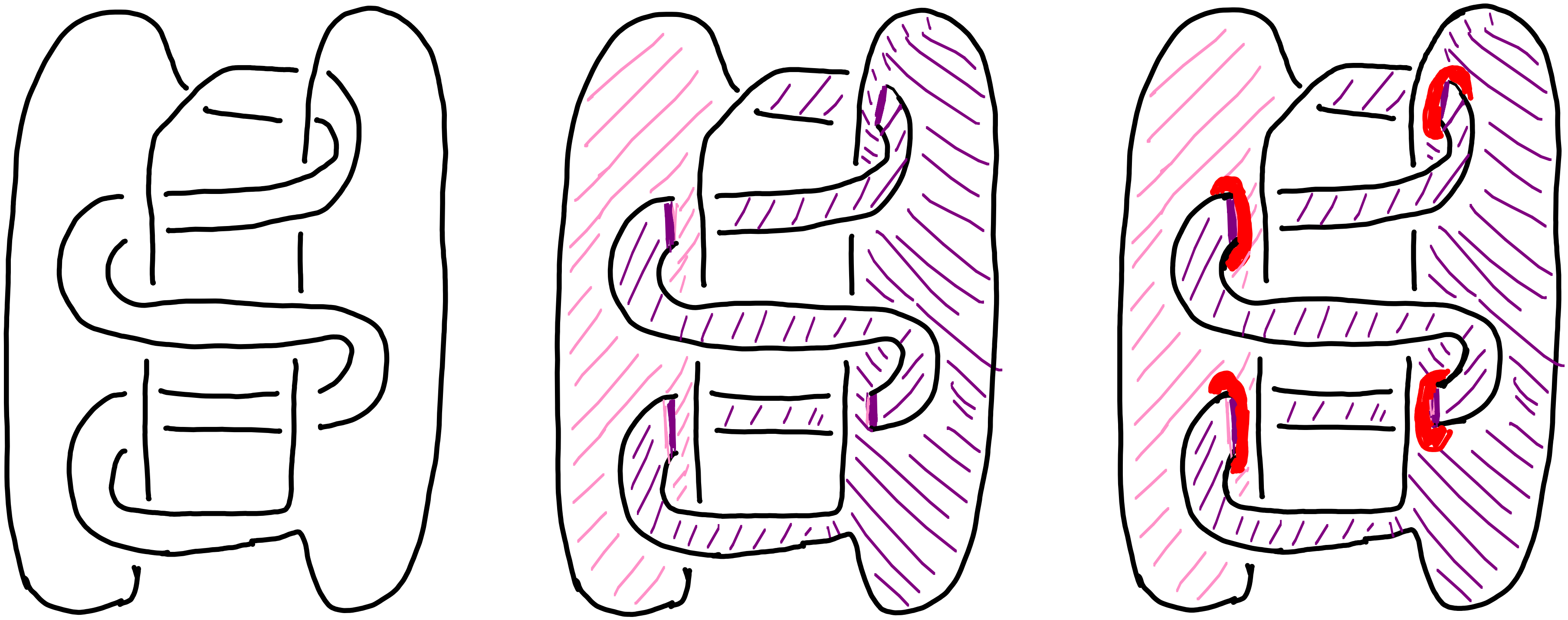
if "link of singularity" is slice, singularity can be resolved.

Examples of slice knots:

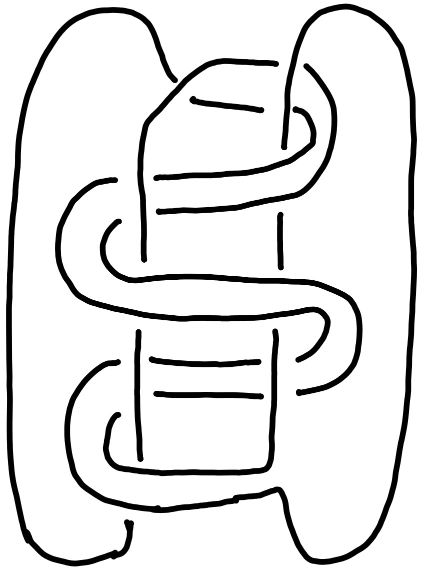
Def: K is ribbon if it bounds an immersed disk in $\mathbb{R}^3$ with only ribbon singularities.



A ribbon knot is slice



Pf: To obtain a disk <u>embedded</u> in $\mathbb{R}^4_+$, push the interior of red disks into interior of $\mathbb{R}^4_+$.

So  is slice.

Ribbon-Slice Conjecture: A knot is a slice knot $\Longleftrightarrow$ it is a ribbon knot

$\mathcal{C}$ = set of concordance classes of knots

Main Problem: Analyze $\mathcal{C}$

Two Goals of this talk:

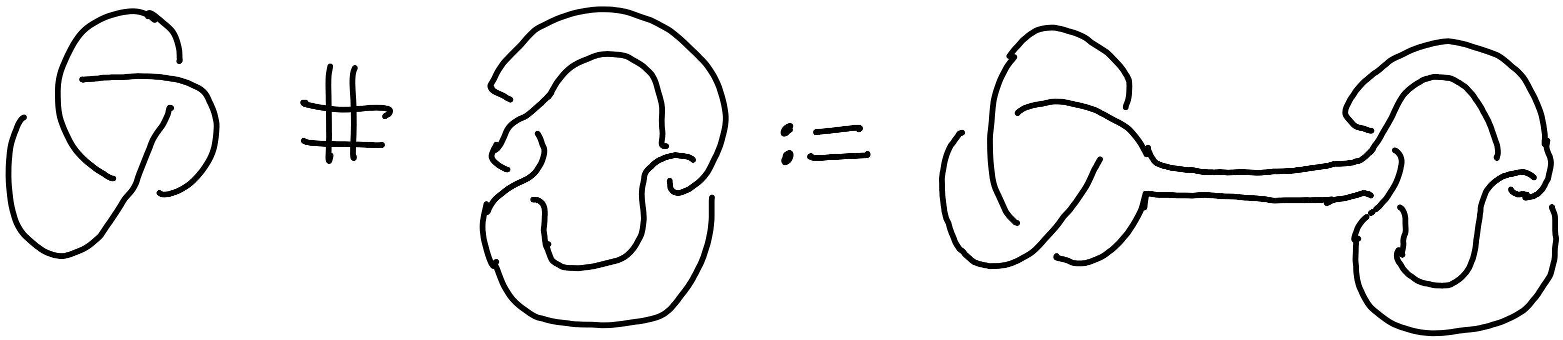
1. $\mathcal{C}$ is an abelian group with lots of structure.

2. $\mathcal{C}$ has natural metrics and self-similarities so is a Fractal Space

What structure does $\mathcal{K}$ have?

Addition:

