Untying Knots in 4-Dimensions

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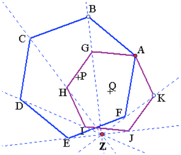
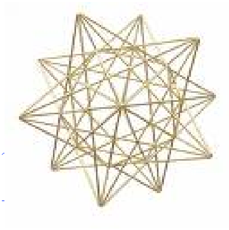
Outline

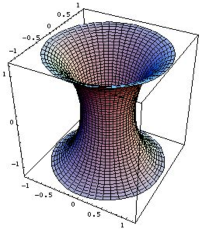
What is Geometry?

Geometry is the study of shape.

1. Euclidean Geometry: study lengths, angles, areas, straight lines, and the curvature of concrete objects in the plane or 3-space
2. Modern geometry: more general objects, generalizations of notions of shape, including the shape of space-time.



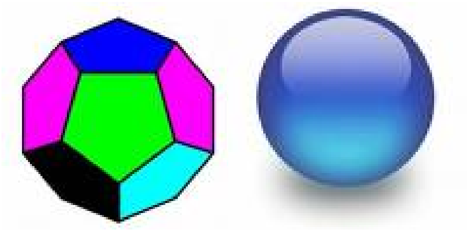




Topology is the crude country cousin of Geometry

Topology is the study of only those (coarse) properties of shape that are unchanged by stretching but not tearing or fusing pieces together.

Any (solid) polyhedron is topologically equivalent to a Solid Ball



Don't ask a Topologist to order breakfast!

A coffee mug is topologically equivalent to a doughnut

since they each have one hole.

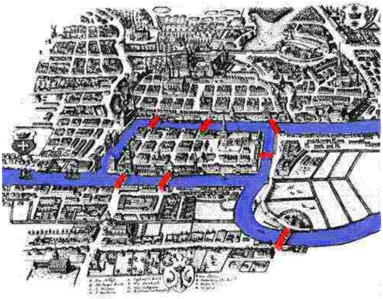
Why Topologists can't read well

Many letters are topologically equivalent.

$$I \sim Z \sim V \sim L \sim W \sim J \sim C$$

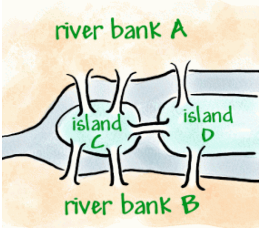
Euler's Königsberg bridge problem 1736

1736 Euler published the first paper on topology: Is it possible to take a stroll and traverse the seven bridges of Königsberg without crossing any bridge twice?

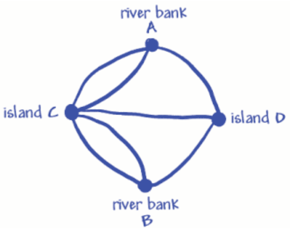


Topology problem not a Geometry problem

The answer doesn't depend on the lengths of the bridges or the sizes of the land masses, just how they are connected.



Birth of Modern Graph Theory

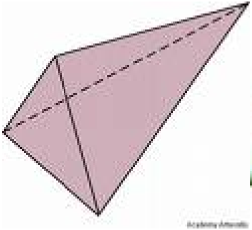
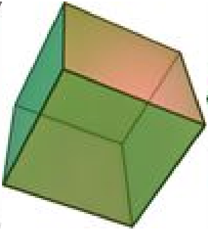
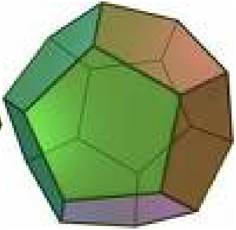


Fun Problem: Characterize those graphs for which you can trace a path (without lifting your pencil) that traverses each edge precisely once?

