LECTURE 6

Grassmannians and Related Varieties

Example 6.6. Grassmannians

Grassmannians are fundamental objects in algebraic geometry: they are simultaneously objects of interest in their own right and basic tools in the construction and study of other varieties. We will be dealing with Grassmannians constantly in the course of this book; here we introduce them and mention a few of their basic properties.

By way of notation, we let $G(k, n)$ denote the set of k-dimensional linear subspaces of the vector space K^n; if we want to talk about the set of k-planes in an abstract vector space V without making a choice of basis for V we also write $G(k, V)$. Of course, a k-dimensional subspace of a vector space K^n is the same thing as a $(k-1)$-plane in the corresponding projective space $\mathbb{P}^{n-1}$, so that we can think of $G(k, n)$ as the set of such $(k-1)$-planes; when we want to think of the Grassmannian this way we will write it $\mathbb{G}(k-1, n-1)$ or $\mathbb{G}(k-1, \mathbb{P}V)$.

In most contexts, Grassmannians are defined initially via coordinate patches or as a quotient of groups; it is then observed that they may be embedded in a projective space. Since our main objects of interest here are projective varieties, we will do it differently, describing the Grassmannian first as a subset of projective space. This is straightforward: if $W \subset V$ is the k-dimensional linear subspace spanned by vectors $v_1, \ldots, v_k$, we can associate to W the multivector

$$\lambda = v_1 \wedge \cdots \wedge v_k \in \bigwedge^k(V).$$

λ is determined up to scalars by W: if we chose a different basis, the corresponding vector λ would simply be multiplied by the determinant of the change of basis matrix. We thus have a well-defined map of sets

$$\psi: G(k, V) \to \mathbb{P}(\wedge^k V).$$

In fact, this is an inclusion: for any $[\omega] = \psi(W)$ in the image, we can recover the corresponding subspace W as the space of vectors $v \in V$ such that $v \wedge \omega = 0 \in \wedge^{k+1} V$. This inclusion is called the *Plücker embedding* of $G(k, V)$.

The homogeneous coordinates on $\mathbb{P}^N = \mathbb{P}(\wedge^k V)$ are called *Plücker coordinates* on $G(k, V)$. Explicitly, if we choose an identification $V \cong K^n$ we can represent the plane W by the $k \times n$ matrix M_W whose rows are the vectors v_i; the matrix M_W is determined up to multiplication on the left by an invertible $k \times k$ matrix. The Plücker coordinates are then just the maximal minors of the matrix M_W.

We have described the Grassmannian $G(k, V)$ as a subset of $\mathbb{P}(\wedge^k V)$; we should now check that it is indeed a subvariety. This amounts to characterizing the subset of *totally decomposable* vectors $\omega \in \wedge^k V$, that is, products $\omega = v_1 \wedge \cdots \wedge v_k$ of linear factors. We begin with a basic observation: given a multivector $\omega \in \wedge^k V$ and a vector $v \in V$, the vector v will divide ω—that is, ω will be expressible as $v \wedge \varphi$ for some $\varphi \in \wedge^{k-1} V$—if and only if the wedge product $\omega \wedge v = 0$. Moreover, a multivector ω will be totally decomposable if and only if the space of vectors v dividing it is k-dimensional. Thus, $[\omega]$ will lie in the Grassmannian if and only if the rank of the map

$$\varphi(\omega): V \to \wedge^{k+1} V$$
$$: v \mapsto \omega \wedge v$$

is $n - k$. Since the rank of $\varphi(\omega)$ is never strictly less than $n - k$, we can say

$$[\omega] \in G(k, V) \Leftrightarrow \operatorname{rank}(\varphi(\omega)) \leq n - k.$$

Now, the map $\wedge^k V \to \operatorname{Hom}(V, \wedge^{k+1} V)$ sending ω to $\varphi(\omega)$ is linear, that is, the entries of the matrix $\varphi(\omega) \in \operatorname{Hom}(V, \wedge^{k+1} V)$ are homogeneous coordinates on $\mathbb{P}(\wedge^k V)$; we can say that $G(k, V) \subset \mathbb{P}(\wedge^k V)$ is the subvariety defined by the vanishing of the $(n - k + 1) \times (n - k + 1)$ minors of this matrix.

