Second Midterm Exam
Math 212 Fall 2010

Instructions: This is a **90 minute** exam. You should work alone, without access to any book or notes. No calculators are allowed. Do not discuss this exam with anyone other than your instructor. When you have completed the exam, write out and sign the Honor Code pledge on the front.

The exam consists of 6 questions. You must show all of your work on each problem to receive full credit, and **be sure to clearly indicate your final answer** to each question.

1. [15 Points] Let $\mathbf{f}(x, y, z) = (xz, \sin(xy))$ and $\mathbf{g}(u, v) = (u^2, e^{uv}, \ln v)$

   (a) Compute the derivatives $\mathbf{Df}$ and $\mathbf{Dg}$.

Solution:

$$\mathbf{Df} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix} = \boxed{\begin{bmatrix} z & 0 & x \\ y\cos xy & x\cos xy & 0 \end{bmatrix}},$$

$$\mathbf{Dg} = \begin{bmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \\ \frac{\partial g_3}{\partial u} & \frac{\partial g_3}{\partial v} \end{bmatrix} = \boxed{\begin{bmatrix} 2u & 0 \\ ve^{uv} & ue^{uv} \\ 0 & 1/v \end{bmatrix}}.$$

   (b) Using the multivariable chain rule, compute the derivative $\mathbf{D}(\mathbf{f} \circ \mathbf{g})$. (Your final answer should be a matrix whose entries are functions of u and v.)

Solution:

$$\begin{aligned} \mathbf{D}(\mathbf{f} \circ \mathbf{g}) &= (\mathbf{Df})(\mathbf{Dg}) \\ &= \begin{bmatrix} z & 0 & x \\ y\cos xy & x\cos xy & 0 \end{bmatrix} \begin{bmatrix} 2u & 0 \\ ve^{uv} & ue^{uv} \\ 0 & 1/v \end{bmatrix} \\ &= \begin{bmatrix} z(2u) & x/v \\ (y\cos xy)(2u) + (x\cos xy)(ve^{uv}) & (x\cos xy)(ue^{uv}) \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 2u\ln v & u^2/v \\ 2ue^{uv}\cos(u^2e^{uv}) + u^2ve^{uv}\cos(u^2e^{uv}) & u^3e^{uv}\cos(u^2e^{uv}) \end{bmatrix}} \end{aligned}$$

(c) Using the multivariable chain rule, compute the derivative $\mathbf{D}(\mathbf{g} \circ \mathbf{f})$. (Your final answer should be a matrix whose entries are functions of x, y, and z.)

Solution:

$$\begin{aligned}
\mathbf{D}(\mathbf{g} \circ \mathbf{f}) &= (\mathbf{Dg})(\mathbf{Df}) \\
&= \begin{bmatrix} 2u & 0 \\ ve^{uv} & ue^{uv} \\ 0 & 1/v \end{bmatrix} \begin{bmatrix} z & 0 & x \\ y\cos xy & x\cos xy & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2uz & 0 & 2ux \\ ve^{uv}z + ue^{uv}y\cos cy & ue^{uv}x\cos xy & ve^{uv}x \\ \frac{y\cos xy}{v} & \frac{x\cos xy}{v} & 0 \end{bmatrix} \\
&= \boxed{\begin{bmatrix} 2xz^2 & 0 & 2x^2z \\ (\sin xy)e^{xz\sin xy}z + xze^{xz\sin xy}y\cos xy & xze^{xz\sin xy}x\cos xy & (\sin xy)e^{xz\sin xy}x \\ \frac{y\cos xy}{\sin xy} & \frac{x\cos xy}{\sin xy} & 0 \end{bmatrix}}
\end{aligned}$$

2. [20 Points] Let $f(x, y) = 2x^2 + y^4 - 4xy$.

(a) Find the critical points of f.

Solution: We set

$$\mathbf{0} = \nabla f = (4x - 4y, 4y^3 - 4x),$$

so $x = y$ and $4y^3 - 4y = 4y(y-1)(y+1) = 0$, and we get three critical points, $\boxed{(-1,-1), (0,0), (1,1)}$.

