First-year grad students should do most of these problems and hand them in by next Wednesday. Other students should do the problems that would benefit them. All graduate students should do and hand in number 2.

PROBLEM SET 1: MATH 541 FALL 2009

1. Suppose G is a group and $H \lhd G$ (normal subgroup). Show that each element $g \in G$ yields an isomorphism $f_g : H/[H,H] \to H/[H,H]$ given by $[x] \mapsto [gxg^{-1}]$ where $x \in H$ and $[x]$ is its coset. Of course $H/[H,H]$ is an abelian group (a $\mathbb{Z}$-module) where $m \star [x] = [x^m]$ for all $m \in \mathbb{Z}$ and $x \in H$. Show that this imposes on $H/[H,H]$ the structure of a left module over the ring $\mathbb{Z}[G]$. In the process describe a general formula for :
$$(\sum_{g \in G} n_g g) \star [x] = ?$$
where $n_g \in \mathbb{Z}$, $x \in H$ and the sum is finite. Now show that H acts trivially, i.e. $h \star [x] = [x]$ so that this can be viewed as a module over $\mathbb{Z}[G/H]$. (Note: if one uses $[x] \mapsto [g^{-1}xg]$ one gets a right module!!)
2. $G = \langle x, y, z \mid xyx = yxy, xzx = zxz \rangle$ is a presentation of the group $\pi_1(E(K))$ where K is the granny knot, where each of x, y, z represent positively-oriented meridians. Using the method of class, calculate the Alexander module of K. Is it cyclic?
3. Let X be the wedge of 2-circles labeled x and y, so $\pi_1(X) \cong F\langle x, y \rangle$. Choose generators for its obvious cellular chain complex. What is the isomorphism between its *abstract* p^{th} chain group $\mathbb{Z}^r$ and its *actual* chain group $C_p(X)$, that is free on the cells? Consider a regular Γ-cover, $\tilde{X}$, where $\Gamma = \mathbb{Z} \times \mathbb{Z}$-cover (so $\pi_1(\tilde{X}) \cong [F, F]$). Draw a picture of this covering space. Lift this cell structure to a cell structure on $\tilde{X}$. There is a choice in how one identifies the group of deck translations with Γ. This leads to either a cellular left or right Γ action on $\tilde{X}$. What is the rank, as free $\mathbb{Z}\Gamma \cong \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$-modules, of the cellular chain groups $C_p(\tilde{X})$? Choose a $\mathbb{Z}\Gamma$-basis? What is the isomorphism between $C_p(\tilde{X})$ and $\mathbb{Z}\Gamma^r$? Construct and understand commutative diagrams where π is induced from the covering map:
$$\begin{array}{ccc} C_p(\tilde{X}) & \xrightarrow{\tilde{\partial}} & C_{p-1}(\tilde{X}) \\ \pi \downarrow & & \pi \downarrow \\ C_p(X) & \xrightarrow{\partial} & C_{p-1}(X) \end{array}$$
Show that after identifying $C_p(X)$ with $\mathbb{Z}^r$ and $C_p(\tilde{X})$ with $\mathbb{Z}\Gamma^r$ that the map π becomes a map $(\mathbb{Z}\Gamma \to \mathbb{Z})^r$ that is the augmentation (set $x = y = 1$) or can be seen as the composition $\mathbb{Z}\Gamma \to \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma} \mathbb{Z} \cong \mathbb{Z}$ (where the $\mathbb{Z}$ in the tensor product is the trivial $\mathbb{Z}\Gamma$-module).
4. Suppose that you are given that the group of any knot has a presentation with n generators and $n-1$ relations. Briefly argue, as in [2.] that the Alexander module of

any knot has a square presentation matrix (has a presentation (as a module) with an equal number of generators and relations).