commutative and associative



K_1 $\qquad$ K_2 $\qquad$ $K_1 \# K_2$

connected sum

Monoid Structure

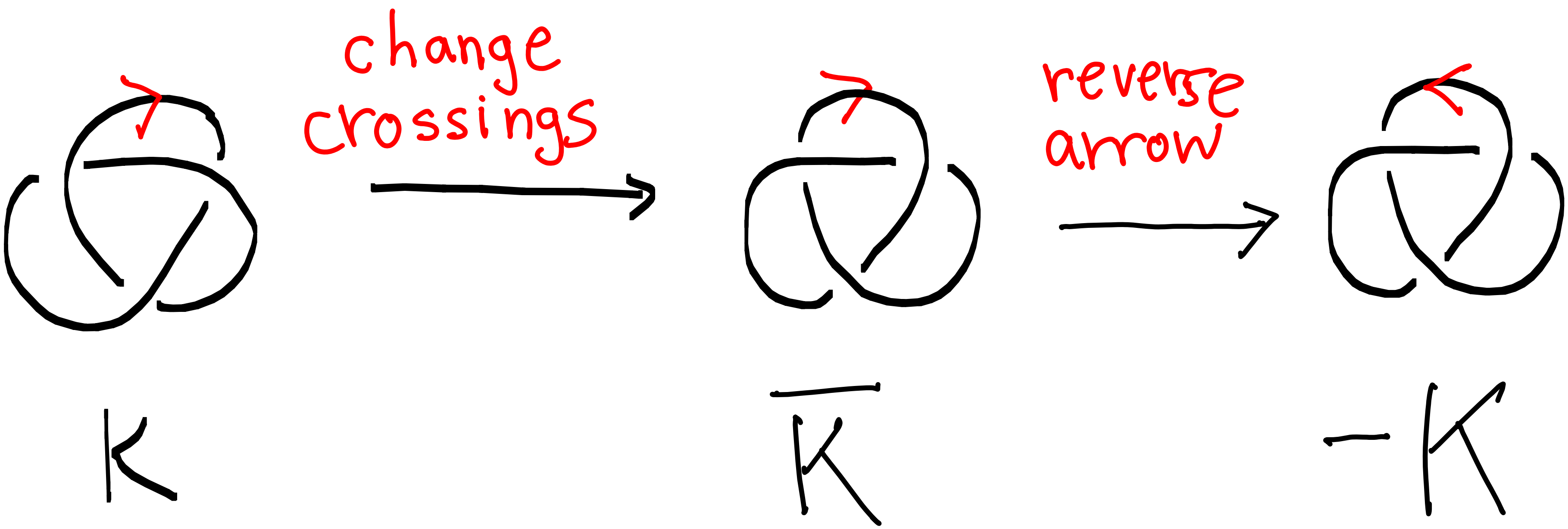
Claim: ($\mathcal{C}$, connected sum) is abelian gp.

- identity = [{slice Knots}] = [trivial Knot]

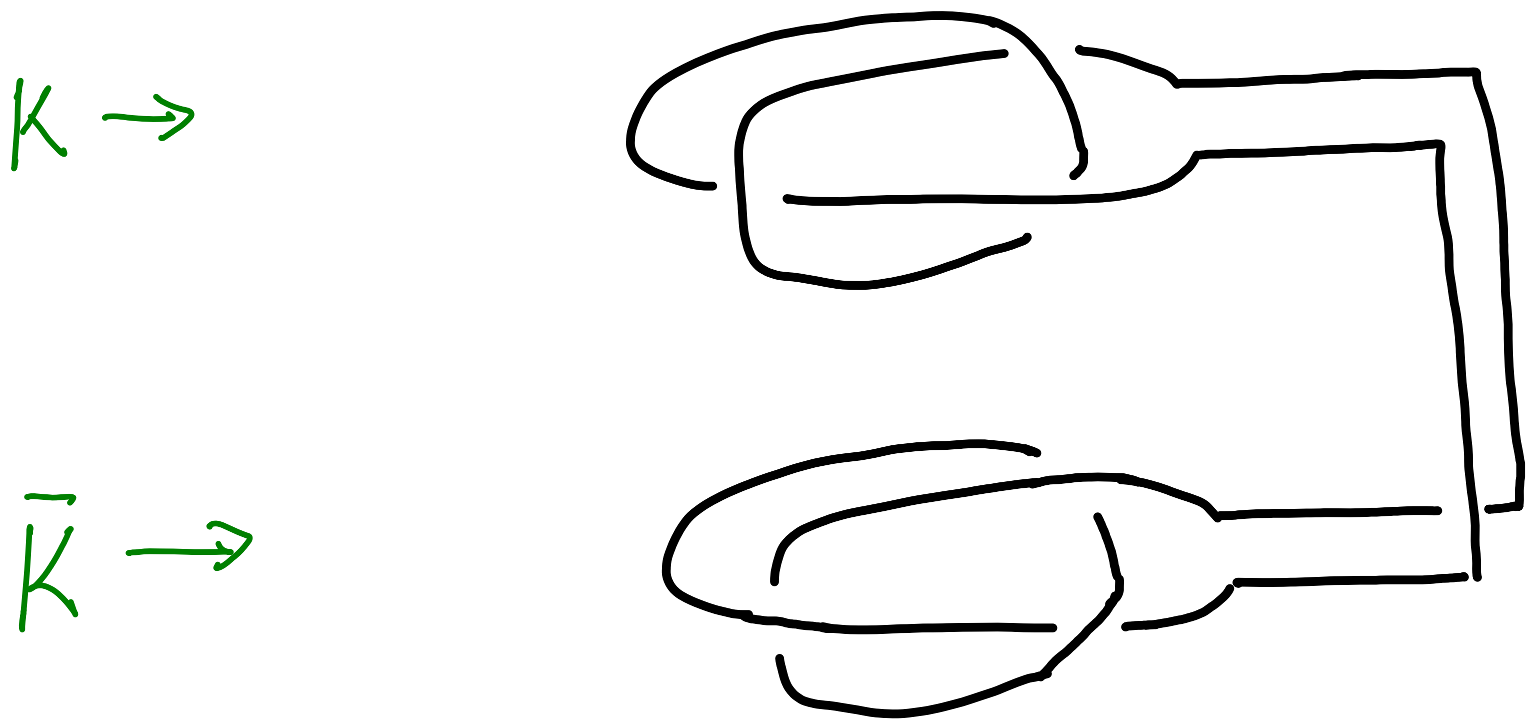
- inverse of K is (reverse of) mirror image of K

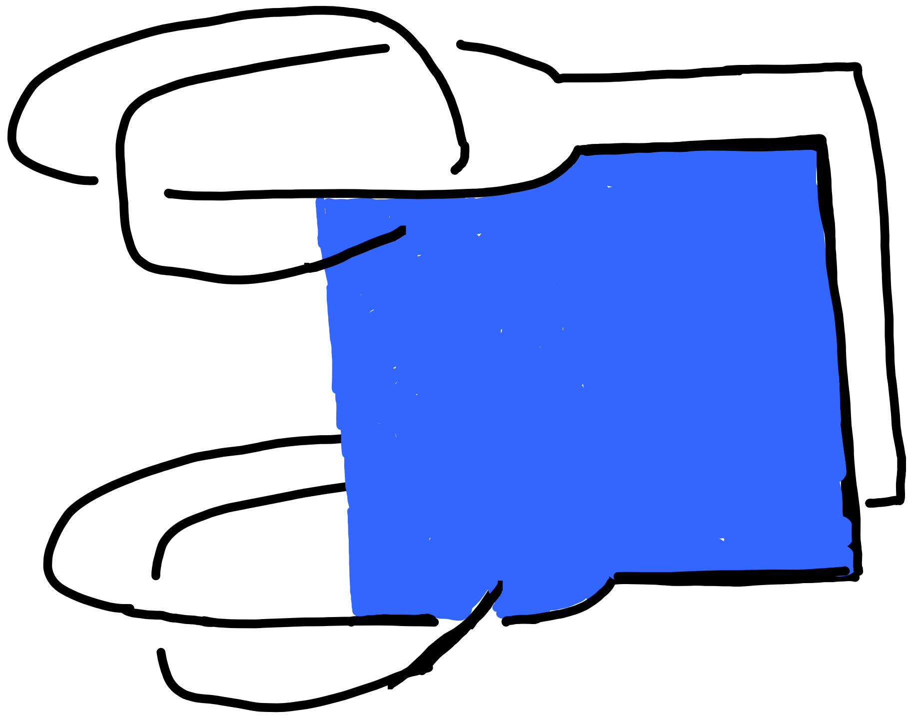
mirror image of K: change all crossings in some projection of K

