10.4 TENSOR PRODUCTS OF MODULES

In this section we study the tensor product of two modules \( M \) and \( N \) over a ring (not necessarily commutative) containing 1. Formation of the tensor product is a general construction that, loosely speaking, enables one to form another module in which one can take "products" \( mn \) of elements \( m \in M \) and \( n \in N \). The general construction involves various left- and right- module actions, and it is instructive, by way of motivation, to first consider an important special case: the question of "extending scalars" or "changing the base."

Suppose that the ring \( R \) is a subring of the ring \( S \). Throughout this section, we always assume that \( 1_R = 1_S \) (this ensures that \( S \) is a unital \( R \)-module).

If \( N \) is a left \( S \)-module, then \( N \) can also be naturally considered as a left \( R \)-module since the elements of \( R \) (being elements of \( S \)) act on \( N \) by assumption. The \( S \)-module axioms for \( N \) include the relations

\[
(s_1 + s_2)n = s_1 n + s_2 n \quad \text{and} \quad s(n_1 + n_2) = sn_1 + sn_2
\]

for all \( s, s_1, s_2 \in S \) and all \( n, n_1, n_2 \in N \), and the relation

\[
(s_1 s_2)n = s_1 (s_2 n) \quad \text{for all} \quad s_1, s_2 \in S, \quad \text{and all} \quad n \in N.
\]

A particular case of the latter relation is

\[
(s r)n = s(rn) \quad \text{for all} \quad s \in S, \quad r \in R \quad \text{and} \quad n \in N.
\]

More generally, if \( f : R \to S \) is a ring homomorphism from \( R \) into \( S \) with \( f(1_R) = 1_S \) (for example the injection map if \( R \) is a subring of \( S \) as above) then it is easy to see that \( N \) can be considered as an \( R \)-module with \( rn = f(r)n \) for \( r \in R \) and \( n \in N \). In this situation \( S \) can be considered as an extension of the ring \( R \) and the resulting \( R \)-module is said to be obtained from \( N \) by restriction of scalars from \( S \) to \( R \).

Suppose now that \( R \) is a subring of \( S \) and we try to reverse this, namely we start with an \( R \)-module \( N \) and attempt to define an \( S \)-module structure on \( N \) that extends the action of \( R \) on \( N \) to an action of \( S \) on \( N \) (hence "extending the scalars" from \( R \) to \( S \)). In general this is impossible, even in the simplest situation: the ring \( R \) itself is an \( R \)-module but is usually not an \( S \)-module for the larger ring \( S \). For example, \( \mathbb{Z} \) is a \( \mathbb{Z} \)-module but it cannot be made into a \( \mathbb{Q} \)-module (if it could, then \( \frac{1}{2} \circ 1 = 1 \in \mathbb{Z} \) would be an element of \( \mathbb{Z} \) with \( z + z = 1 \), which is impossible). Although \( \mathbb{Z} \) itself cannot be made into a \( \mathbb{Q} \)-module it is contained in a \( \mathbb{Q} \)-module, namely \( \mathbb{Q} \) itself. Put another way, there is an injection (also called an embedding) of the \( \mathbb{Z} \)-module \( \mathbb{Z} \) into the \( \mathbb{Q} \)-module \( \mathbb{Q} \) (and similarly the ring \( R \) can always be embedded as an \( R \)-submodule of the \( S \)-module \( S \)). This raises the question of whether an arbitrary \( R \)-module \( N \) can be embedded as an \( R \)-submodule of some \( S \)-module, or more generally, the question of what \( R \)-module homomorphisms exist from \( N \) to \( S \)-modules. For example, suppose \( N \) is a nontrivial finite abelian group, say \( N = \mathbb{Z}/2\mathbb{Z} \), and consider possible \( \mathbb{Z} \)-module homomorphisms (i.e., abelian group homomorphisms) of \( N \) into some \( \mathbb{Q} \)-module. A \( \mathbb{Q} \)-module is just a vector space over \( \mathbb{Q} \) and every nonzero element in a vector space over \( \mathbb{Q} \) has infinite (additive) order. Since every element of \( N \) has finite order, every element of \( N \) must map to 0 under such a homomorphism. In other words there are no nonzero \( \mathbb{Z} \)-module homomorphisms from this \( N \) to any \( \mathbb{Q} \)-module, much less embeddings of \( N \) identifying

Sec. 10.4  Tensor Products of Modules 359
$N$ as a submodule of a $\mathbb{Q}$-module. The two $\mathbb{Z}$-modules $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ exhibit extremely different behaviors when we try to "extend scalars" from $\mathbb{Z}$ to $\mathbb{Q}$: the first module maps injectively into some $\mathbb{Q}$-module, the second always maps to 0 in a $\mathbb{Q}$-module.

We now construct for a general $R$-module $N$ an $S$-module that is the "best possible" target in which to try to embed $N$. We shall also see that this module determines all of the possible $R$-module homomorphisms of $N$ into $S$-modules, in particular determining when $N$ is contained in some $S$-module (cf. Corollary 9). In the case of $R = \mathbb{Z}$ and $S = \mathbb{Q}$ this construction will give us $\mathbb{Q}$ when applied to the module $N = \mathbb{Z}$, and will give us 0 when applied to the module $N = \mathbb{Z}/2\mathbb{Z}$ (Examples 2 and 3 following Corollary 9).

If the $R$-module $N$ were already an $S$-module then of course there is no difficulty in "extending" the scalars from $R$ to $S$, so we begin the construction by returning to the basic module axioms in order to examine whether we can define "products" of the form $sn$, for $s \in S$ and $n \in N$. These axioms start with an abelian group $N$ together with a map from $S \times N$ to $N$, where the image of the pair $(s, n)$ is denoted by $sn$. It is therefore natural to consider the free $\mathbb{Z}$-module (i.e., the free abelian group) on the set $S \times N$, i.e., the collection of all finite commuting sums of elements of the form $(s_i, n_i)$ where $s_i \in S$ and $n_i \in N$. This is an abelian group where there are no relations between any distinct pairs $(s, n)$ and $(s', n')$, i.e., no relations between the "formal products" $sn$, and in this abelian group the original module $N$ has been thoroughly distinguished from the new "coefficients" from $S$. To satisfy the relations necessary for an $S$-module structure imposed in equation (1) and the compatibility relation with the action of $R$ on $N$ in (2'), we must take the quotient of this abelian group by the subgroup $H$ generated by all elements of the form

$$(s_1 + s_2, n) - (s_1, n) - (s_2, n),$$

$$(s, n_1 + n_2) - (s, n_1) - (s, n_2),$$

and

$$(sr, n) - (s, rn),$$

for $s, s_1, s_2 \in S$, $n, n_1, n_2 \in N$ and $r \in R$, where $rn$ in the last element refers to the $R$-module structure already defined on $N$.

