

SOLUTIONS/ANSWERS TO MATH 222 PRACTICE  
PROBLEMS

3. a.  $\int_C f \cdot d\alpha = \int_0^1 (t^3, t^4, t^3) \cdot (2t, 1, 2t) dt = \int_0^1 5t^4 dt = \boxed{1}$

b. Solve  $f = \nabla g$  to get  $g(x, y, z) = xyz$ . Then  
 $\int_C f \cdot d\alpha = \int_C \nabla g \cdot d\alpha = g(t=1) - g(t=0) = t^2 \cdot t \cdot t^2 \Big|_0^1 = \boxed{1}$

7. By Stoke's Thm,  $\iint_S \operatorname{curl} F \cdot dS = \oint_{\partial S} F \cdot d\gamma$ .



Parametrize  $\partial S$  by  $t \rightarrow (\cos t, \sin t, 0)$  so

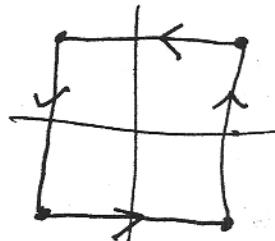
$$\gamma'(t) = (-\sin t, \cos t, 0)$$

$$F(\gamma(t)) = \text{...} (-\sin t, \cos t, 1)$$

Thus  $\oint_{\partial S} F \cdot d\gamma = \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t) dt = \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = \boxed{2\pi}$

8. a.  $\int_C f_1 \cdot \vec{T} ds = \text{...} = \int_{-1}^1 f_1 \cdot \gamma' dt$

$$= \int_{-1}^1 f_1 \cdot d\gamma$$



since  $\vec{T}$  is unit tangent vector  $= \frac{\gamma'}{\|\gamma'\|}$  and

$ds = \|\gamma'\| dt$  is arc-length differential.

By Green's Thm  $= \iint_{\text{square}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dx dy$   $f_1 = (P, Q) = (x+y^2, y+x^2)$

$$= \iint_{-1}^1 (2x+2y) dx dy = \int_{-1}^1 x^2 + 2yx \Big|_{-1}^1 dy$$

$$= \int_{-1}^1 -4y dy = -2y^2 \Big|_{-1}^1 = 0$$

b. This may be easily done without using Green's Thm.  
 However to use Green's Thm, one employs a  
 trick (alternate form of Green's Theorem), that  
 we did not emphasize. It goes as follows:

Suppose you want to calculate  $\oint \mathbf{F} \cdot \vec{n} ds$  around a closed curve parametrized by  $\gamma(t)$ . Here the unit tangent vector is  $\gamma'/\|\gamma'\| = \vec{T}$ . Suppose  $\frac{\gamma'}{\|\gamma'\|} = (A(t), B(t)) = (A, B)$ .

Suppose  $\mathbf{F} = (P, Q)$ . Then  $\vec{n} = (B, -A)$  since  $\vec{n} \cdot \vec{T} = 0$  and  $\vec{n} \times \vec{T} = +\hat{k}$ .



$$\begin{aligned} \text{Thus } \oint \mathbf{F} \cdot \vec{n} ds &= \oint (P, Q) \cdot (B, -A) \|\gamma'\| dt \\ &= \oint PB - AQ \|\gamma'\| dt \end{aligned}$$

Hmm.... how can we use Green's theorem? Consider a vector field  $\vec{G} = (Q, P)$ . Then the above is

$$\oint \vec{G} \cdot \gamma' dt \quad \text{since } PB - AQ = (-Q, P) \cdot (A, B)$$

so by Green's Theorem

$$\boxed{\begin{aligned} \oint \mathbf{F} \cdot \vec{n} ds &= \oint \vec{G} \cdot d\gamma = \iint_{\text{Region}} \frac{\partial P}{\partial x} - \left( -\frac{\partial Q}{\partial y} \right) \\ &= \iint_{\text{Region}} \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} dx dy \end{aligned}}$$

In our particular case  $\vec{F} = f_2 = (y^2 - x^2, x^2 + y^2) = (P, Q)$

$$\text{so } \oint f_2 \cdot \vec{n} ds = \iint_{-1-1}^{1,1} -2x + 2y dx dy = \int_{-1}^1 -x^2 + 2yx \Big|_1^1 dy$$

$$= \int_{-1}^1 4y dy = \boxed{0}$$

9. Let  $B$  be the region bounded by  $x + 2y = \pi$ ,  $x + 2y = 0$ ,  $x - 2y = 0$ , and  $x - 2y = \pi$ . Use change of variable formula to rewrite the following integral in  $(u, v)$  variables so that all the integration limits are constant numbers. Evaluate the integral.

$$\int \int_B (x - 2y)^2 \sin(x + 2y) dA = \int_?^? \int_?^? (?) du dv$$

Answer: Let  $u = x + 2y$  and  $v = x - 2y$ , then  $\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{4}$ . According to the change of variable formula, the new integral is

$$\frac{1}{4} \int_0^\pi \int_0^\pi v^2 \sin u du dv = \frac{1}{4} \left( \frac{v^3}{3} \Big|_0^\pi \right) (-\cos u \Big|_0^\pi) = \frac{\pi^3}{6}$$

4. (a) Let  $\mathbf{f} = (P, Q, R) = (x + z, ax + y, bx + cy - z)$  be a vector field where  $a, b$ , and  $c$  are constants. For what value(s) of  $a, b$ , and  $c$  is  $\mathbf{f}$  a gradient field?  
(b) Find a potential function for the vector field  $\mathbf{g} = (\ln y, x/y, \sin z)$  on the set  $\{(x, y, z) \in \mathbb{R}^3, y > 0\}$ .

Answer:

- (a) We check  $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ ,  $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$ , and  $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$  and solve for  $a = 0, b = 1$  and  $c = 0$ .  
(b)  $F(x, y, z) = x \ln y - \cos z + C$
- b. Convert the following integral into an integral in polar coordinate system and evaluate it.

$$\int_0^2 \int_x^{\sqrt{8-x^2}} (x^2 + y^2)^{\frac{3}{2}} dy dx = \int_?^? \int_?^? (?) dr d\theta$$

Answer:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\sqrt{2}} r^4 dr d\theta = \frac{\pi}{4} \frac{1}{5} r^5 \Big|_0^{2\sqrt{2}} = \frac{32\sqrt{2}\pi}{5}$$

7. Let  $L$  represent the line segment joining points  $(0, 0, a)$  and  $(0, b, 0)$  for  $a > 0$  and  $b > 0$ . Rotate  $L$  about the  $z$  axis to obtain a surface. Sketch the surface. Set up a double integral to compute the area of the surface and evaluate it.

Answer: the surface is a cone and the equation of the surface is given by

$$z = f(x, y) = a - \frac{a}{b} \sqrt{x^2 + y^2} = a - \frac{a}{b} r$$

We consider the surface area integral

$$\int \int_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dx dy$$

where  $D = \{x^2 + y^2 \leq b^2\}$ . Compute  $\frac{\partial f}{\partial x} = -\frac{ax}{br}$  and  $\frac{\partial f}{\partial y} = -\frac{ay}{br}$ . Therefore

$$\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} = \frac{\sqrt{a^2 + b^2}}{b}$$

and the area of the surface is  $b\sqrt{a^2 + b^2}\pi$ .