reverse of K: change circle orientation



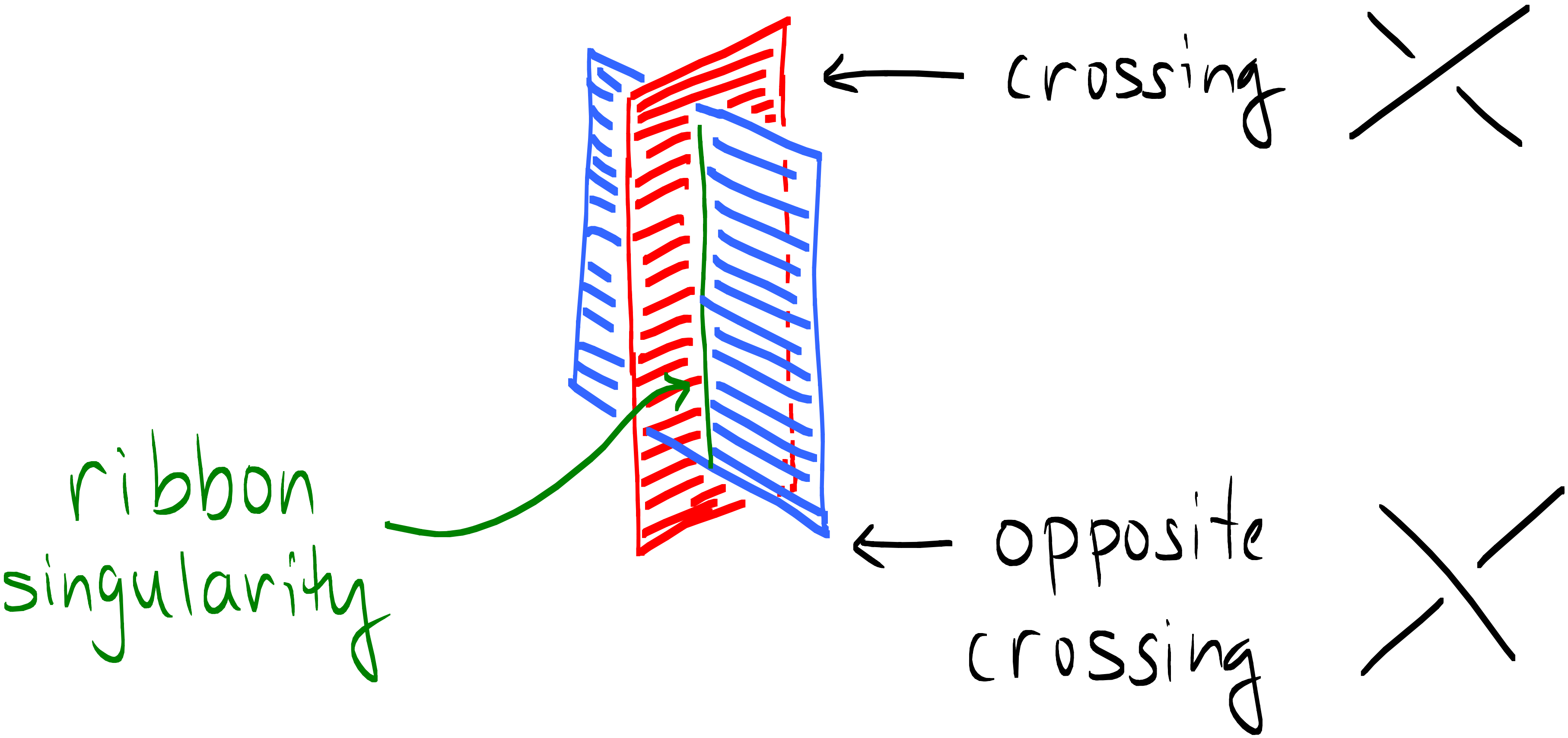
$K \# r\bar{K}$ is a ribbon knot ($\Rightarrow$ slice)





Continue this "vertical drape" around the arc. Immersed Rectangle

Intersections look like



$\Rightarrow K \# r\bar{K}$ is ribbon!

Main Questions about abelian group $\mathcal{C}$:

1. Finitely-generated? (NO)
2. What are possible torsion elements?
3. Are there any infinitely divisible elements?
4. How does the group structure interact with other operations on knots?
5. Find invariants sufficient to classify $\mathcal{C}$.

Example: Amphichiral Knots yield 2-torsion:

K: 

$K = r\overline{K}$ but K not slice

$$2[K] = [K \# K] = [K \# r\overline{K}]$$

$$= [\bigcirc]$$

No other torsion is Known!!!

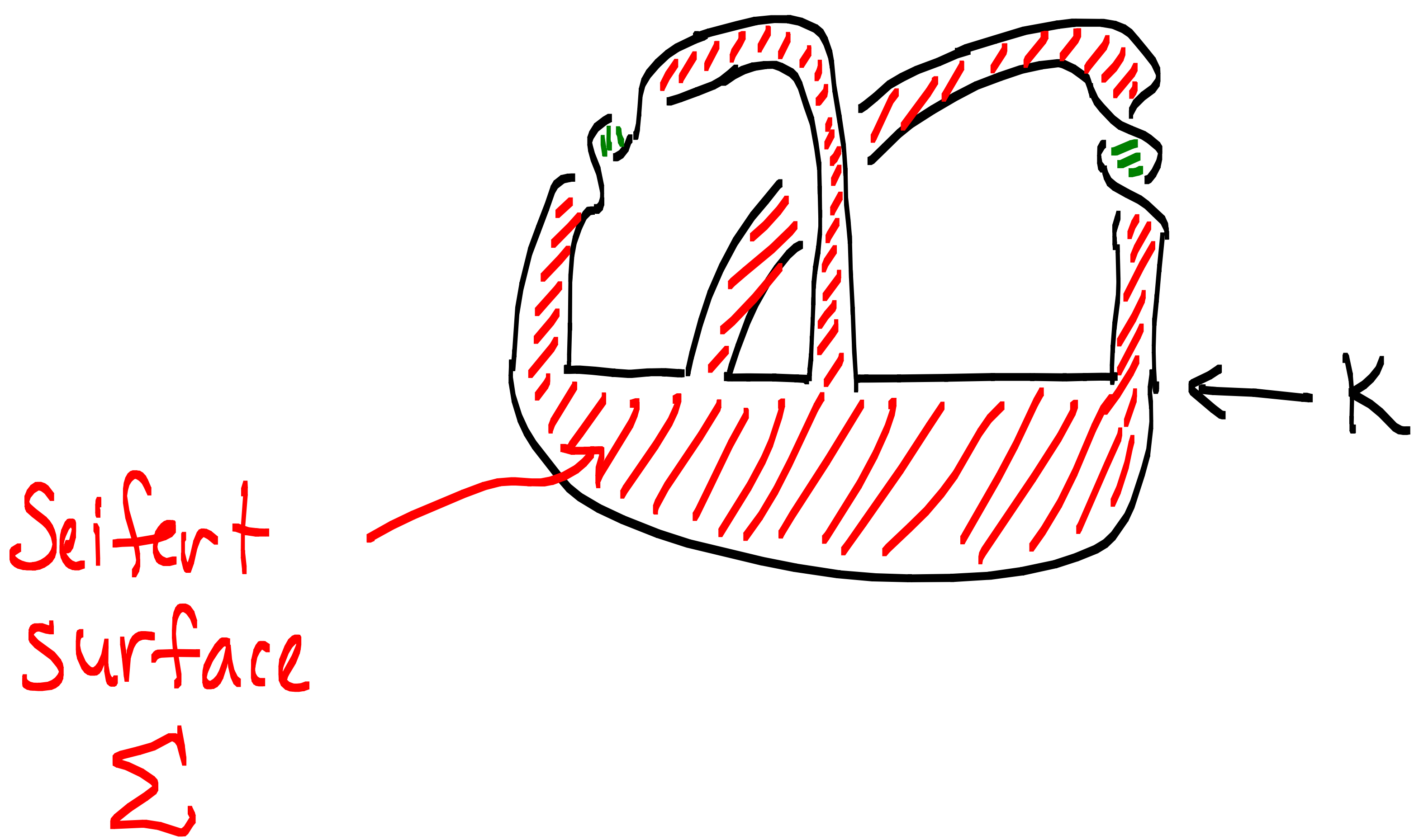
Theorem (Levine 60's): There is epimorphism

$$\pi : \mathcal{C} \twoheadrightarrow \mathbb{Z}^{\infty} \times \mathbb{Z}_2^{\infty} \times \mathbb{Z}_4^{\infty}$$