The resulting quotient group is denoted by $S \otimes_R N$ (or just $S \otimes N$ if $R$ is clear from the context) and is called the tensor product of $S$ and $N$ over $R$. If $s \otimes n$ denotes the coset containing $(s, n)$ in $S \otimes_R N$ then by definition of the quotient we have forced the relations

$$(s_1 + s_2) \otimes n = s_1 \otimes n + s_2 \otimes n,$$

$$s \otimes (n_1 + n_2) = s \otimes n_1 + s \otimes n_2,$$ and

$$sr \otimes n = s \otimes rn.$$ 

The elements of $S \otimes_R N$ are called tensors and can be written (non-uniquely in general) as finite sums of "simple tensors" of the form $s \otimes n$ with $s \in S, n \in N$.

We now show that the tensor product $S \otimes_R N$ is naturally a left $S$-module under the action defined by

$$s \left( \sum_{\text{finite}} s_i \otimes n_i \right) = \sum_{\text{finite}} (ss_i) \otimes n_i.$$ 

(10.5)
We first check this is well defined, i.e., independent of the representation of the element of \( S \otimes_R N \) as a sum of simple tensors. Note first that if \( s' \) is any element of \( S \) then

\[
(s'(s_1 + s_2), n) - (s's_1, n) - (s's_2, n) \quad (= (s's_1 + s's_2, n) - (s's_1, n) - (s's_2, n),)
\]

\[
(s's, n_1 + n_2) - (s's, n_1) - (s's, n_2),
\]

and

\[
(s'(sr), n) - (s's, rn) \quad (= ((s's)r, n) - (s's, rn))
\]

each belongs to the set of generators in (3), so in particular each lies in the subgroup \( H \). This shows that multiplying the first entries of the generators in (3) on the left by \( s' \) gives another element of \( H \) (in fact another generator). Since any element of \( H \) is a sum of elements as in (3), it follows that for any element \( \sum (s_i, n_i) \) in \( H \) also \( \sum (s's_i, n_i) \) lies in \( H \). Suppose now that \( \sum s_i \otimes n_i = \sum s'_i \otimes n'_i \) are two representations for the same element in \( S \otimes_R N \). Then \( \sum (s_i, n_i) - \sum (s'_i, n'_i) \) is an element of \( H \), and by what we have just seen, for any \( s \in S \) also \( \sum (ss_i, n_i) - \sum (ss'_i, n'_i) \) is an element of \( H \). But this means that \( \sum ss_i \otimes n_i = \sum ss'_i \otimes n'_i \) in \( S \otimes_R N \), so the expression in (5) is indeed well defined.

It is now straightforward using the relations in (4) to check that the action defined in (5) makes \( S \otimes_R N \) into a left \( S \)-module. For example, on the simple tensor \( s_i \otimes n_i \),

\[
(s + s') (s_i \otimes n_i) = ((s + s')s_i) \otimes n_i \quad \text{by definition (5)}
\]

\[
= (ss_i + s's_i) \otimes n_i
\]

\[
= ss_i \otimes n_i + s's_i \otimes n_i \quad \text{by the first relation in (4)}
\]

\[
= s (s_i \otimes n_i) + s' (s_i \otimes n_i) \quad \text{by definition (5)}.
\]

The module \( S \otimes_R N \) is called the (left) \( S \)-module obtained by extension of scalars from the (left) \( R \)-module \( N \).

There is a natural map \( \iota : N \to S \otimes_R N \) defined by \( n \mapsto 1 \otimes n \) (i.e., first map \( n \in N \) to the element \( (1, n) \) in the free abelian group and then pass to the quotient group). Since \( 1 \otimes rn = r \otimes n = r(1 \otimes n) \) by (4) and (5), it is easy to check that \( \iota \) is an \( R \)-module homomorphism from \( N \) to \( S \otimes_R N \). Since we have passed to a quotient group, however, \( \iota \) is not injective in general. Hence, while there is a natural \( R \)-module homomorphism from the original left \( R \)-module \( N \) to the left \( S \)-module \( S \otimes_R N \), in general \( S \otimes_R N \) need not contain (an isomorphic copy of) \( N \). On the other hand, the relations in equation (3) were the minimal relations that we had to impose in order to obtain an \( S \)-module, so it is reasonable to expect that the tensor product \( S \otimes_R N \) is the “best possible” \( S \)-module to serve as target for an \( R \)-module homomorphism from \( N \). The next theorem makes this more precise by showing that any other \( R \)-module homomorphism from \( N \) factors through this one, and is referred to as the universal property for the tensor product \( S \otimes_R N \). The analogous result for the general tensor product is given in Theorem 10.

Sec. 10.4 Tensor Products of Modules 361
Theorem 8. Let \( R \) be a subring of \( S \), let \( N \) be a left \( R \)-module and let \( \iota : N \to S \otimes_R N \) be the \( R \)-module homomorphism defined by \( \iota(n) = 1 \otimes n \). Suppose that \( L \) is any left \( S \)-module (hence also an \( R \)-module) and that \( \varphi : N \to L \) is an \( R \)-module homomorphism from \( N \) to \( L \). Then there is a unique \( S \)-module homomorphism \( \Phi : S \otimes_R N \to L \) such that \( \varphi \) factors through \( \Phi \), i.e., \( \varphi = \Phi \circ \iota \) and the diagram

\[
\begin{array}{c}
N \\
\downarrow \Phi \\
L
\end{array}
\]

\( \Phi \)

commutes. Conversely, if \( \Phi : S \otimes_R N \to L \) is an \( S \)-module homomorphism then \( \varphi = \Phi \circ \iota \) is an \( R \)-module homomorphism from \( N \) to \( L \).

Proof: Suppose \( \varphi : N \to L \) is an \( R \)-module homomorphism to the \( S \)-module \( L \). By the universal property of free modules (Theorem 6 in Section 3) there is a \( \mathbb{Z} \)-module homomorphism from the free \( \mathbb{Z} \)-module \( F \) on the set \( S \times N \) to \( L \) that sends each generator \( (s, n) \) to \( s \varphi(n) \). Since \( \varphi \) is an \( R \)-module homomorphism, the generators of the subgroup \( H \) in equation (3) all map to zero in \( L \). Hence this \( \mathbb{Z} \)-module homomorphism factors through \( H \), i.e., there is a well defined \( \mathbb{Z} \)-module homomorphism \( \Phi \) from \( F/H = S \otimes_R N \) to \( L \) satisfying \( \Phi(s \otimes n) = s \varphi(n) \). Moreover, on simple tensors we have

\[
s' \Phi(s \otimes n) = s'(s \varphi(n)) = (s's) \varphi(n) = \Phi((s's) \otimes n) = \Phi(s'(s \otimes n)).
\]

for any \( s' \in S \). Since \( \Phi \) is additive it follows that \( \Phi \) is an \( S \)-module homomorphism, which proves the existence statement of the theorem. The module \( S \otimes_R N \) is generated as an \( S \)-module by elements of the form \( 1 \otimes n \), so any \( S \)-module homomorphism is uniquely determined by its values on these elements. Since \( \Phi(1 \otimes n) = \varphi(n) \), it follows that the \( S \)-module homomorphism \( \Phi \) is uniquely determined by \( \varphi \), which proves the uniqueness statement of the theorem. The converse statement is immediate.