Euler's characteristic number

1752: In studying convex polyhedra, Euler found a surprising universal formula satisfied by the number of vertices, V, the number of edges, E, and the number of faces, F. For any such polyhedron

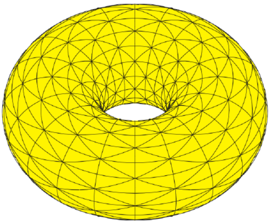
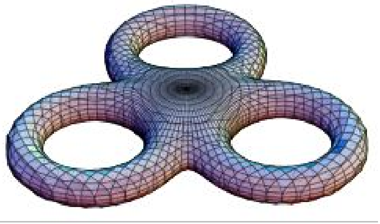
$$V - E + F = 2$$

The Euler characteristic

In 1813 Lhuilier noticed that Euler's formula was wrong for surfaces of solids with holes in them. He showed that if a solid has g holes then

$$V - E + F = 2 - 2g$$

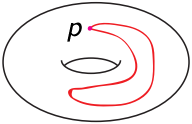
The quantity $V - E + F$, or $2 - 2g$, is called the Euler characteristic. It is a topological invariant because it depends not on the geometry, but just the topology, in this the case just the number of holes.

Poincaré: Father of Topology

1895: Henri Poincaré put Topology on a rigorous foundation in a series of papers. In particular, he established a rigorous way to talk about the holes in an object.

Given a topological space X consider the loops or circuits that begin and end at some fixed point p. The constant loop is the one that never leaves the point p. A loop is said to be trivial if it can be continuously deformed to the constant loop. Two loops are equivalent if one can be continuously deformed to the other staying inside the space X.

This loop on the torus X is trivial.



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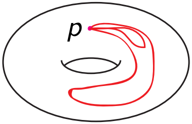
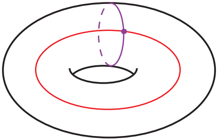
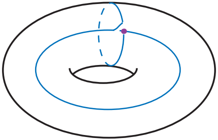


The loops a and b are not trivial. They cannot be deformed to the constant loop because of holes in the torus X.



Moreover loops can be multiplied by concatenation. Here is the loop ab.



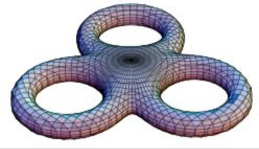
Therefore the set of (deformation classes of) loops of X has has the algebraic structure of a group (like real numbers or matrices).

Definition (Poincaré)

The group of deformation classes of loops in a space X is called the the fundamental group of X, denoted $\pi_1(X)$.

$\pi_1(X)$ measures the complexity of the 1-dimensional circuits (or 'holes') in a topological space. The simplest spaces are the ones whose fundamental group is trivial, i.e. every loop can be contracted to the constant loop.

A 2-**dimensional manifold** (surface) is a space that is 'locally' like the plane, i.e. can be described by 2 independent coordinates. Among surfaces, the only one with $\pi_1(X) = 0$, i.e. "no holes" is the 2-dimensional sphere S^2.

A 3-**dimensional manifold** is a space that is 'locally' like the space in this room, i.e. can be described by 3 independent coordinates.

Imagine gluing opposite sides of this dodecahedron together to get a 3-dimensional manifold.

The Poincaré Conjecture

The Poincaré Conjecture

The only 3-dimensional manifold (compact with no boundary) that has $\pi_1(X) = 0$ is the 3-dimensional sphere (the boundary of the 4-dimensional ball)

$$S^3 = \{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 = 1\}$$

The recent proof (by Perelman) uses topology, geometry and analysis.

Topologists and Geometers study **Topological spaces**.

Definition

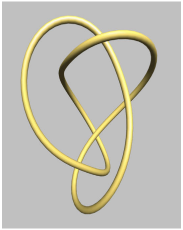
A Topological Space is a set of points together with a notion of the distance between two points.

In **this** talk almost all of the topological spaces can be pictured in ordinary 3-dimensional space. But most spaces studied cannot be pictured in 3-dimensional space.

1. the space of real solutions of the equation $x^2 + y^2 + z^2 + w^2 + t^2 = 1$;
2. the space of all solutions of some differential equation;
3. the space of all $n \times n$ matrices;
4. the space of all sequences of real numbers;
5. the space of all possible configurations of a mechanical apparatus.

What is a Knot?

A mathematical knot is a closed loop in $\mathbb{R}^3$. Two knots are equivalent if one can be smoothly deformed to the other.