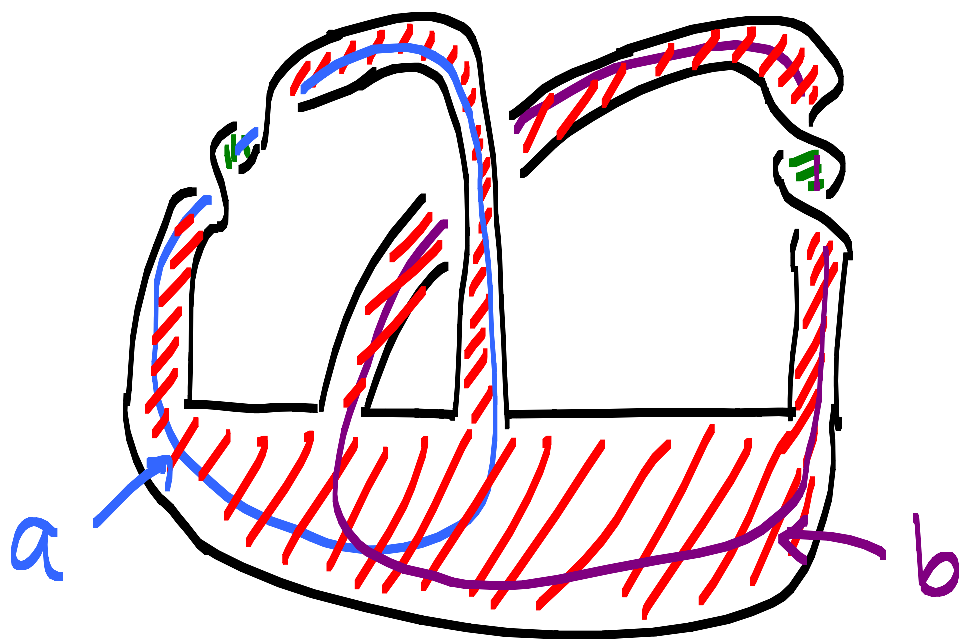
$$\{\text{Knots}\} \longrightarrow \{\text{MATRICES}\} \longrightarrow \{\text{quadratic forms}\} \longrightarrow \text{signatures and discriminants}$$

$$K \longrightarrow \text{“Seifert Matrix of } K\text{”}$$

Def: A Seifert surface Σ for K is a 2-sided surface embedded in S^3 with $\partial\Sigma = K$.



From a Seifert surface ⇝ Seifert matrix



$$V = \begin{pmatrix} lk(a,a^+) & lk(a,b^+) \\ lk(b,a^+) & lk(b,b^+) \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

b^+ = push b off Σ into + direction

$lk(a,b^+)$ = linking number of a and b^+.

For $\omega \in \mathbb{C}$, $\|\omega\| = 1$

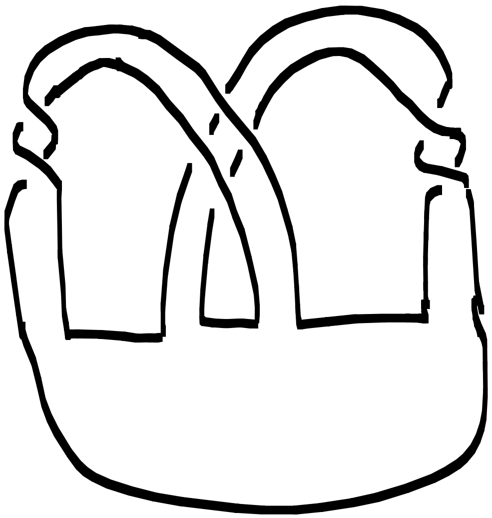
$(1-\omega)V + (1-\bar{\omega})V^T$ is a Hermitian matrix

Def: $\sigma_\omega(K) := \text{signature of} \left((1-\omega)V + (1-\bar{\omega})V^T \right)$

If K is slice and $\omega = (p^k)^{th}$ root of unity $\Rightarrow \sigma_\omega(K) = 0$. (Tristram)

$\rightsquigarrow \quad \bigoplus \sigma_\omega : \mathcal{C} \longrightarrow \mathbb{Z}^\infty$

Ex:



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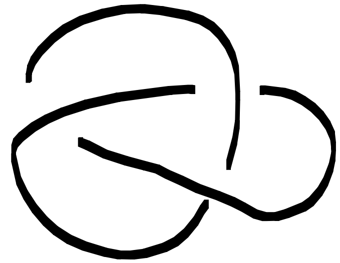
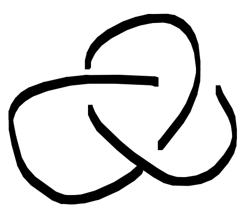


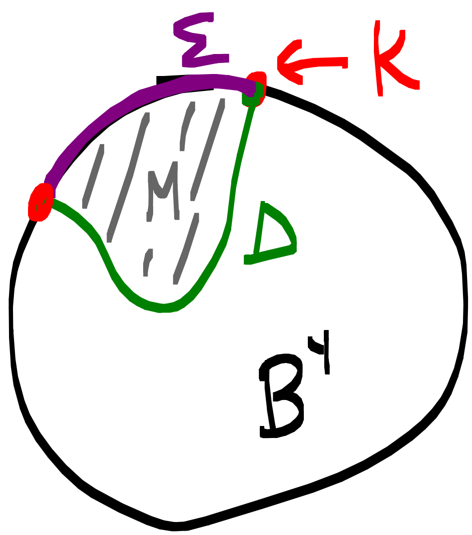
$$V = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$w = -1 \Rightarrow \sigma_w = -2$$

$\Rightarrow$  is not slice

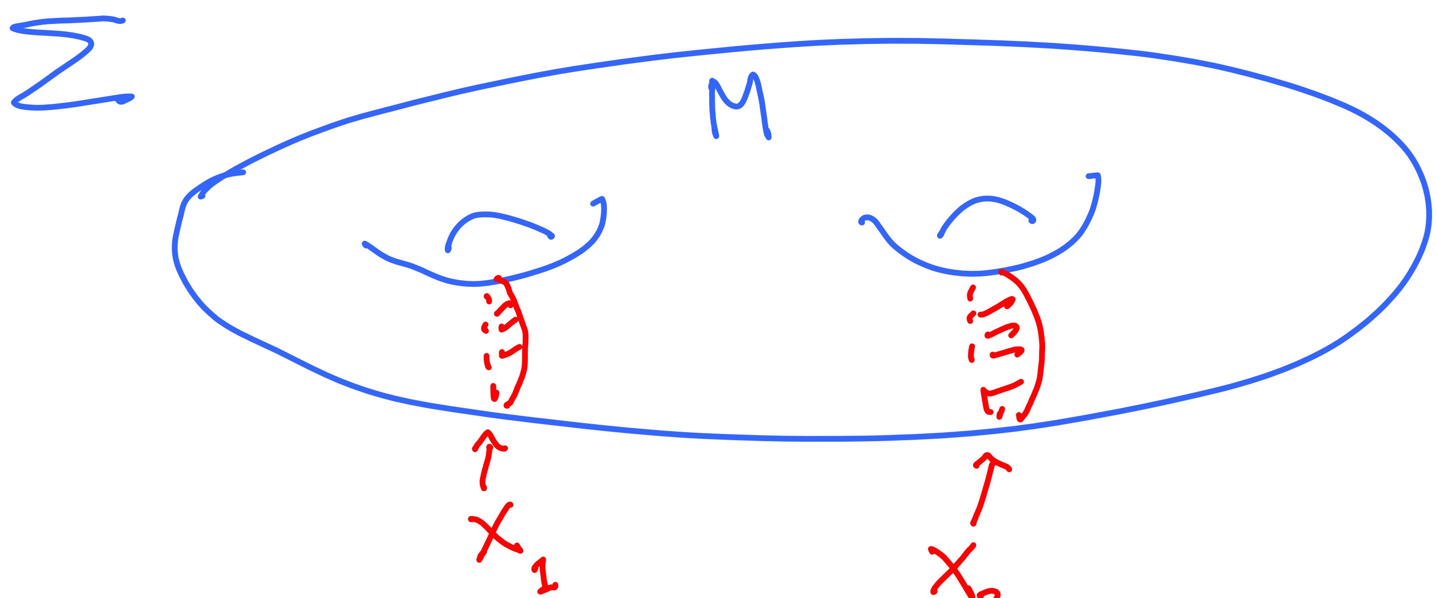
Lemma: If K is slice then its Seifert matrix has the form $\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$. ($\Rightarrow$ signatures are zero)

Pf: Let Σ be a Seifert surface for K and Δ = slice disk for K. Consider $\hat{\Sigma} = \Sigma \cup \Delta$.