The universal property of \( S \otimes_R N \) in Theorem 8 shows that \( R \)-module homomorphisms of \( N \) into \( S \)-modules arise from \( S \)-module homomorphisms from \( S \otimes_R N \). In particular this determines when it is possible to map \( N \) injectively into some \( S \)-module:

Corollary 9. Let \( \iota : N \to S \otimes_R N \) be the \( R \)-module homomorphism in Theorem 8. Then \( N/\ker \iota \) is the unique largest quotient of \( N \) that can be embedded in any \( S \)-module. In particular, \( N \) can be embedded as an \( R \)-submodule of some left \( S \)-module if and only if \( \iota \) is injective (in which case \( N \) is isomorphic to the \( R \)-submodule \( \iota(N) \) of the \( S \)-module \( S \otimes_R N \)).

Proof: The quotient \( N/\ker \iota \) is mapped injectively (by \( \iota \)) into the \( S \)-module \( S \otimes_R N \). Suppose now that \( \varphi \) is an \( R \)-module homomorphism injecting the quotient \( N/\ker \varphi \) of \( N \) into an \( S \)-module \( L \). Then, by Theorem 8, \( \ker \iota \otimes \varphi \) is mapped to \( 0 \) by \( \varphi \), i.e., \( \ker \iota \subseteq \ker \varphi \). Hence \( N/\ker \varphi \) is a quotient of \( N/\ker \iota \) (namely, the quotient by the submodule \( \ker \varphi/\ker \iota \)). It follows that \( N/\ker \iota \) is the unique largest quotient of \( N \) that can be embedded in any \( S \)-module. The last statement in the corollary follows immediately.
Examples

(1) For any ring \( R \) and any left \( R \)-module \( N \) we have \( R \otimes_R N \cong N \) (so "extending scalars from \( R \) to \( R' \)" does not change the module). This follows by taking \( \varphi \) to be the identity map from \( N \) to itself (and \( S = R' \)) in Theorem 8: \( i \) is then an isomorphism with inverse isomorphism given by \( \Phi \). In particular, if \( A \) is any abelian group (i.e., a \( \mathbb{Z} \)-module), then \( \mathbb{Z} \otimes_{\mathbb{Z}} A = A \).

(2) Let \( R = \mathbb{Z} \), \( S = \mathbb{Q} \) and let \( A \) be a finite abelian group of order \( n \). In this case the \( \mathbb{Q} \)-module \( \mathbb{Q} \otimes_{\mathbb{Z}} A \) obtained by extension of scalars from the \( \mathbb{Z} \)-module \( A \) is \( 0 \). To see this, observe first that in any tensor product \( 1 \otimes 0 = 1 \otimes (0 + 0) = 1 \otimes 0 + 1 \otimes 0 \), by the second relation in (4), so

\[
1 \otimes 0 = 0.
\]

Now, for any simple tensor \( q \otimes a \) we can write the rational number \( q \) as \( (q/n)n \). Then since \( na = 0 \) in \( A \) by Lagrange's Theorem, we have

\[
q \otimes a = \left( \frac{q}{n} \cdot n \right) \otimes a = \frac{q}{n} \otimes (na) = (q/n) \otimes 0 = (q/n)(1 \otimes 0) = 0.
\]

It follows that \( \mathbb{Q} \otimes_{\mathbb{Z}} A = 0 \). In particular, the map \( i : A \rightarrow S \otimes_R A \) is the zero map. By Theorem 8, we see again that any homomorphism of a finite abelian group into a rational vector space is the zero map. In particular, if \( A \) is nontrivial, then the original \( \mathbb{Z} \)-module \( A \) is not contained in the \( \mathbb{Q} \)-module obtained by extension of scalars.

(3) Extension of scalars for free modules: If \( N \cong R^n \) is a free module of rank \( n \) over \( R \) then \( S \otimes_R N \cong S^n \) is a free module of rank \( n \) over \( S \). We shall prove this shortly (Corollary 18) when we discuss tensor products of direct sums. For example, \( \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q}^n \).

In this case the module obtained by extension of scalars contains (an isomorphic copy of) the original \( R \)-module \( N \). For example, \( \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q}^n \) and \( \mathbb{Z}^n \) is a subgroup of the abelian group \( \mathbb{Q}^n \).

(4) Extension of scalars for vector spaces: As a special case of the previous example, let \( F \) be a subfield of the field \( K \) and let \( V \) be an \( n \)-dimensional vector space over \( F \) (i.e., \( V \cong F^n \)). Then \( K \otimes_F V \cong K^n \) is a vector space over the larger field \( K \) of the same dimension, and the original vector space \( V \) is contained in \( K \otimes_F V \) as an \( F \)-vector subspace.

(5) Induced modules for finite groups: Let \( R \) be a commutative ring with 1, let \( G \) be a finite group and let \( H \) be a subgroup of \( G \). As in Section 7.2 we may form the group ring \( RG \) and its subring \( RH \). For any \( RH \)-module \( N \) define the induced module \( RG \otimes_{RH} N \). In this way we obtain an \( RG \)-module for each \( RH \)-module \( N \). We shall study properties of induced modules and some of their important applications to group theory in Chapters 17 and 19.

The general tensor product construction follows along the same lines as the extension of scalars above, but before describing it we make two observations from this special case. The first is that the construction of \( S \otimes_R N \) as an abelian group involved only the elements in equation (3), which in turn only required \( S \) to be a right \( R \)-module and \( N \) to be a left \( R \)-module. In a similar way we shall construct an abelian group \( M \otimes_R N \) for any right \( R \)-module \( M \) and any left \( R \)-module \( N \). The second observation is that the \( S \)-module structure on \( S \otimes_R N \) defined by equation (5) required only a left \( S \)-module structure on \( S \) together with a "compatibility relation"

\[
s'(sr) = (s's)r \quad \text{for } s, s' \in S, r \in R,
\]
between this left \( S \)-module structure and the right \( R \)-module structure on \( S \) (this was needed in order to deduce that (5) was well defined). We first consider the general construction of \( M \otimes_R N \) as an abelian group, after which we shall return to the question of when this abelian group can be given a module structure.