(b) Compute the Hessian matrix of f.

$$Hf = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \boxed{\begin{bmatrix} 4 & -4 \\ -4 & 12y^2 \end{bmatrix}}$$

(c) Using the multivariable second derivative test, classify the critical points of f.

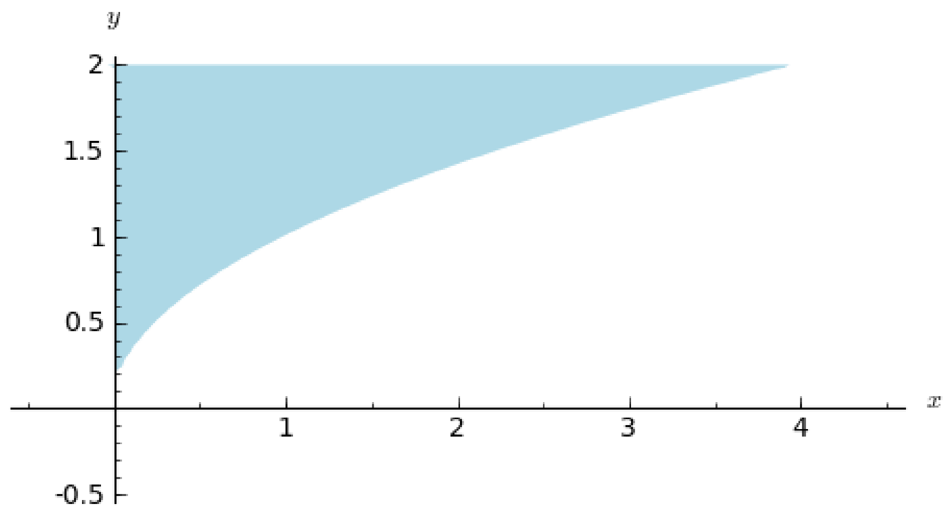
Solution: We have $\frac{\partial^2 f}{\partial x^2} = 4 > 0$ everywhere, and the determinant of Hf is $16(3y^2 - 1)$, so by the second derivative test, $\boxed{(1,1) \text{ and } (-1,-1) \text{ are local minima and } (0,0) \text{ is a saddle point.}}$

(d) Starting from the point $(3, 2)$, in which direction does the function f decrease most rapidly?

Solution: The negative of the gradient points in the direction of greatest decrease: $-\nabla f(3,2) = (-4, -20)$, so the direction is $\boxed{\frac{1}{\sqrt{4^2 + 20^2}}(-4, -20)}$.

3. [15 Points] Let $I = \int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5+1} dy\ dx$

(a) Write I as the double integral over a region D in the plane. Sketch the region D.



(b) Reverse the order of integration, i.e. rewrite I as an iterated integral "$dx\ dy$".

Solution:

$$\boxed{\int_0^2 \int_0^{y^2} \frac{x}{y^5+1} dx\ dy}$$

(c) Compute I.

Solution:

$$\begin{aligned}
\int_0^2 \int_0^{y^2} \frac{x}{y^5+1} dx\ dy &= \int_0^2 \left(\frac{\frac{1}{2}x^2}{y^5+1} \Bigg|_{x=0}^{y^2} \right) dy \\
&= \frac{1}{2} \int_0^2 \frac{y^4}{y^5+1} dy \\
&= \frac{1}{10} \int_1^{33} \frac{1}{u} du \qquad \text{where } u = y^5+1,\ du = 5ydy \\
&= \boxed{\frac{1}{10} \ln 33}
\end{aligned}$$