$\hat{\Sigma} = \partial M$, M = 2-sided orientable 3-manifold

Lemma ($\frac{1}{2}$-lives, $\frac{1}{2}$-dies) If a 2-manifold bounds a 3-manifold then the kernel of $H_1(\Sigma;\mathbb{Q}) \to H_1(M;\mathbb{Q})$ has ½-rank.



Then show $lk(x_i, x_j) = 0$ so $V = \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$.

Techniques applied to study $\mathcal{C}$:

1. Invariants of Hermitian forms
2. $\ell^{(2)}$-methods, group von Neumann algebras
3. localization of noncommutative rings
4. gauge theory
5. Heegard Floer Homology
6. Khovanov Homology

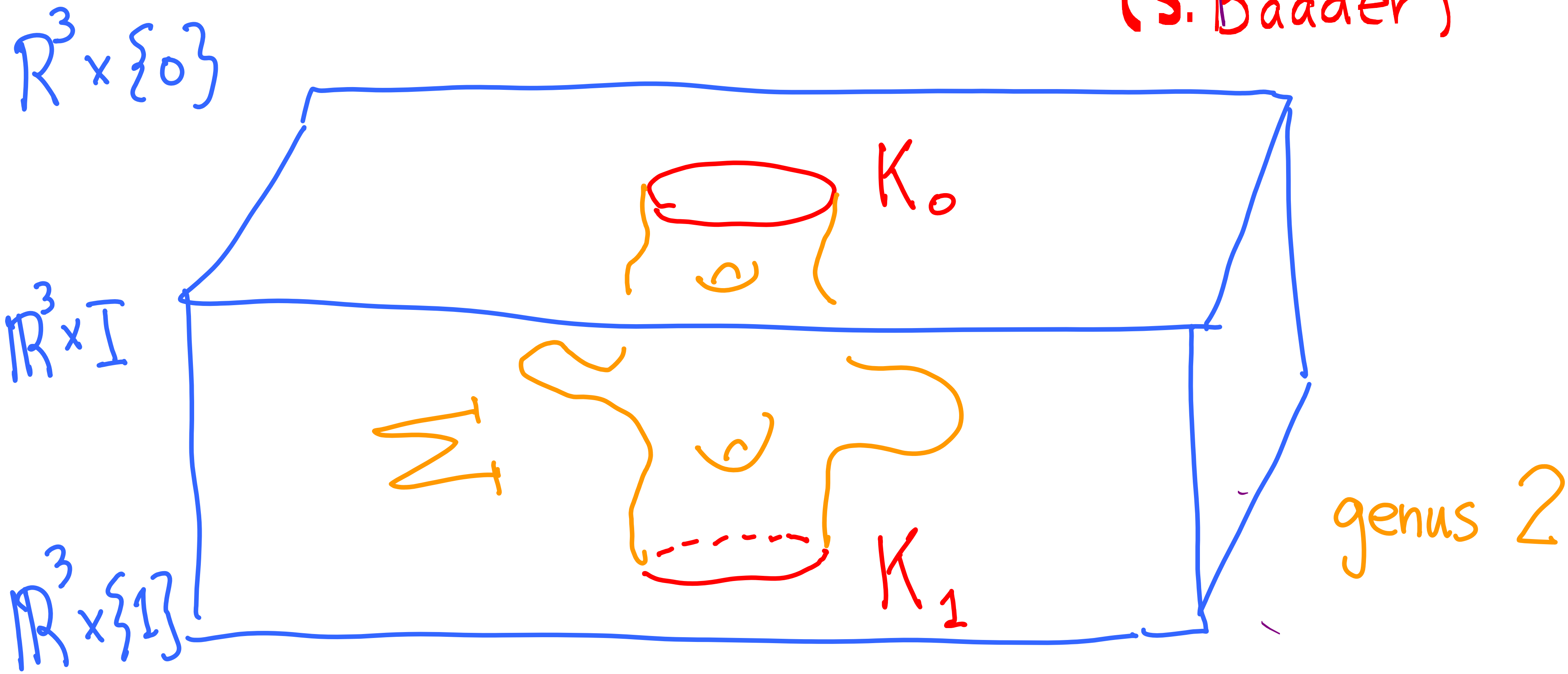
How is one led to these sophisticated techiques?

NEW Perspective on $\mathcal{C}$: $\mathcal{C}$ as a metric space

$\mathcal{C}$ admits many natural metrics.

Example: $d(K_0, K_1)$ = minimal genus of cobordism Σ in $\mathbb{R}^3 \times [0,1]$

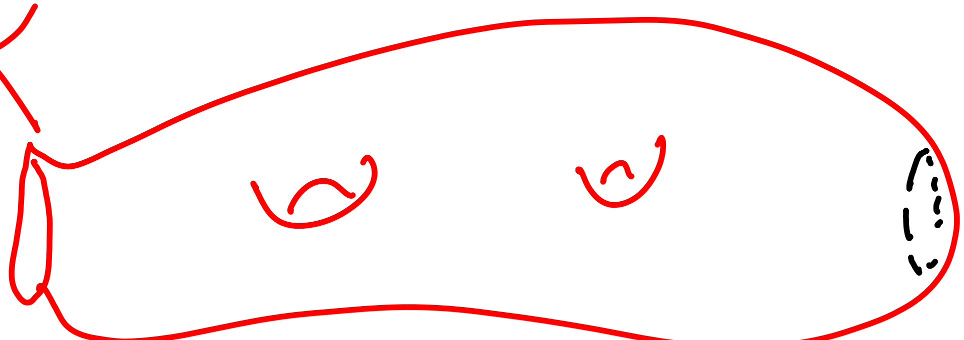
(S. Baader)



- K_0 concordant to $K_1 \Longrightarrow \Sigma$ is annulus
  $\Longrightarrow d(K_0, K_1) = 0$

- $d([K], [\text{unknot}]) =$ slice genus $\leq$ genus(K)
  $\quad = $ minimal genus of all Seifert surfaces

K



unknot

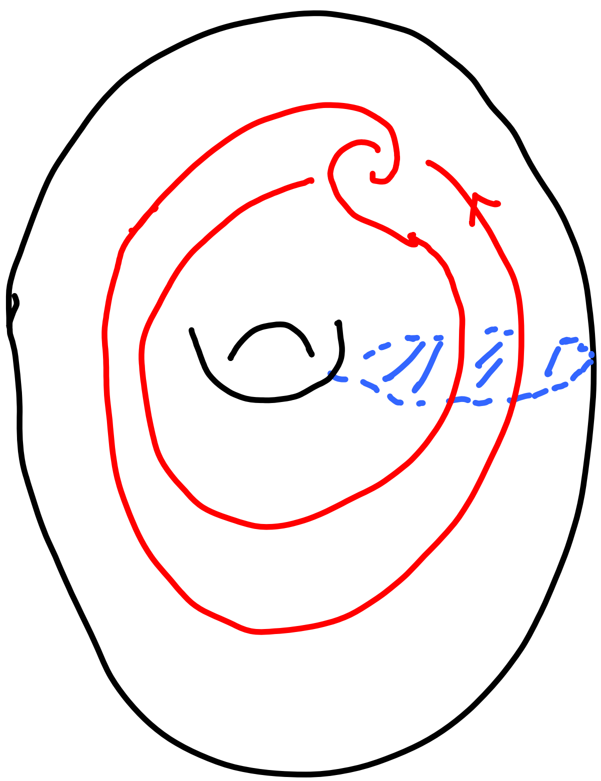
Seifert Surface

Fractal is a set that "exhibits approximate self-similarity at many scales".