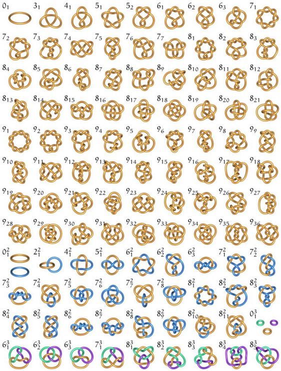
Scottish Mathematical Physicists

1860-1888 Thompson (Lord Kelvin), Maxwell and Tait became heavily involved with knot theory because it arose out of physical considerations.

Thompson: Many believed that space was filled with 'aether'. Based on analogies from Helmholz' papers on perfect fluids, Thompson hypothesized that atoms are knotted vortices of aether. The permanence of matter was to be explained by the stability of knots under distortion. The variety of elements was mirrored by the infinitude of knots and links.

This viewpoint held up for 20 years until the Michelson-Morley experiment. But in the meantime the scientific study of knots was taken up on a large scale.

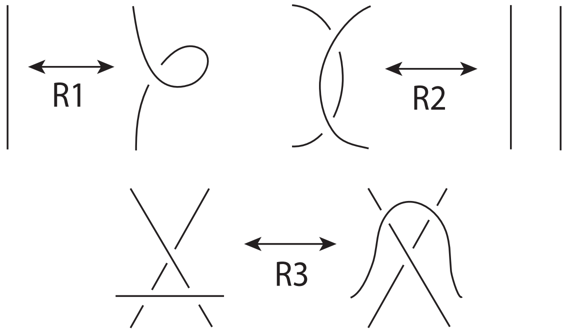
Tait: Tait compiled vast tables of planar diagrams of knots and links, thinking that he was compiling an atomic table!



Remarkably, Tait's table contained only a few errors and became the basis for modern knot tables.
The crossing number of a knot is the minimum number of crossings in a planar diagram.

For the example the crossing number of the knot above is 0 since it is a trivial knot!!
Tait's Conjectures about knot diagrams: only finally proven in 1990.

Maxwell: introduced moves for planar diagrams of knots:



Theorem (1927 Alexander-Briggs, Reidemeister)

Two different planar diagrams represent equivalent knots if and only if one diagram can be transformed to the other by a finite number of the Reidemeister moves.

Birth of Combinatorial and Algebraic Knot Theory: Associate quantities to the combinatorics of a knot diagram and find those quantities that are unchanged by the Reidemeister moves. These will then be knot invariants.

1. 1911 **Alexander polynomial**
2. 1985 **Jones polynomial**
3. 2000 **Khovanov Homology Groups**

Fruitful interactions with statistical mechanics and many areas of mathematical physics.

Sample Problems/Questions in Knot Theory?

1. Find numerical invariants of knots. Find structure. Classify all knots.

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Sample Problems/Questions in Knot Theory?

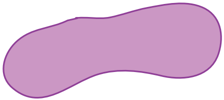
1. Find numerical invariants of knots. Find structure. Classify all knots.
2. Which knots are equivalent to their own mirror images (amphicheiral) ?
3. There is an algorithm to decide if a knot is equivalent to the trivial knot. Is there an effective algorithm?
4. What is the unknotting number of a knot, i.e. the minimum number of crossings that must be changed from over to under to transform the knot to a trivial knot?

What kind of knots can be "unknotted in 4-dimensions"?

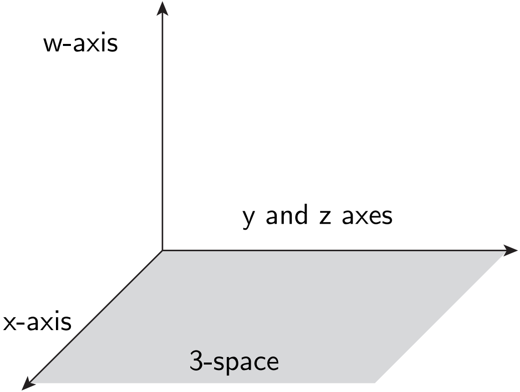
Knots live in 3-dimensional space

$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \text{ any real numbers}\}$

A **trivial knot** can be characterized as one that is the boundary of a disk in 3-space.