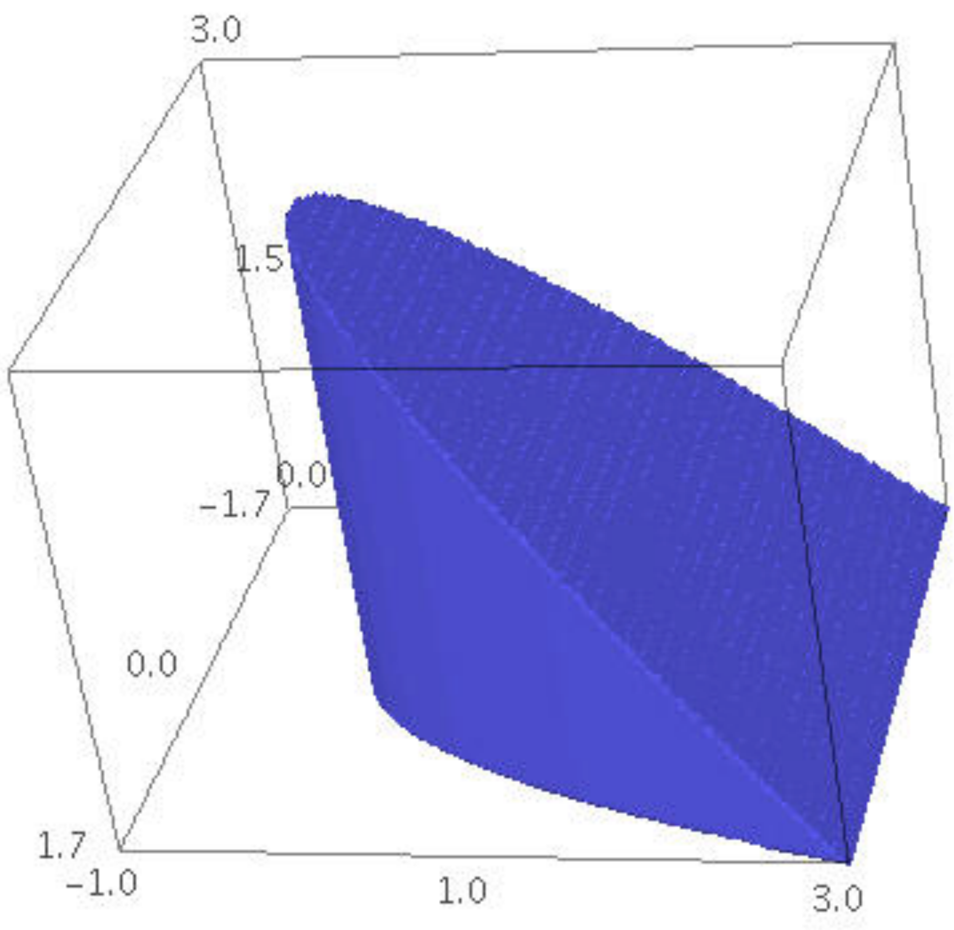
4. [15 Points] Find the dimensions of a right circular cylinder of the maximum possible volume that can be inscribed in a sphere of radius 16. (Make sure to clearly explain what your variables mean.)

Solution: Let the sphere by given by $x^2+y^2+z^2 = 16^2$. By symmetry, we may as well assume that the axis of the cylinder is the z-axis. Let (x, y, z) be a point on both the surface of the sphere and the surface of the cylinder. The volume of the inscribed cylinder is then $(\pi r^2)(2z)$, where $r = \sqrt{x^2 + y^2}$ is the radius and $2z$ is the height. Thus, we are trying to maximize the function $f(r, z) = r^2 z$, subject to the constraint $g(r, z) = r^2 + z^2 = 16^2$.

To do so, we solve the Lagrange multiplier equation $\nabla f = \lambda \nabla g$, where $\nabla f = (2rz, r^2)$ and $\nabla g = (2r, 2z)$. This gives $\frac{2rz}{2r} = \frac{r^2}{2z}$, so $r^2 = 2z^2$. Substituting this into the constraint equation yields $3z^2 = 16^2$, so $z = \frac{16}{\sqrt{3}}$, and $r = \sqrt{2}z = \frac{16\sqrt{2}}{\sqrt{3}}$. Thus the largest volume cylinder has $\boxed{\text{height } \frac{32}{\sqrt{3}}}$ and $\boxed{\text{radius } \frac{16\sqrt{2}}{\sqrt{3}}}$.