Claim: $\mathcal{C}$ is a fractal (metric) space

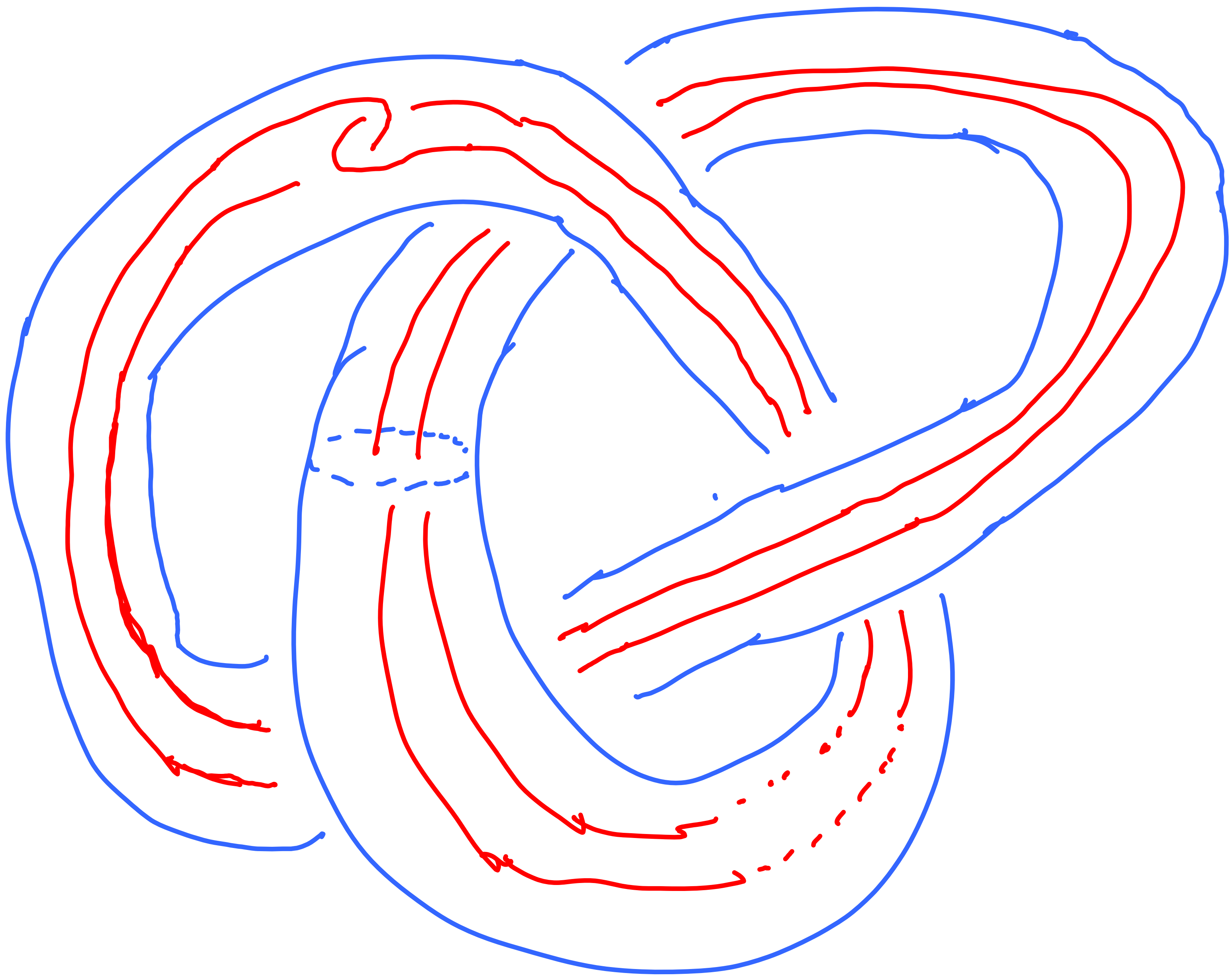
Discuss operations $\mathcal{C} \xrightarrow{P} \mathcal{C}$ that appear to be self-similarities (at least approximately); called satellite operations.

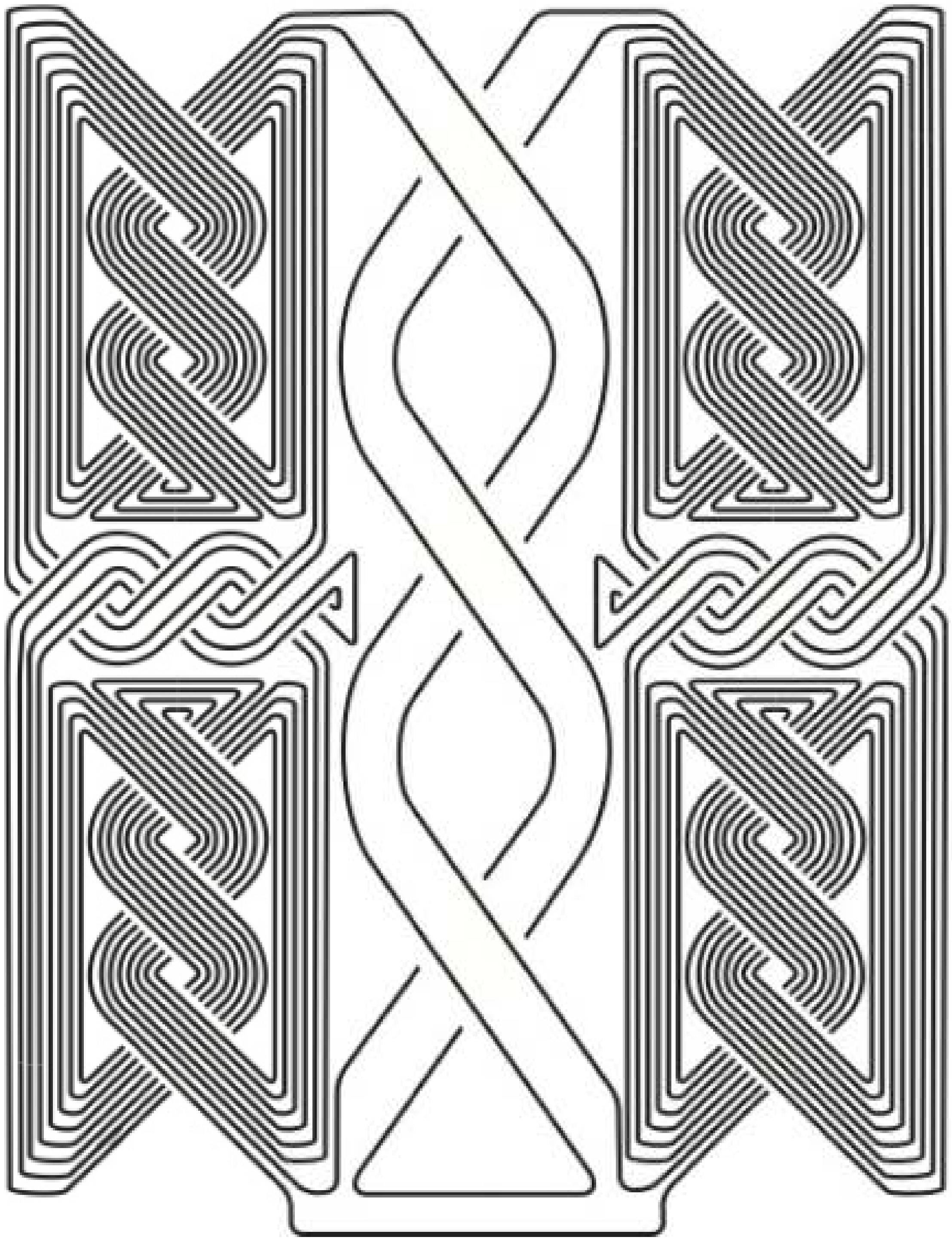
Let P be a knot inside a solid torus called pattern knot. Algebraic intersection with meridional 2-disk is winding number of P.



P with winding # 0

given any knot K form new knot P(K)
satellite of K with pattern P.





P(P(Trefoil))

Patterns are operators on Knots:

$$P : \mathcal{C} \longrightarrow \mathcal{C}$$

- injective? surjective? (nearly so)
- contractions? isometries?
  quasi-isometries?
- what can they tell us about $\mathcal{C}$?