This is the simplest way to see that $G(k, V)$ is a subvariety of $\mathbb{P}(\wedge^k V)$, but the polynomials we get in this way are far from the simplest possible; in particular, they do not generate the homogeneous ideal of $G(k, V)$. To find the actual generators of the ideal, we need to invoke also the natural identification of $\wedge^k V$ with the exterior power $\wedge^{n-k} V^*$ of the dual space V^* (this is natural only up to scalars, but that's okay for our purposes). In particular, an element $\omega \in \wedge^k V$ corresponding to $\omega^* \in \wedge^{n-k} V^*$ gives rise in this way to a map

$$\psi(\omega): V^* \to \wedge^{n-k+1} V^*$$
$$: v^* \mapsto v^* \wedge \omega^*;$$

by the same argument ω will be totally decomposable if and only if the map $\psi(\omega)$ has rank at most k. What's more, in case ω is totally decomposable, the kernel of the map $\varphi(\omega)$—the subspace W itself—will be exactly the annihilator of the kernel of $\psi(\omega)$; equivalently, the images of the transpose maps

$${}^t\varphi(\omega): \wedge^{k+1} V^* \to V^*$$

and

$$ {}^t\psi(\omega)\colon \textstyle\bigwedge^{n-k+1} V \to V $$

annihilate each other. In sum, then, we see that $[\omega] \in G(k, V)$ if and only if for every pair $\alpha \in \bigwedge^{k+1} V^*$ and $\beta \in \bigwedge^{n-k+1} V$, the contraction

$$ \Xi_{\alpha,\beta}(\omega) = \langle {}^t\varphi(\omega)(\alpha), {}^t\psi(\omega)(\beta) \rangle = 0. $$

The $\Xi_{\alpha,\beta}$ are thus quadratic polynomials whose common zero locus is the Grassmannian $G(k, V)$. They are called the *Plücker relations*, and they do in fact generate the homogeneous ideal of $G(k, V)$, though we will not prove that here.

Exercise 6.2. In the special case $k = 2$, assuming $\mathrm{char}(K) \neq 2$ show directly that a vector $\omega \in \bigwedge^2 V$ is decomposable if and only if $\omega \wedge \omega = 0$ and hence that the Grassmannian $G(2, V) \subset \mathbb{P}(\bigwedge^2 V)$ is a variety cut out by quadrics. (In fact, the equation $\omega \wedge \omega = 0$ represents $\binom{n}{4}$ independent quadratic relations, which are exactly the span of the Plücker relations.)

Observe in particular that the first nontrivial Grassmannian—the first one that is not a projective space—is $G(2, 4)$, and this sits as a quadric hypersurface in $\mathbb{P}(\bigwedge^2 K^4) \cong \mathbb{P}^5$.

We can get another picture of the Grassmannian by looking at certain special affine open subsets. To describe these first intrinsically, let $\Gamma \subset V$ be a subspace of dimension $n - k$, corresponding to a multivector $\omega \in \bigwedge^{n-k} V = \bigwedge^k V^*$. We can think of ω as a homogeneous linear form on $\mathbb{P}(\bigwedge^k V)$; let $U \subset \mathbb{P}(\bigwedge^k V)$ be the affine open subset where $\omega \neq 0$. Then the intersection of $G(k, V)$ with U is just the set of k-dimensional subspaces $\Lambda \subset V$ complementary to Γ. Any such subspace can be viewed as the graph of a map from V/Γ to Γ and vice versa, so that we have an identification

$$ G(k, V) \cap U \cong \mathrm{Hom}(V/\Gamma, \Gamma) \cong K^{k(n-k)}. $$

To see this in coordinates, identify V with K^n and say the subspace Γ is spanned by the last $n - k$ basis vectors $e_{k+1}, \ldots, e_n \in K^n$. Then $U \cap G(k, n)$ is the subset of spaces Λ such that the $k \times n$ matrix M_Λ whose first $k \times k$ minor is nonzero. It follows that any $\Lambda \in G(k, V) \cap U$ is represented as the row space of a unique matrix of the form