Suppose then that \( N \) is a left \( R \)-module and that \( M \) is a right \( R \)-module. The quotient of the free \( \mathbb{Z} \)-module on the set \( M \times N \) by the subgroup generated by all elements of the form

\[
(m_1 + m_2, n) - (m_1, n) - (m_2, n),
\]

\[
(m, n_1 + n_2) - (m, n_1) - (m, n_2), \quad \text{and}
\]

\[
(m, n) - (m, rn),
\]

for \( m, m_1, m_2 \in M, n, n_1, n_2 \in N \) and \( r \in R \) is an abelian group, denoted by \( M \otimes_R N \), or simply \( M \otimes N \) if the ring \( R \) is clear from the context, and is called the \textit{tensor product of \( M \) and \( N \) over \( R \)}. The elements of \( M \otimes_R N \) are called tensors, and the coset, \( m \otimes n \), of \( (m, n) \) in \( M \otimes_R N \) is called a simple tensor. We have the relations

\[
(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n,
\]

\[
m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2, \quad \text{and}
\]

\[
mr \otimes n = m \otimes rn.
\]

Every tensor can be written (non-uniquely in general) as a finite sum of simple tensors.

\textbf{Remark:} We emphasize that care must be taken when working with tensors, since each \( m \otimes n \) represents a coset in some quotient group, and so we may have \( m \otimes n = m' \otimes n' \) where \( m \neq m' \) or \( n \neq n' \). More generally, an element of \( M \otimes N \) may be expressible in many different ways as a sum of simple tensors. In particular, care must be taken when defining maps from \( M \otimes_R N \) to another group or module, since a map from \( M \otimes N \) which is described on the generators \( m \otimes n \) in terms of \( m \) and \( n \) is not well defined unless it is shown to be independent of the particular choice of \( m \otimes n \) as a coset representative.

Another point where care must be exercised is in reference to the element \( m \otimes n \) when the modules \( M \) and \( N \) or the ring \( R \) are not clear from the context. The first two examples of extension of scalars give an instance where \( M \) is a submodule of a larger module \( M' \), and for some \( m \in M \) and \( n \in N \) we have \( m \otimes n = 0 \) in \( M' \otimes_R N \) but \( m \otimes n \) is nonzero in \( M \otimes_R N \). This is possible because the symbol "\( m \otimes n \)" represents different cosets, hence possibly different elements, in the two tensor products. In particular, these two examples show that \( M \otimes_R N \) need not be a subgroup of \( M' \otimes_R N \) even when \( M \) is a submodule of \( M' \) (cf. also Exercise 2).

Mapping \( M \times N \) to the free \( \mathbb{Z} \)-module on \( M \times N \) and then passing to the quotient defines a map \( \iota : M \times N \rightarrow M \otimes_R N \) with \( \iota(m, n) = m \otimes n \). This map is in general not a group homomorphism, but it is additive in both \( m \) and \( n \) separately and satisfies \( \iota(mr, n) = mr \otimes n = m \otimes rn = \iota(m, rn) \). Such maps are given a name:
**Definition.** Let $M$ be a right $R$-module, let $N$ be a left $R$-module and let $L$ be an abelian group (written additively). A map $\varphi : M \times N \rightarrow L$ is called $R$-balanced or middle linear with respect to $R$ if

$$\varphi(m_1 + m_2, n) = \varphi(m_1, n) + \varphi(m_2, n)$$

$$\varphi(m, n_1 + n_2) = \varphi(m, n_1) + \varphi(m, n_2)$$

$$\varphi(m, rn) = \varphi(m, n)$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, and $r \in R$.

With this terminology, it follows immediately from the relations in (7) that the map $\iota : M \times N \rightarrow M \otimes_R N$ is $R$-balanced. The next theorem proves the extremely useful universal property of the tensor product with respect to balanced maps.

**Theorem 10.** Suppose $R$ is a ring with 1, $M$ is a right $R$-module, and $N$ is a left $R$-module. Let $M \otimes_R N$ be the tensor product of $M$ and $N$ over $R$ and let $\iota : M \times N \rightarrow M \otimes_R N$ be the $R$-balanced map defined above.

1. If $\Phi : M \otimes_R N \rightarrow L$ is any group homomorphism from $M \otimes_R N$ to an abelian group $L$ then the composite map $\varphi = \Phi \circ \iota$ is an $R$-balanced map from $M \times N$ to $L$.

2. Conversely, suppose $L$ is an abelian group and $\varphi : M \times N \rightarrow L$ is any $R$-balanced map. Then there is a unique group homomorphism $\Phi : M \otimes_R N \rightarrow L$ such that $\varphi$ factors through $\iota$, i.e., $\varphi = \Phi \circ \iota$ as in (1).

Equivalently, the correspondence $\varphi \leftrightarrow \Phi$ in the commutative diagram

$$M \times N \xrightarrow{\iota} M \otimes_R N \xrightarrow{\Phi} L$$

establishes a bijection

$$\left\{ \text{R-balanced maps} \right\} \leftrightarrow \left\{ \text{group homomorphisms} \right\}$$


**Proof:** The proof of (1) is immediate from the properties of $\iota$ above. For (2), the map $\varphi$ defines a unique $\mathbb{Z}$-module homomorphism $\tilde{\varphi}$ from the free group on $M \times N$ to $L$ (Theorem 6 in Section 3) such that $\tilde{\varphi}(m, n) = \varphi(m, n) \in L$. Since $\varphi$ is $R$-balanced, $\tilde{\varphi}$ maps each of the elements in equation (6) to 0; for example

$$\tilde{\varphi}((mr, n) - (m, rn)) = \varphi(mr, n) - \varphi(m, rn) = 0.$$

It follows that the kernel of $\tilde{\varphi}$ contains the subgroup generated by these elements, hence $\tilde{\varphi}$ induces a homomorphism $\Phi$ on the quotient group $M \otimes_R N$ to $L$. By definition we then have

$$\Phi(m \otimes n) = \tilde{\varphi}(m, n) = \varphi(m, n),$$

i.e., $\varphi = \Phi \circ \iota$. The homomorphism $\Phi$ is uniquely determined by this equation since the elements $m \otimes n$ generate $M \otimes_R N$ as an abelian group. This completes the proof.