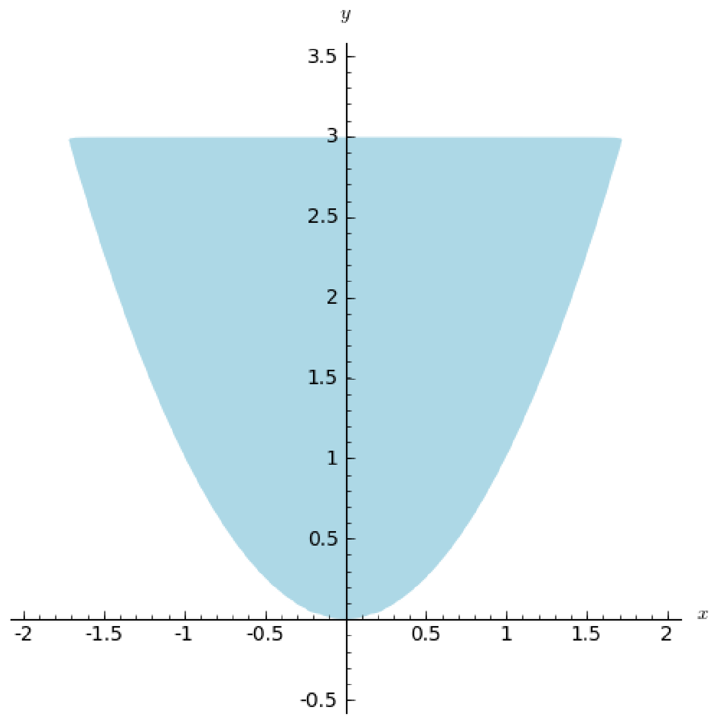
5. [20 Points] Let W be the region in $\mathbb{R}^3$ defined by the inequalities $y \geq x^2$, $z \leq 3 - y$, and $z \geq 0$. Let $f(x, y, z)$ be a continuous function on W.

   (a) Sketch the region W.



(b) Write $\iiint_W f(x,y,z)\, dV$ as an iterated integral in the order "$dz\ dy\ dx$".

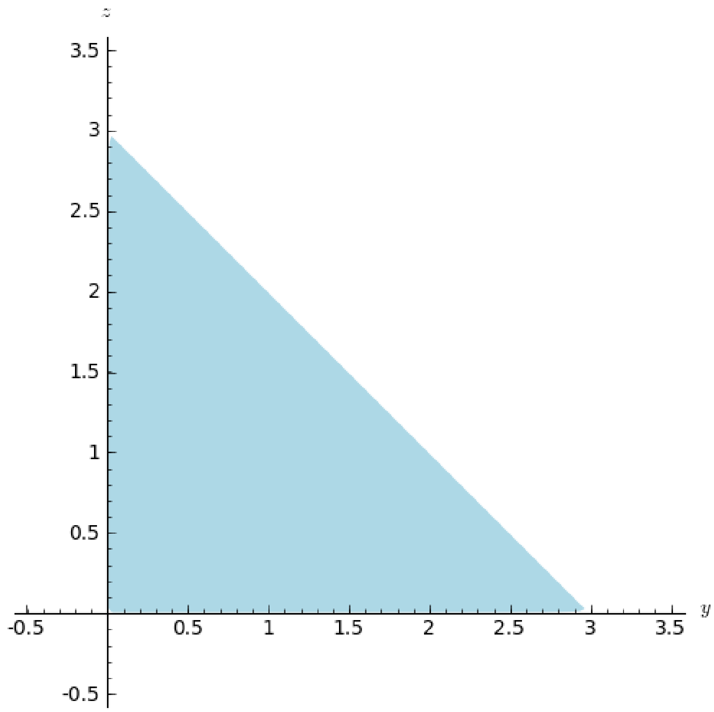
Solution:



$$\iiint_W f(x,y,z)\, dV = \boxed{\int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2}^{3} \int_{0}^{3-y} f(x,y,z) dz\ dy\ dx}.$$

(c) Write $\iiint_W f(x,y,z)\, dV$ as an iterated integral in the order "$dx\ dy\ dz$".