$$ \begin{bmatrix} 1 & 0 & 0 & . & . & 0 & a_{1,1} & a_{1,2} & . & . & . & a_{1,n-k} \\ 0 & 1 & 0 & . & . & 0 & a_{2,1} & a_{2,2} & . & . & . & a_{2,n-k} \\ \vdots & & & & & & & & & & & \\ 0 & 0 & . & . & 0 & 1 & a_{k,1} & a_{k,2} & . & . & . & a_{k,n-k} \end{bmatrix} $$

and vice versa. The entries $a_{i,j}$ of this matrix then give the bijection of $U \cap G(k, V)$ with $K^{k(n-k)}$.

Note that the affine coordinates on the affine open subset of $G(k, V)$ are just the $k \times k$ minors of this matrix, which is to say the minors of all sizes of the $(n - k) \times k$ matrix $(a_{i,j})$. In particular, expansion of any of these determinants along any row or column yields a quadratic relation among these minors; thus, for example,

$$a_{1,1} \cdot a_{2,2} - a_{1,2} \cdot a_{2,1} = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}$$

is a relation among the affine coordinates on $\mathbb{P}(\wedge^k K^n)$ restricted to $G(k, n)$. In this way, we can write down all the Plücker relations explicitly in coordinates.

There is, finally, another way to describe the affine coordinates on the open subset $U \cap G(k, n)$ of k-planes Λ complementary to a given $(n - k)$-plane Γ: we take vectors $v_1, \ldots, v_k \in K^n$ that, together with Γ, span all of K^n, and set

$$v_i(\Lambda) = \Lambda \cap (\Gamma + v_i).$$

The vectors $v_i(\Lambda)$ then give a basis for Λ, for all $\Lambda \in U$; and the k-tuple of vectors $v_i(\Lambda) - v_i \in \Gamma$ gives an identification of $U \cap G(k, n)$ with Γ^k.

Subvarieties of Grassmannians

To begin with, an inclusion of vector spaces $W \hookrightarrow V$ induces an inclusion of Grassmannians $G(k, W) \hookrightarrow G(k, V)$; likewise, a quotient map $V \to V/U$ to the quotient of V by an l-dimensional subspace U induces an inclusion $G(k - l, V/U) \hookrightarrow G(k, V)$. More generally, if $U \subset W \subset V$, we have an inclusion $G(k - l, W/U) \hookrightarrow G(k, V)$. The images of such maps are called sub-Grassmannians and are subvarieties of $G(k, V)$ (in terms of the Plücker embedding $G(k, V) \hookrightarrow \mathbb{P}(\wedge^k V)$, they are just the intersection of $G(k, V)$ with linear subspaces in $\mathbb{P}(\wedge^k V)$, as we will see in the following paragraph).

If we view the Grassmannian as the set of linear subspaces in a projective space $\mathbb{P}V$, the sub-Grassmannians are just the subsets of planes contained in a fixed subspace and/or containing a fixed subspace. We can also consider the subset $\Sigma(\Lambda) \subset \mathbb{G}(k, \mathbb{P}V)$ of k-planes that meet a given m-dimensional linear subspace $\Lambda \subset \mathbb{P}V$, or more generally the subset $\Sigma_l(\Lambda)$ of k-planes that meet a given Λ in a subspace of dimension of at least l. These are again subvarieties of the Grassmannian; $\Sigma_l(\Lambda)$ may be described as the locus

$$\Sigma_l(\Lambda) = \{[\omega]: \omega \wedge v_1 \wedge \cdots \wedge v_{m-l+1} = 0 \qquad \forall v_1, \ldots, v_{m-l+1} \in \Lambda\}$$

from which we see in particular that it, like the sub-Grassmannians, is the intersection of the Grassmannian with a linear subspace of $\mathbb{P}(\wedge^k V)$. These are in turn special cases of a class of subvarieties of $\mathbb{G}(k, \mathbb{P}V)$ called *Schubert cycles*, about which we will write more later.