Sec. 10.4 Tensor Products of Modules 365
Theorem 10 is extremely useful in defining homomorphisms on $M \otimes_R N$ since it replaces the often tedious check that maps defined on simple tensors $m \otimes n$ are well defined with a check that a related map defined on ordered pairs $(m, n)$ is balanced.

The first consequence of the universal property in Theorem 10 is a characterization of the tensor product $M \otimes_R N$ as an abelian group:

**Corollary 11.** Suppose $D$ is an abelian group and $i' : M \times N \to D$ is an $R$-balanced map such that

(i) the image of $i'$ generates $D$ as an abelian group, and

(ii) every $R$-balanced map defined on $M \times N$ factors through $i'$ as in Theorem 10.

Then there is an isomorphism $f : M \otimes_R N \cong D$ of abelian groups with $i' = f \circ i$.

**Proof:** Since $i' : M \times N \to D$ is a balanced map, the universal property in (2) of Theorem 10 implies there is a (unique) homomorphism $f : M \otimes_R N \to D$ with $i' = f \circ i$. In particular $i'(m, n) = f(m \otimes n)$ for every $m \in M$, $n \in N$. By the first assumption on $i'$, these elements generate $D$ as an abelian group, so $f$ is a surjective map. Now, the balanced map $\iota : M \times N \to M \otimes_R N$ together with the second assumption on $i'$ implies there is a (unique) homomorphism $g : D \to M \otimes_R N$ with $\iota = g \circ i'$. Then $m \otimes n = (g \circ f)(m \otimes n)$. Since the simple tensors $m \otimes n$ generate $M \otimes_R N$, it follows that $g \circ f$ is the identity map on $M \otimes_R N$ and so $f$ is injective, hence an isomorphism. This establishes the corollary.

We now return to the question of giving the abelian group $M \otimes_R N$ a module structure. As we observed in the special case of extending scalars from $R$ to $S$ for the $R$-module $N$, the $S$-module structure on $S \otimes_R N$ required only a left $S$-module structure on $S$ together with the compatibility relation $s'(sr) = (s's)r$ for $s, s' \in S$ and $r \in R$. In this special case this relation was simply a consequence of the associative law in the ring $S$. To obtain an $S$-module structure on $M \otimes_R N$ more generally we impose a similar structure on $M$:

**Definition.** Let $R$ and $S$ be any rings with 1. An abelian group $M$ is called an $(S, R)$-bimodule if $M$ is a left $S$-module, a right $R$-module, and $s(mr) = (sm)r$ for all $s \in S$, $r \in R$ and $m \in M$.

**Examples**

1. Any ring $S$ is an $(S, R)$-bimodule for any subring $R$ with $1_R = 1_S$ by the associativity of the multiplication in $S$. More generally, if $f : R \to S$ is any ring homomorphism with $f(1_R) = 1_S$ then $S$ can be considered as a right $R$-module with the action $s \cdot r = sf(r)$, and with respect to this action $S$ becomes an $(S, R)$-bimodule.

2. Let $I$ be an ideal (two-sided) in the ring $R$. Then the quotient ring $R/I$ is an $(R/I, R)$-bimodule. This is easy to see directly and is also a special case of the previous example (with respect to the canonical projection homomorphism $R \to R/I$).

3. Suppose that $R$ is a commutative ring. Then a left (respectively, right) $R$-module $M$ can always be given the structure of a right (respectively, left) $R$-module by defining $mr = rm$ (respectively, $rm = mr$), for all $m \in M$ and $r \in R$, and this makes $M$ into
an \((R, R)\)-bimodule. Hence every module (right or left) over a commutative ring \(R\) has at least one natural \((R, R)\)-bimodule structure.

(4) Suppose that \(M\) is a left \(S\)-module and \(R\) is a subring contained in the center of \(S\) (for example, if \(S\) is commutative). Then in particular \(R\) is commutative so \(M\) can be given a right \(R\)-module structure as in the previous example. Then for any \(s \in S, r \in R\) and \(m \in M\) by definition of the right action of \(R\) we have

\[
(sm)r = r(sm) = (rs)m = (sr)m = srm = s(mr)
\]

(note that we have used the fact that \(r\) commutes with \(s\) in the middle equality). Hence \(M\) is an \((S, R)\)-bimodule with respect to this definition of the right action of \(R\).

Since the situation in Example 3 occurs so frequently, we give this bimodule structure a name:

**Definition.** Suppose \(M\) is a left (or right) \(R\)-module over the commutative ring \(R\). Then the \((R, R)\)-bimodule structure on \(M\) defined by letting the left and right \(R\)-actions coincide, i.e., \(rm = rm\) for all \(m \in M\) and \(r \in R\), will be called the standard \(R\)-module structure on \(M\).

Suppose now that \(N\) is a left \(R\)-module and \(M\) is an \((S, R)\)-bimodule. Then just as in the example of extension of scalars the \((S, R)\)-bimodule structure on \(M\) implies that

\[
s\left(\sum_{\text{finite}} m_i \otimes n_i\right) = \sum_{\text{finite}} (sm_i) \otimes n_i \tag{10.8}
\]

gives a well defined action of \(S\) under which \(M \otimes_R N\) is a left \(S\)-module. Note that Theorem 10 may be used to give an alternate proof that (8) is well defined, replacing the direct calculations on the relations defining the tensor product with the easier check that a map is \(R\)-balanced, as follows. It is very easy to see that for each fixed \(s \in S\) the map \((m, n) \mapsto sm \otimes n\) is an \(R\)-balanced map from \(M \times N\) to \(M \otimes_R N\). By Theorem 10 there is a well defined group homomorphism \(\lambda_s\) from \(M \otimes_R N\) to itself such that \(\lambda_s(m \otimes n) = sm \otimes n\). Since the right side of (8) is then \(\lambda_s(\sum m_i \otimes n_i)\), the fact that \(\lambda_s\) is well defined shows that this expression is indeed independent of the representation of the tensor \(\sum m_i \otimes n_i\) as a sum of simple tensors. Because \(\lambda_s\) is additive, equation (8) holds.

By a completely parallel argument, if \(M\) is a right \(R\)-module and \(N\) is an \((R, S)\)-bimodule then the tensor product \(M \otimes_R N\) has the structure of a right \(S\)-module, where \((\sum m_i \otimes n_i) s = \sum m_i \otimes (n_i s)\).

Before giving some more examples of tensor products it is worthwhile to highlight one frequently encountered special case of the previous discussion, namely the case when \(M\) and \(N\) are two left modules over a commutative ring \(R\) and \(S = R\) (in some works on tensor products this is the only case considered). Then the standard \(R\)-module structure on \(M\) defined previously gives \(M\) the structure of an \((R, R)\)-bimodule, so in this case the tensor product \(M \otimes_R N\) always has the structure of a left \(R\)-module.