Solution:



$$\iiint_W f(x,y,z)\ dV = \boxed{\int_0^3 \int_0^{3-z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z) dx\ dy\ dz}.$$

(d) Evaluate $\iiint_W y\ dV$

Solution:

$$\begin{aligned}
\int_0^3 \int_0^{3-z} \int_{-\sqrt{y}}^{\sqrt{y}} y\ dx\ dy\ dz &= \int_0^3 \int_0^{3-z} y \int_{-\sqrt{y}}^{\sqrt{y}} 1\ dx\ dy\ dz \\
&= \int_0^3 \int_0^{3-z} 2y\sqrt{y}\ dy\ dz \\
&= \int_0^3 \left(\tfrac{4}{5} y^{5/2} \Big|_{y=0}^{3-z} \right) dz \\
&= \int_0^3 \tfrac{4}{5}(3-z)^{5/2} dz \\
&= -\tfrac{8}{35}(3-z)^{7/2} \Big|_{z=0}^{3} \\
&= \boxed{\tfrac{8}{35} 3^{7/2} = \tfrac{216}{35}\sqrt{3}}.
\end{aligned}$$

6. [15 Points] Let $f(x,y) = x^2 + y^2 - 2x - 4y$. Find the absolute maximum and minimum of f, subject to the constraints $x^2 + y^2 \le 2$ and $y \ge 0$.

Solution: First, we set $\mathbf{0} = \nabla f = (2x - 2, 2y - 4)$, which tells us that $(1, 2)$ is the only critical point of f, but it is outside the region defined by the constraints. Thus the minimum and maximum of f will be on the boundary. The curve bounding the region has two smooth components, meeting non-smoothly at the points $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$.

The first component is given implicitly by the equation $g_1(x,y) = x^2+y^2 = 2$. Thus we set $\nabla f = \lambda \nabla g_1$, giving $(2x - 2, 2y - 4) = \lambda(2x, 2y)$. After checking that there are no solutions with $x = 0$ or $y = 0$, we get $\frac{2x-2}{2x} = \frac{2y-4}{2y}$, so $\frac{1}{x} = \frac{2}{y}$, and $y = 2x$. The line $y = 2x$ meets the circle $x^2 + y^2 = 2$ in two points, $\pm\left(\sqrt{\frac{2}{5}}, 2\sqrt{\frac{2}{5}}\right)$, but only $\left(\sqrt{\frac{2}{5}}, 2\sqrt{\frac{2}{5}}\right)$ satisfies $y \ge 0$.

The second boundary component is given implicitly by the equation $g_2(x,y) = y = 0$. We set $\nabla f = \lambda \nabla g_2$, giving $(2x - 2, 2y - 4) = \lambda(0, 1)$, so that $x = 1$, giving us the point $(1, 0)$.

We now have four candidates for extrema of f on our region, the two points $(\pm\sqrt{2}, 0)$ where the boundary components meet, and the two constrained critical points from the Lagrange multiplier calculations, $\left(\sqrt{\frac{2}{5}}, 2\sqrt{\frac{2}{5}}\right)$ and $(1, 0)$. To find the absolute maximum and minimum, we just evaluate f at these four points:

$$f\left(\sqrt{2}, 0\right) = 2 - 2\sqrt{2}$$

$$f\left(-\sqrt{2}, 0\right) = 2 + 2\sqrt{2} \quad \boxed{\text{absolute maximum}}$$

$$f\left(\sqrt{\frac{2}{5}}, 2\sqrt{\frac{2}{5}}\right) = 2 - 2\sqrt{10} \quad \boxed{\text{absolute minimum}}$$

$$f(1, 0) = -1$$