There are also analogs for Grassmannians of projection maps on projective space. Specifically, suppose $W \subset V$ is a subspace of codimension l in the n-dimensional vector space V. For $k \leq l$, we have a map $\pi: U \to G(k, V/W)$ defined on the open set $U \subset G(k, V)$ of k-planes meeting W only in (0) simply by taking the image; for $k \geq l$ we have a map $\eta: U' \to G(k - l, W)$ defined on the open subset $U' \subset G(k, V)$ of planes transverse to W by taking the intersection. Note that both these maps may be realized, via the Plücker embeddings of both target and domain, by a linear projection on the ambient projective space $\mathbb{P}(\wedge^k V)$—for example, the map π is the restriction to $G(k, V)$ of the linear map $\mathbb{P}(\wedge^k V) \to \mathbb{P}(\wedge^k(V/W))$ induced by the projection $V \to V/W$.

Example 6.3. The Grassmannian $\mathbb{G}(1, 3)$

The next few exercises deal specifically with the geometry of the Grassmannian $\mathbb{G} = \mathbb{G}(1, 3)$ parametrizing lines in $\mathbb{P}^3$, which as we have seen may be realized (via the Plücker embedding) as a quadric hypersurface in $\mathbb{P}^5$.

Exercise 6.4. For any point $p \in \mathbb{P}^3$ and plane $H \subset \mathbb{P}^3$ containing p, let $\Sigma_{p,H} \subset \mathbb{G}$ be the locus of lines in $\mathbb{P}^3$ passing through p and lying in H. Show that under the Plücker embedding $\mathbb{G} \to \mathbb{P}^5$, $\Sigma_{p,H}$ is carried to a line, and that conversely every line in $\mathbb{P}^5$ lying on $\mathbb{G}$ is of the form $\Sigma_{p,H}$ for some p and H.

Exercise 6.5. For any point $p \in \mathbb{P}^3$, let $\Sigma_p \subset \mathbb{G}$ be the locus of lines in $\mathbb{P}^3$ passing through p; for any plane $H \subset \mathbb{P}^3$, let $\Sigma_H \subset \mathbb{G}$ be the locus of lines in $\mathbb{P}^3$ lying in H. Show that under the Plücker embedding, both Σ_p and Σ_H are carried into two-planes in $\mathbb{P}^5$, and that conversely any two-plane $\Lambda \cong \mathbb{P}^2 \subset G \subset \mathbb{P}^5$ is either equal to Σ_p for some p or to Σ_H for some H.

Exercise 6.6. Let $l_1, l_2 \subset \mathbb{P}^3$ be skew lines. Show that the set $Q \subset \mathbb{G}$ of lines in $\mathbb{P}^3$ meeting both is the intersection of $\mathbb{G}$ with a three-plane $\mathbb{P}^3 \subset \mathbb{P}^5$, and so is a quadric surface. Deduce yet again that $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. What happens if l_1 and l_2 meet?

Exercise 6.7. Now let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Show that the two families of lines on Q correspond to plane conic curves on $\mathbb{G}$ lying in complementary two-planes $\Lambda_1, \Lambda_2 \subset \mathbb{P}^5$. Show that, conversely, the lines in $\mathbb{P}^3$ corresponding to a plane conic curve $C \subset \mathbb{G}$ sweep out a smooth quadric surface if and only if the plane Λ spanned by C is not contained in $\mathbb{G}$. What happens to this correspondence if either the quadric becomes a cone or the plane Λ lies in G?

The next exercise is a direct generalization of the preceding one; it deals with Segre varieties other than $\mathbb{P}^1 \times \mathbb{P}^1$.

Exercise 6.8. Let $\Sigma_{1,k} \cong \mathbb{P}^1 \times \mathbb{P}^k \subset \mathbb{P}^{2k+1}$ be the Segre variety, and for each $p \in \mathbb{P}^1$ let Λ_p be the fiber of $\Sigma_{1,k}$ over p. We have seen that Λ_p is a k-plane in $\mathbb{P}^{2k+1}$; show that the assignment $p \mapsto \Lambda_p$ defines a regular map of $\mathbb{P}^1$ to the Grassmannian $\mathbb{G}(k, 2k + 1)$ whose image is a rational normal curve lying in a $(k + 1)$-plane in $\mathbb{P}(\wedge^{k+1} K^{2k+2})$.