Theorem (C - C. Davis - A. Ray) If P has strong winding number 1 then

$$P : \mathcal{C} \longrightarrow \mathcal{C}$$

is injective, modulo the smooth 4D Poincaré conjecture. For any non-zero winding number m, $P(K_0) = P(K_1)$ implies K_0 is concordant to K_1 in a $\mathbb{Z}[\frac{1}{m}]$-homology $\mathbb{R}^3 \times [0,1]$.

Proposition (C - S. Harvey) under same hypotheses, these are isometric embeddings with respect to a natural metric.

Winding number zero satellite operators are approximate contractions.

Conjecture: many winding number zero satellite operators (not all) are embeddings.

Techniques applied to study $\mathcal{C}$:

1. Invariants of Hermitian forms
2. $\ell^{(2)}$-methods, group von Neumann algebras
3. localization of noncommutative rings
4. gauge theory
5. Heegard Floer Homology
6. Khovanov Homology

How is one led to these sophisticated techiques?

Consider more refined version of Seifert matrix with entries in $G = \pi_1(S^3 \setminus K)$. Make it Hermitian as before.

What are invariants of Hermitian matrices over a noncommutative ring?

determinant? Signature?

Given a unitary representation

$$\rho : G \longrightarrow U(n) \text{ or } U(\ell^2 G)$$

form a bigger matrix

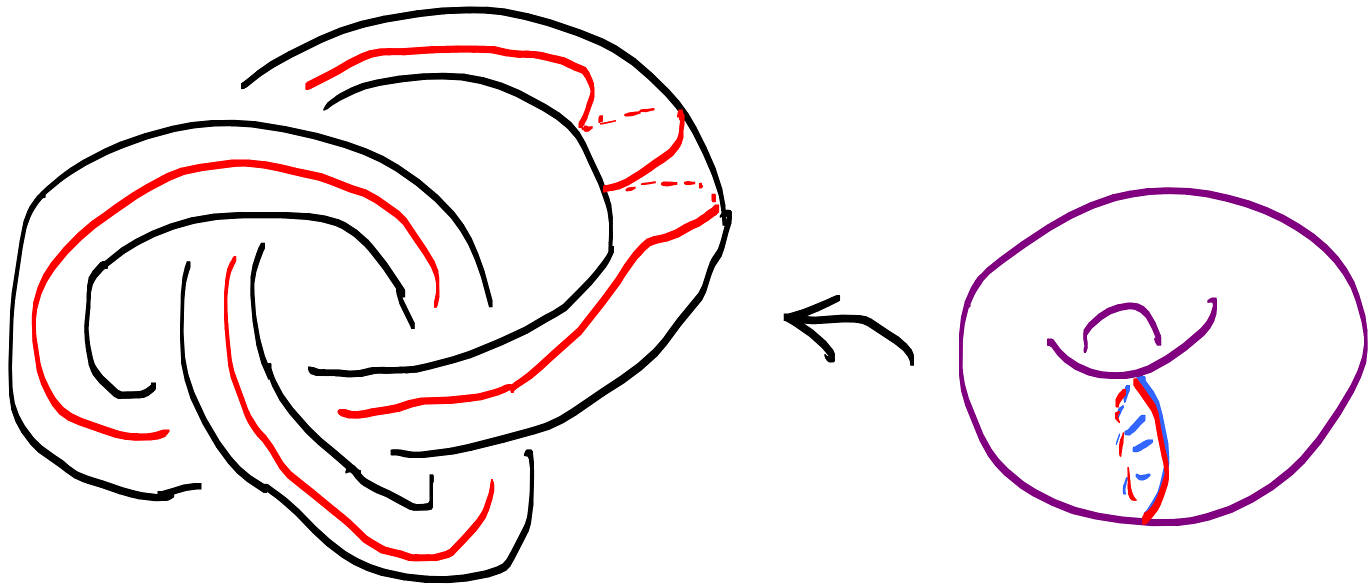
$$\begin{pmatrix} g & h \\ * & * \end{pmatrix} \longrightarrow \begin{pmatrix} \rho(g) & \rho(h) \\ * & * \end{pmatrix}$$

result is Hermitan matrix over $\mathbb{C}$ or $(\ell^2 G)^m$

How does gauge theory enter the picture?

1. $\{\text{Knots}\} \xrightarrow{\text{surgery}} \{\text{closed 3-manifolds}\}$

$$K \longrightarrow S^3 \setminus (K \times D^2) \cup S^1 \times D^2$$



$S^3 \setminus (K \times D^2)$

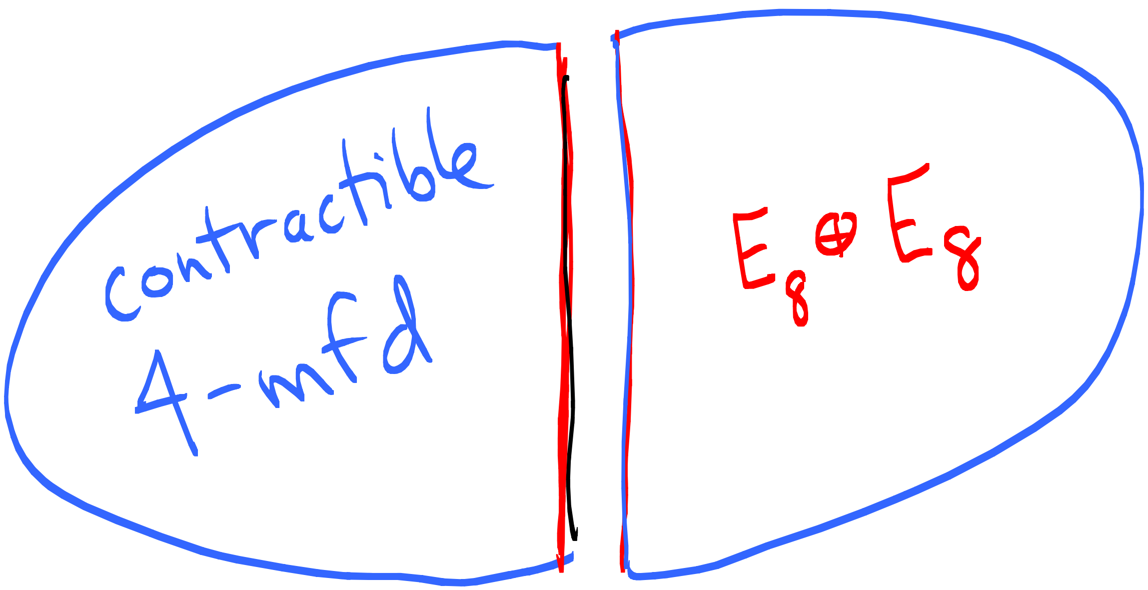
$\{\text{Knots}\} \xrightarrow{\text{surgery}} \{\text{3-manifolds}\}$

concordant knots $\longrightarrow$ 3-manifolds that are homology coborant

Slice Knot $\xmapsto{+1}$ homology 3-sphere that is boundary of contractible 4-manifold

SHOWING K is NOT SLICE by contradiction:

So if $+1$ surgery on K happens to bound a specific 4-manifold with positive definite intersection form $E_8 \oplus E_8$



= contradiction to Donaldson's theorem about smooth 4-mfds

Other work on $\mathcal{C}$:

There exists a filtration of $\mathcal{C}$ by subgroups

$$\{0\} \subset \cdots \subset \mathcal{C}_2 \subset \mathcal{C}_1 \subset \mathcal{C}_0 \subset \mathcal{C}$$

and for each n

$$\exists\ \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \subset \mathcal{C}_n / \mathcal{C}_{n+1}$$

C-Orr-Teichner, C-Harvey-Leidy,
C-Harvey-Horn

Let W be a smooth 4-dimensional manifold with $\partial W = S^3$. If K is any knot, we can ask if K is slice in W, i.e. does K bound a smoothly embedded disk in W.