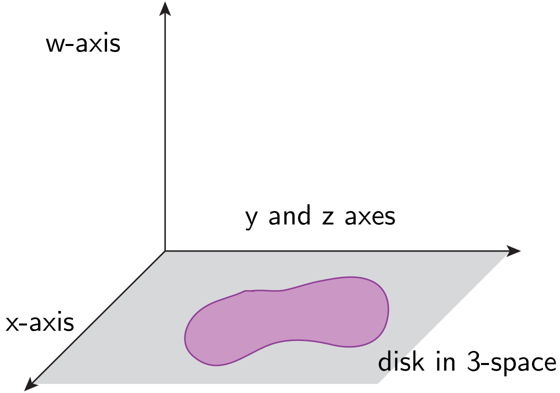


Now consider one-half of 4-space: $\{(x, y, z, w) \mid w \geq 0\} = \mathbb{R}^4_+$



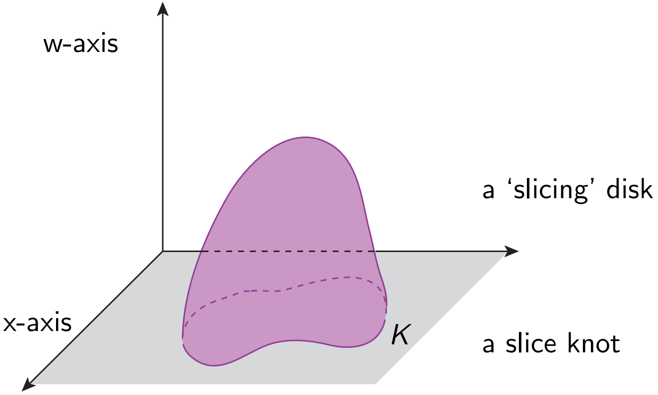
The grey 'plane' represents 3-space $\mathbb{R}^3 = \{(x, y, z, 0) \mid w = 0\}$

A knot in 3-space is trivial if it bounds a disk in $\mathbb{R}^3$, as shown.

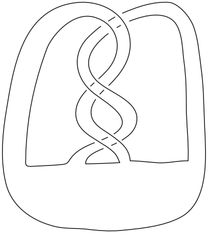


Definition

A slice knot K is one that is the boundary of a disk in $\mathbb{R}^4_+$.

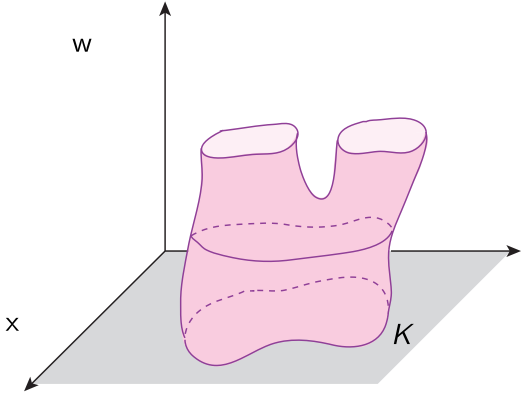


An Example of a Non-Trivial Slice Knot



Let this knot be called K. I need to show you a disk in $\mathbb{R}^4_+$ whose boundary is K. *****

I will show you a disk in 4-space that schematically looks like this:



Pair of Pants

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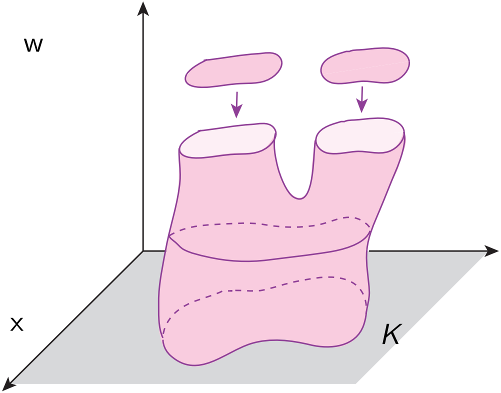