The corresponding map \(\iota : M \times N \to M \otimes_R N\) maps \(M \times N\) into an \(R\)-module and is additive in each factor. Since \(r(m \otimes n) = rm \otimes n = mr \otimes n = m \otimes rn\) it also satisfies

\[
\iota(rm, n) = r \iota(m, n) = \iota(m, rn).
\]
Such maps are given a name:

**Definition.** Let $R$ be a commutative ring with 1 and let $M$, $N$, and $L$ be left $R$-modules. The map $\varphi : M \times N \to L$ is called $R$-bilinear if it is $R$-linear in each factor, i.e., if

$$\varphi(r_1 m_1 + r_2 m_2, n) = r_1 \varphi(m_1, n) + r_2 \varphi(m_2, n),$$

and

$$\varphi(m, r_1 n_1 + r_2 n_2) = r_1 \varphi(m, n_1) + r_2 \varphi(m, n_2)$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r_1, r_2 \in R$.

With this terminology Theorem 10 gives

**Corollary 12.** Suppose $R$ is a commutative ring. Let $M$ and $N$ be two left $R$-modules and let $M \otimes_R N$ be the tensor product of $M$ and $N$ over $R$, where $M$ is given the standard $R$-module structure. Then $M \otimes_R N$ is a left $R$-module with

$$r(m \otimes n) = (rm) \otimes n = (mr) \otimes n = m \otimes (rn),$$

and the map $\iota : M \times N \to M \otimes_R N$ with $\iota(m, n) = m \otimes n$ is an $R$-bilinear map. If $L$ is any left $R$-module then there is a bijection

$$\left\{ \begin{array}{l}
\text{R-bilinear maps} \\
\varphi : M \times N \to L
\end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l}
\text{R-module homomorphisms} \\
\Phi : M \otimes_R N \to L
\end{array} \right\}$$

where the correspondence between $\varphi$ and $\Phi$ is given by the commutative diagram

$$\begin{array}{ccc}
M \times N & \xrightarrow{\iota} & M \otimes_R N \\
\downarrow \varphi & & \downarrow \Phi \\
L & & L
\end{array}$$

**Proof:** We have shown $M \otimes_R N$ is an $R$-module and that $\iota$ is bilinear. It remains only to check that in the bijective correspondence in Theorem 10 the bilinear maps correspond with the $R$-module homomorphisms. If $\varphi : M \times N \to L$ is bilinear then it is an $R$-balanced map, so the corresponding $\Phi : M \otimes_R N$ is a group homomorphism. Moreover, on simple tensors $\Phi((rm) \otimes n) = \varphi(rm, n) = r\varphi(m, n) = r\Phi(m \otimes n)$, where the middle equality holds because $\varphi$ is $R$-linear in the first variable. Since $\Phi$ is additive this extends to sums of simple tensors to show $\Phi$ is an $R$-module homomorphism. Conversely, if $\Phi$ is an $R$-module homomorphism it is an exercise to see that the corresponding balanced map $\varphi$ is bilinear.

**Examples**

1. In any tensor product $M \otimes_R N$ we have $m \otimes 0 = m \otimes (0 + 0) = (m \otimes 0) + (m \otimes 0)$, so $m \otimes 0 = 0$. Likewise $0 \otimes n = 0$.

2. We have $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$, since $3a = a$ for $a \in \mathbb{Z}/2\mathbb{Z}$ so that

$$a \otimes b = 3a \otimes b = a \otimes 3b = a \otimes 0 = 0$$

and every simple tensor is reduced to 0. In particular $1 \otimes 1 = 0$. It follows that there are no nonzero balanced (or bilinear) maps from $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ to any abelian group.

368 Chap. 10 Introduction to Module Theory
On the other hand, consider the tensor product \( \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \), which is generated as an abelian group by the elements \( 0 \otimes 0 = 1 \otimes 0 = 0 \otimes 1 = 0 \) and \( 1 \otimes 1 \). In this case \( 1 \otimes 1 \neq 0 \) since, for example, the map \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \) defined by \( (a, b) \mapsto ab \) is clearly nonzero and linear in both \( a \) and \( b \). Since \( 2(1 \otimes 1) = 2 \otimes 1 = 0 \otimes 1 = 0 \), the element \( 1 \otimes 1 \) is of order 2. Hence \( \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \).

(3) In general,

\[
\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z},
\]

where \( d \) is the g.c.d. of the integers \( m \) and \( n \). To see this, observe first that

\[
a \otimes b = a \otimes (b \cdot 1) = (ab) \otimes 1 = ab(1 \otimes 1),
\]

from which it follows that \( \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \) is a cyclic group with \( 1 \otimes 1 \) as generator. Since \( m(1 \otimes 1) = m \otimes 1 = 0 \otimes 1 = 0 \) and similarly \( n(1 \otimes 1) = 1 \otimes n = 0 \), we have \( d(1 \otimes 1) = 0 \), so the cyclic group has order dividing \( d \). The map \( \varphi : \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z} \) defined by \( \varphi(a \mod m, b \mod n) = ab \mod d \) is well defined since \( d \) divides both \( m \) and \( n \). It is clearly \( \mathbb{Z} \)-bilinear. The induced map \( \Phi : \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z} \) from Corollary 12 maps \( 1 \otimes 1 \) to the element \( 1 \in \mathbb{Z}/d\mathbb{Z} \), which is an element of order \( d \). In particular \( \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \) has order at least \( d \). Hence \( 1 \otimes 1 \) is an element of order \( d \) and \( \Phi \) gives an isomorphism \( \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z} \).

(4) In \( \mathbb{Q}/\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z} \) a simple tensor has the form \( (a/b \mod \mathbb{Z}) \otimes (c/d \mod \mathbb{Z}) \) for some rational numbers \( a/b \) and \( c/d \). Then

\[
\left( \frac{a}{b} \mod \mathbb{Z} \right) \otimes \left( \frac{c}{d} \mod \mathbb{Z} \right) = d \left( \frac{a}{bd} \mod \mathbb{Z} \right) \otimes \left( \frac{c}{d} \mod \mathbb{Z} \right)
\]

\[
= \left( \frac{a}{bd} \mod \mathbb{Z} \right) \otimes \left( \frac{c}{d} \mod \mathbb{Z} \right) = \left( \frac{a}{bd} \mod \mathbb{Z} \right) \otimes 0 = 0
\]

and so

\[
\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z} = 0.
\]

In a similar way, \( A \otimes_{\mathbb{Z}} B = 0 \) for any divisible abelian group \( A \) and torsion abelian group \( B \) (an abelian group in which every element has finite order). For example

\[
\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0.
\]

(5) The structure of a tensor product can vary considerably depending on the ring over which the tensors are taken. For example \( \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \) and \( \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \) are isomorphic as left \( \mathbb{Q} \)-modules (both are one dimensional vector spaces over \( \mathbb{Q} \)) — cf. the exercises. On the other hand we shall see at the end of this section that \( \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \) and \( \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \) are not isomorphic \( \mathbb{C} \)-modules (the former is a 1-dimensional vector space over \( \mathbb{C} \) and the latter is 2-dimensional over \( \mathbb{C} \)).