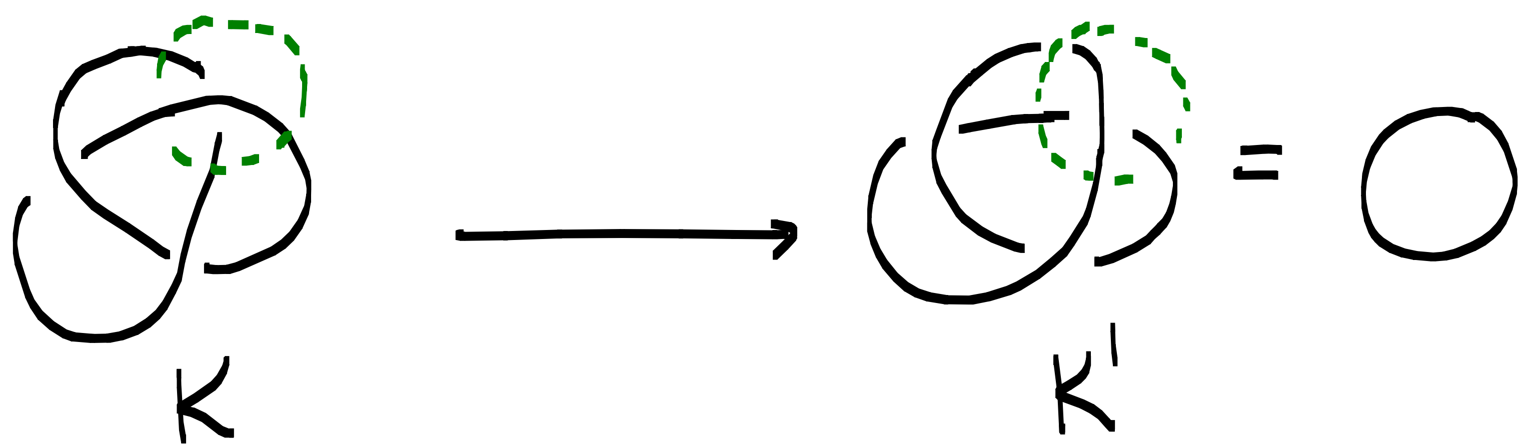
Then we could try to filter $\mathcal{C}$ by considering various 4-manifolds W_i.

Prop: Every knot is slice in $\left(\#_k \mathbb{C}P^2\right) \# \left(\#_\ell \overline{\mathbb{C}P^2}\right) \setminus B^4$.

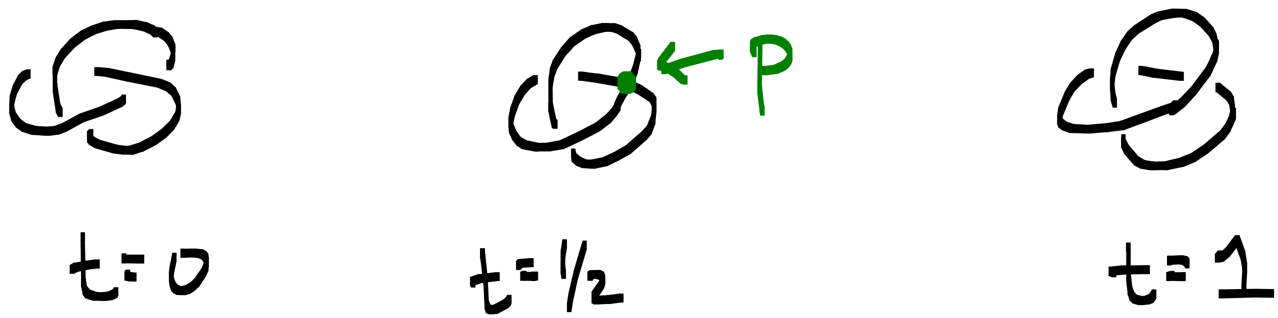
Recall $\mathbb{C}P^n = \{\text{complex lines in } \mathbb{C}^{n+1}\}$.

$\overline{\mathbb{C}P^n} = \mathbb{C}P^n$ w/ opposite orientation

Pf: Every knot can be changed into the unknot by changing crossings.

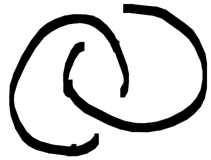


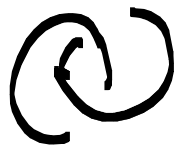
A crossing change gives a homotopy $S^1 \times I \longrightarrow S^3 \times I$ starting at $K \times \{0\}$ and ending at $K' \times \{1\}$ which is an embedding except at one point p.



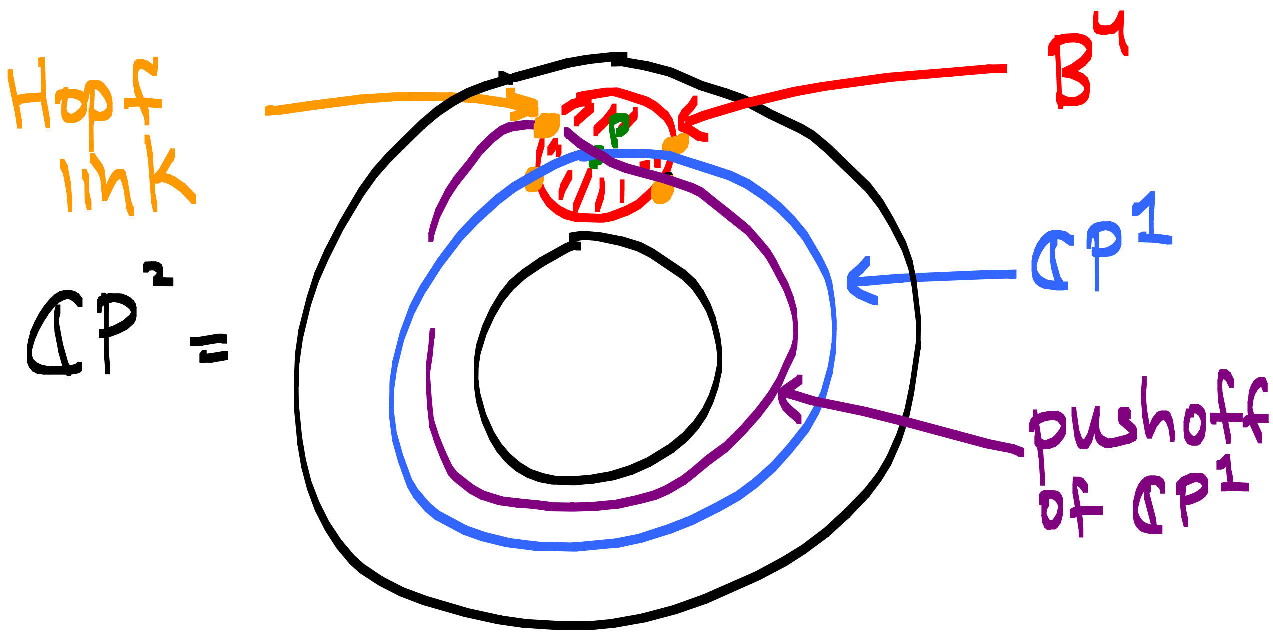
Take a 4-ball $B \subset S^3 \times I$ around p.
Let A = image of $S^1 \times I$. Then $A \cap B$

is two complex disks in $B^4 \subset \mathbb{C}^2$ and

$\partial(A \cap B) =$ , Hopf link.

We can replace B with $\mathbb{CP}^2 \setminus B^4$ or $(\overline{\mathbb{CP}}^2 \setminus B^4)$ and then  will bound disjointly embedded disks (blowing up at p).

$\mathbb{CP}^2 =$



replace B^4 with $\mathbb{CP}^2 \setminus B^4$

Def: A knot K is n-positive if $\exists$ a smooth 4-manifold W with $\partial W = S^3$ and a disk Δ smoothly embedded in W with $\partial \Delta = K$ s.t.

(1) $H_1(W) = 0$

(2) $\exists$ disjointly embedded surfaces $S_1, \ldots, S_j$ freely generating $H_2(W)$ with $S_i \cap S_i^+ = \{pt\}$, a positive intersection $\forall\, i$.

(3) $S_i \cap \Delta = \phi \quad \forall\, i$

(4) $\pi_1(S_i) \subset \pi_1(W \setminus \Delta)^{(n)} \quad \forall\, i$