Pair of Pants + Two Little Disks

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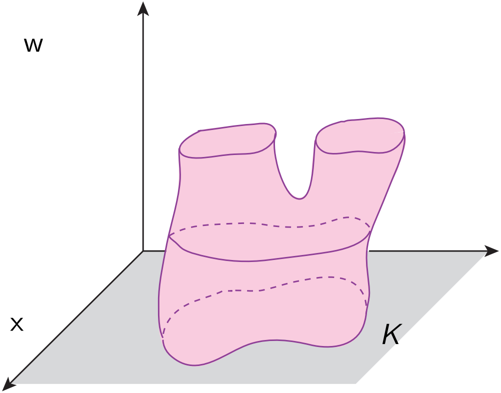
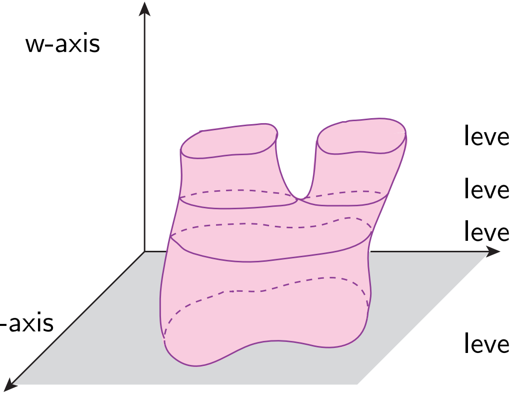


Pair of Pants + Two Little Disks = BIG SLICING DISK

I will prove the existence of this disk by showing you the different 3-dimensional level sets for differing values of w.