(6) General extension of scalars or change of base: Let \( f : R \to S \) be a ring homomorphism with \( f(1_R) = 1_S \). Then \( s \cdot r = sf(r) \) gives \( S \) the structure of a right \( R \)-module with respect to which \( S \) is an \( (S, R) \)-bimodule. Then for any left \( R \)-module \( N \), the resulting tensor product \( S \otimes_R N \) is a left \( S \)-module obtained by changing the base from \( R \) to \( S \). This gives a slight generalization of the notion of extension of scalars (where \( R \) was a subring of \( S \)).

(7) Let \( f : R \to S \) be a ring homomorphism as in the preceding example. Then we have \( S \otimes_R R \cong S \) as left \( S \)-modules, as follows. The map \( \varphi : S \times R \to S \) defined by \( (s, r) \mapsto sr \) (where \( sr = sf(r) \) by definition of the right \( R \)-action on \( S \)), is an \( R \)-balanced map, as is easily checked. For example,

\[
\varphi(s_1 + s_2, r) = (s_1 + s_2)r = s_1 r + s_2 r = \varphi(s_1, r) + \varphi(s_2, r)
\]
and

\[ \varphi(sr, r') = (sr)r' = s(rr') = \varphi(s, rr'). \]

By Theorem 10 we have an associated group homomorphism \( \Phi : S \otimes_R R \to S \) with \( \Phi(s \otimes r) = sr. \) Since \( \Phi(s(s \otimes r)) = \Phi(s's \otimes r) = s'sr = s'\Phi(s \otimes r), \) it follows that \( \Phi \) is also an \( S \)-module homomorphism. The map \( \Phi' : S \to S \otimes_R R \) with \( s \mapsto s \otimes 1 \) is an \( S \)-module homomorphism that is inverse to \( \Phi \) because \( \Phi \circ \Phi'(s) = \Phi(s \otimes 1) = s \) gives \( \Phi \Phi' = 1, \) and

\[ \Phi' \circ \Phi(s \otimes r) = \Phi'(sr) = sr \otimes 1 = s \otimes r \]

shows that \( \Phi' \Phi \) is the identity on simple tensors, hence \( \Phi' \Phi = 1. \)

(8) Let \( R \) be a ring (not necessarily commutative), let \( I \) be a two sided ideal in \( R, \) and let \( N \) be a left \( R \)-module. Then as previously mentioned, \( R/I \) is an \( (R/I, R) \)-bimodule, so the tensor product \( R/I \otimes_R N \) is a left \( R/I \)-module. This is an example of “extension of scalars” with respect to the natural projection homomorphism \( R \to R/I. \)

Define

\[ IN = \left\{ \sum_{finite} a_i n_i \mid a_i \in I, n_i \in N \right\}, \]

which is easily seen to be a left \( R \)-submodule of \( N \) (cf. Exercise 5, Section 1). Then

\[ (R/I) \otimes_R N \cong N/IN, \]

as left \( R \)-modules, as follows. The tensor product is generated as an abelian group by the simple tensors \( (r \mod I) \otimes n = r(1 \otimes n) \) for \( r \in R \) and \( n \in N \) (viewing the \( R/I \)-module tensor product as an \( R \)-module on which \( I \) acts trivially). Hence the elements \( 1 \otimes n \) generate \( (R/I) \otimes_R N \) as an \( R/I \)-module. The map \( N \to (R/I) \otimes_R N \) defined by \( n \mapsto 1 \otimes n \) is a left \( R \)-module homomorphism and, by the previous observation, is surjective. Under this map \( a_i n_i \) with \( a_i \in I \) and \( n_i \in N \) maps to \( 1 \otimes a_i n_i = a_i \otimes n_i = 0, \) and so \( IN \) is contained in the kernel. This induces a surjective \( R \)-module homomorphism \( f : N/IN \to (R/I) \otimes_R N \) with \( f(n \mod I) = 1 \otimes n. \) We show \( f \) is an isomorphism by exhibiting its inverse. The map \( (R/I) \times N \to N/IN \) defined by mapping \( (r \mod I, n) \) to \( (rn \mod I) \) is well defined and easily checked to be \( R \)-balanced. It follows by Theorem 10 that there is an associated group homomorphism \( g : (R/I) \otimes N \to N/IN \) with \( g((r \mod I) \otimes n) = rn \mod IN. \) As usual, \( fg = 1 \) and \( gf = 1, \) so \( f \) is a bijection and \( (R/I) \otimes_R N \cong N/IN, \) as claimed.

As an example, let \( R = \mathbb{Z} \) with ideal \( I = m\mathbb{Z} \) and let \( N \) be the \( \mathbb{Z} \)-module \( \mathbb{Z}/n\mathbb{Z}. \)

Then \( IN = m(\mathbb{Z}/n\mathbb{Z}) = (m\mathbb{Z} + n\mathbb{Z})/n\mathbb{Z} = d\mathbb{Z}/n\mathbb{Z}, \) where \( d \) is the g.c.d. of \( m \) and \( n. \)

Then \( N/IN \cong \mathbb{Z}/d\mathbb{Z} \) and we recover the isomorphism \( \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z} \) of Example 3 above.

We now establish some of the basic properties of tensor products. Note the frequent application of Theorem 10 to establish the existence of homomorphisms.

**Theorem 13. (The “Tensor Product” of Two Homomorphisms)** Let \( M, M' \) be right \( R \)-modules, let \( N, N' \) be left \( R \)-modules, and suppose \( \varphi : M \to M' \) and \( \psi : N \to N' \) are \( R \)-module homomorphisms.

1. There is a unique group homomorphism, denoted by \( \varphi \otimes \psi, \) mapping \( M \otimes_R N \)

   into \( M' \otimes_R N' \) such that \( (\varphi \otimes \psi)(m \otimes n) = \varphi(m) \otimes \psi(n) \) for all \( m \in M \) and \( n \in N. \)
(2) If $M, M'$ are also $(S, R)$-bimodules for some ring $S$ and $\varphi$ is also an $S$-module homomorphism, then $\varphi \otimes \psi$ is a homomorphism of left $S$-modules. In particular, if $R$ is commutative then $\varphi \otimes \psi$ is always an $R$-module homomorphism for the standard $R$-module structures.