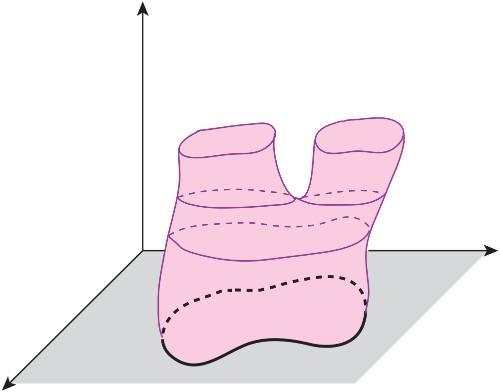


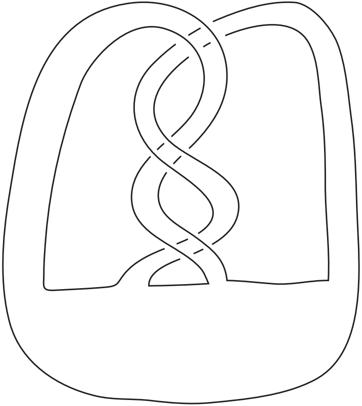
level w = 3 two trivial knots

level w = 2

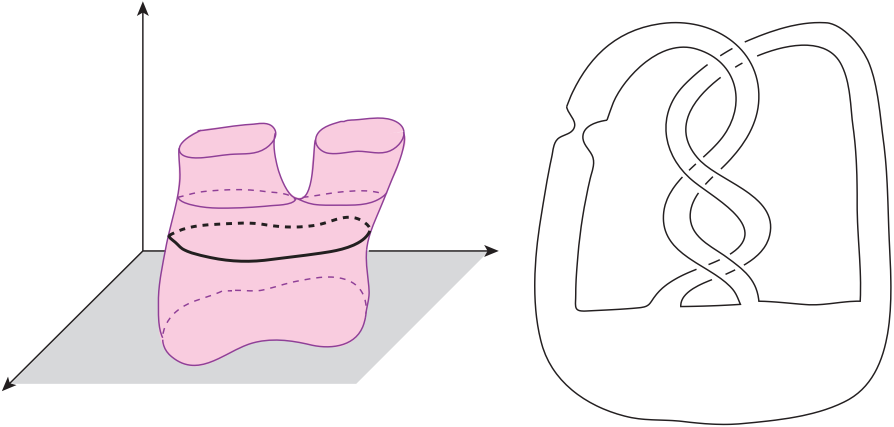
level w = 1

level w = 0 the knot K

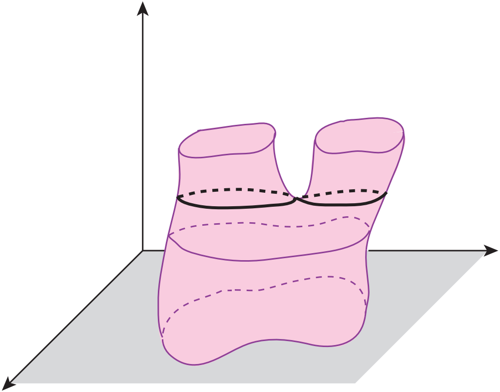


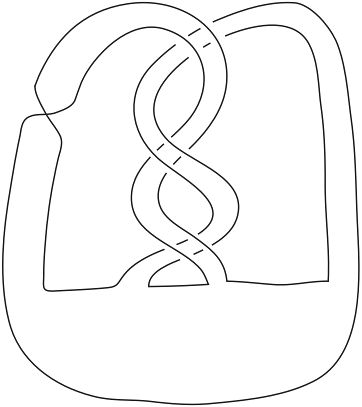


level set $w = 0$

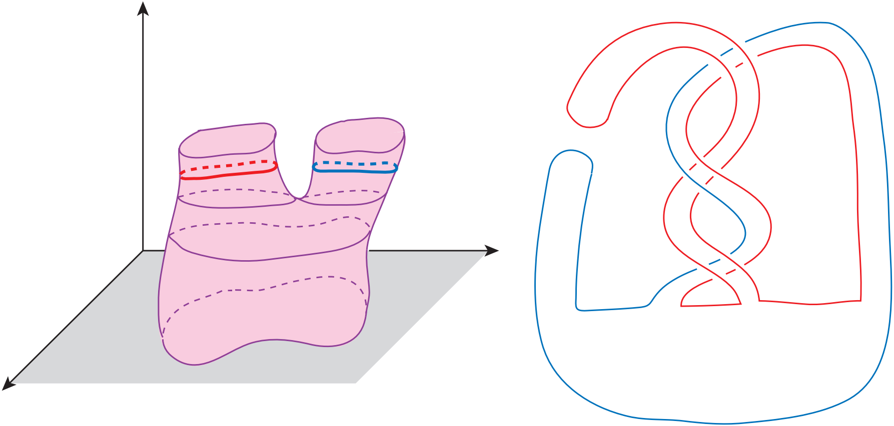


level set $w = 1$

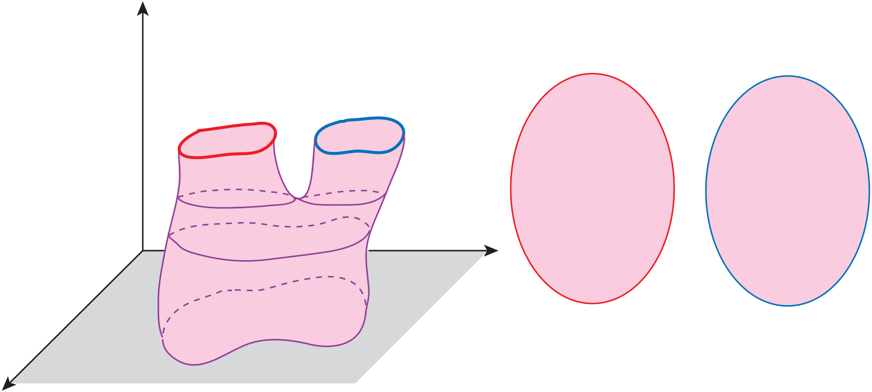




level set $w = 2$



level set $w = 2.5$



level set $w = 3$

It is not known if this knot is a slice knot !!

There is no known algorithm to decide if a knot is a slice knot

Paranormal Knot Tying in 4-Dimensions

Henry Slade a famous "spiritualist" of the nineteenth century; He performed seances for royalty of Europe; his feats were "verified" by some scientists. In order to PROVE that he could communicate with the fourth dimension, he would miraculously cause knots to appear in an unknotted piece of string.

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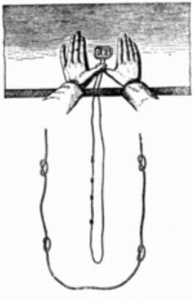
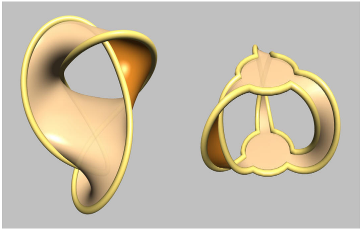


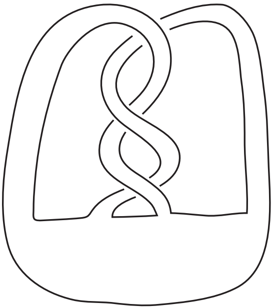
How Can We Prove a Knot is NOT a Slice Knot?

We need a knot invariant that has a special behavior for slice knots. I will describe the signature of a knot.

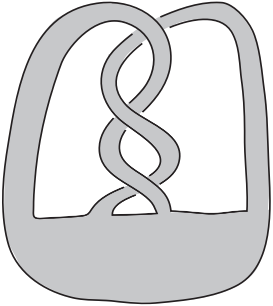
Fact: Every knot is the boundary of a surface in 3-space.



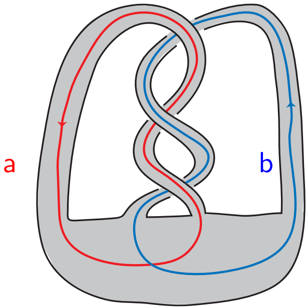
To each knot



To each knot we can associate a surface.



To each knot we can associate a surface. On the surface, there are some special curves that form its spine.



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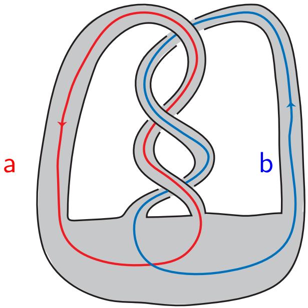

$$V = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

From certain linking numbers of these spinal curves

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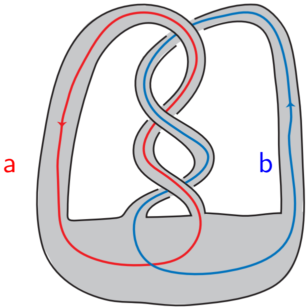

$$V = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

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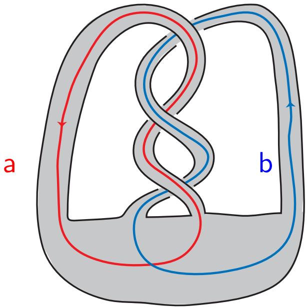

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$$M = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$$

From certain linking numbers of these spinal curves we form a symmetric matrix of integers, M.

Definition

The signature of the knot, $\sigma(K)$, is the number of positive eigenvalues of M minus the number of negative eigenvalues of M.

This turns out to be invariant of all the choices involved. Moreover

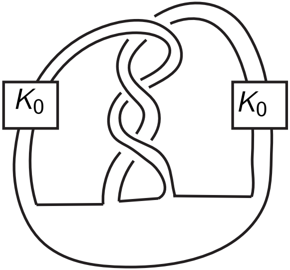
Theorem (Murasugi 1960)

The signature of a slice knot is zero.

Thus the trefoil knot T is NOT a slice knot since $\sigma(T) = 2$.

Is the signature the only obstruction to a knot's being slice? No, let K_0 be the trefoil knot.

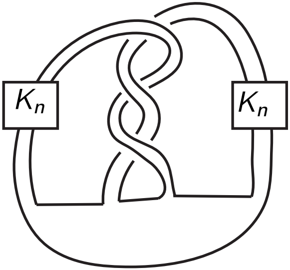
$K_1 =$



This knot has the same associated matrix M as the slice knot, so has $\sigma = 0$, but is not slice due to a higher-order signature invariant, σ_1, (Casson-Gordon 1973).

More generally, consider the family of 'fractal' knots.

K_{n+1} =

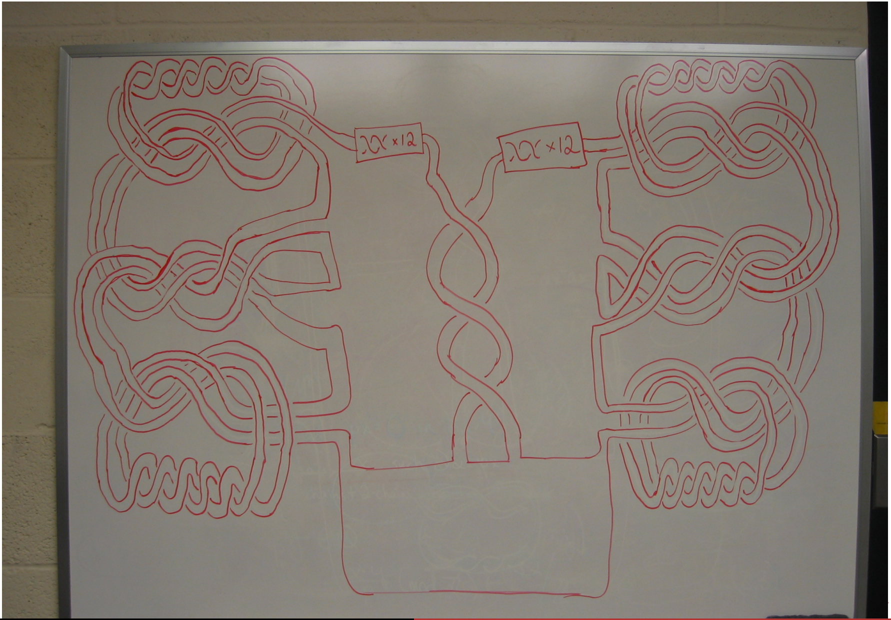


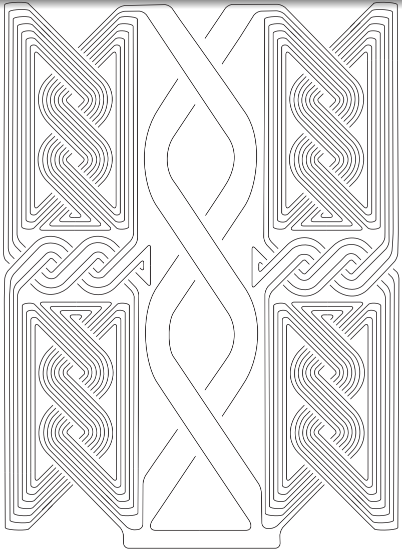
For $n \geq 1$, these have $\sigma = 0$ AND $\sigma_1 = 0$.

Question (1973)(Casson-Gordon (U.Texas), Gilmer (LSU)

Are these knots slice knots?.

$K_2 =$



$K_3 =$ 

The best computer programs to compute knot invariants can handle up to 14 crossings. K_3 has over 2412 crossings.

Theorem (2006)(T.Cochran-Shelly Harvey(Rice)-Constance Leidy (U. Penn.))

If we start with a knot K_0 with $\sigma(K_0) \neq 0$ then none of the knots $K_0, K_1, K_2, ...$ is a slice knot.

We use some "higher-order" signatures, $\sigma_n, n \geq 1$, show that they are knot invariants that should be zero if K_n were a slice knot, and then show that $\sigma_n(K_n)$ is a function of $\sigma(K_0)$.

Higher-Order Signatures

These are defined (loosely) as follows:

- Define more discriminating notions of linking number whose values are not integers but instead are formal linear combinations of elements of the fundamental group $\pi_1(\mathbb{R}^3 - K_n)$ of the knot exterior. These formal linear combinations are like polynomials, they form a noncommutative ring R.

$$1 + x + xyx^{-1}, \;\; xy \neq yx$$

- Associate a self-adjoint matrix M_n to the spinal curves of the surface as before. The entries of this matrix are now not integers but elements of the ring R.

$$M_n = \begin{pmatrix} 2 - x - x^{-1} & 3 - xy^{-1} \\ 3 - xy^{-1} & 2 - y - y^{-1} \end{pmatrix}$$

- map $\pi_1(\mathbb{R}^3 - K_n)$ into $U(\mathcal{H})$, the group of unitary operators on a Hilbert space, allowing us to replace the entries of M_n by complex matrices (infinite size!)
- Use techniques of functional analysis and von Neumann algebras to define a real-valued signature of this matrix, as the dimension of the positive eigenspace minus the dimension of the positive eigenspace(used in previous work of Cochran-Orr-Teichner, 2000).