(3) If $\lambda : M' \to M''$ and $\mu : N' \to N''$ are $R$-module homomorphisms then $(\lambda \otimes \mu) \circ (\varphi \otimes \psi) = (\lambda \circ \varphi) \otimes (\mu \circ \psi)$.

Proof: The map $(m, n) \mapsto \varphi(m) \otimes \psi(n)$ from $M \times N$ to $M' \otimes_R N'$ is clearly $R$-balanced, so (1) follows immediately from Theorem 10.

In (2) the definition of the (left) action of $S$ on $M$ together with the assumption that $\varphi$ is an $S$-module homomorphism imply that on simple tensors

$$(\varphi \otimes \psi)(s(m \otimes n)) = (\varphi \otimes \psi)(sm \otimes n) = \varphi(sm) \otimes \psi(n) = s\varphi(m) \otimes \psi(n).$$

Since $\varphi \otimes \psi$ is additive, this extends to sums of simple tensors to show that $\varphi \otimes \psi$ is an $S$-module homomorphism. This gives (2).

The uniqueness condition in Theorem 10 implies (3), which completes the proof.

The next result shows that we may write $M \otimes N \otimes L$, or more generally, an $n$-fold tensor product $M_1 \otimes M_2 \otimes \cdots \otimes M_n$, unambiguously whenever it is defined.

**Theorem 14. (Associativity of the Tensor Product)** Suppose $M$ is a right $R$-module, $N$ is an $(R, T)$-bimodule, and $L$ is a left $T$-module. Then there is a unique isomorphism

$$(M \otimes_R N) \otimes_T L \cong M \otimes_R (N \otimes_T L)$$

of abelian groups such that $(m \otimes n) \otimes l \mapsto m \otimes (n \otimes l)$. If $M$ is an $(S, R)$-bimodule, then this is an isomorphism of $S$-modules.

Proof: Note first that the $(R, T)$-bimodule structure on $N$ makes $M \otimes_R N$ into a right $T$-module and $N \otimes_T L$ into a left $R$-module, so both sides of the isomorphism are well defined. For each fixed $l \in L$, the mapping $(m, n) \mapsto m \otimes (n \otimes l)$ is $R$-balanced, so by Theorem 10 there is a homomorphism $M \otimes_R N \to M \otimes_R (N \otimes_T L)$ with $m \otimes n \mapsto m \otimes (n \otimes l)$. This shows that the map from $(M \otimes_R N) \times L$ to $M \otimes_R (N \otimes_T L)$ given by $(m \otimes n, l) \mapsto m \otimes (n \otimes l)$ is well defined. Since it is easily seen to be $T$-balanced, another application of Theorem 10 implies that it induces a homomorphism $(M \otimes_R N) \otimes_T L \to M \otimes_R (N \otimes_T L)$ such that $(m \otimes n) \otimes l \mapsto m \otimes (n \otimes l)$. In a similar way we can construct a homomorphism in the opposite direction that is inverse to this one. This proves the group isomorphism.

Assume in addition $M$ is an $(S, R)$-bimodule. Then for $s \in S$ and $t \in T$ we have

$$s \left((m \otimes n) t\right) = (s(m \otimes n)t) = sm \otimes nt = (sm \otimes n)t = (s(m \otimes n)) t$$

so that $M \otimes_R N$ is an $(S, T)$-bimodule. Hence $(M \otimes_R N) \otimes_T L$ is a left $S$-module. Since $N \otimes_T L$ is a left $R$-module, also $M \otimes_R (N \otimes_T L)$ is a left $S$-module. The group isomorphism just established is easily seen to be a homomorphism of left $S$-modules by the same arguments used in previous proofs: it is additive and is $S$-linear on simple tensors since $s \left((m \otimes n) \otimes l\right) = (s(m \otimes n) \otimes l) = (sm \otimes n) \otimes l$ maps to the element $sm \otimes (n \otimes l) = s \left(m \otimes (n \otimes l)\right)$. The proof is complete.
Corollary 15. Suppose $R$ is commutative and $M$, $N$, and $L$ are left $R$-modules. Then

$$(M \otimes N) \otimes L \cong M \otimes (N \otimes L)$$

as $R$-modules for the standard $R$-module structures on $M$, $N$ and $L$.

There is a natural extension of the notion of a bilinear map:

**Definition.** Let $R$ be a commutative ring with 1 and let $M_1, M_2, \ldots, M_n$ and $L$ be $R$-modules with the standard $R$-module structures. A map $\varphi : M_1 \times \cdots \times M_n \to L$ is called $n$-**multilinear over $R$** (or simply *multilinear* if $n$ and $R$ are clear from the context) if it is an $R$-module homomorphism in each component when the other component entries are kept constant, i.e., for each $i$

$$\varphi(m_1, \ldots, m_{i-1}, rm_i + r'm'_i, m_{i+1}, \ldots, m_n)$$

$$= r\varphi(m_1, \ldots, m_i, \ldots, m_n) + r'\varphi(m_1, \ldots, m'_i, \ldots, m_n)$$

for all $m_i, m'_i \in M_i$ and $r, r' \in R$. When $n = 2$ (respectively, 3) one says $\varphi$ is *bilinear* (respectively *trilinear*) rather than $2$-multilinear (or $3$-multilinear).

One may construct the $n$-fold tensor product $M_1 \otimes M_2 \otimes \cdots \otimes M_n$ from first principles and prove its analogous universal property with respect to multilinear maps from $M_1 \times \cdots \times M_n$ to $L$. By the previous theorem and corollary, however, an $n$-fold tensor product may be obtained unambiguously by iterating the tensor product of pairs of modules since any bracketing of $M_1 \otimes \cdots \otimes M_n$ into tensor products of pairs gives an isomorphic $R$-module. The universal property of the tensor product of a pair of modules in Theorem 10 and Corollary 12 then implies that multilinear maps factor uniquely through the $R$-module $M_1 \otimes \cdots \otimes M_n$, i.e., this tensor product is the universal object with respect to multilinear functions:

Corollary 16. Let $R$ be a commutative ring and let $M_1, \ldots, M_n$, $L$ be $R$-modules. Let $M_1 \otimes M_2 \otimes \cdots \otimes M_n$ denote any bracketing of the tensor product of these modules and let

$$\iota : M_1 \times \cdots \times M_n \to M_1 \otimes \cdots \otimes M_n$$

be the map defined by $\iota(m_1, \ldots, m_n) = m_1 \otimes \cdots \otimes m_n$. Then

1. For every $R$-module homomorphism $\Phi : M_1 \otimes \cdots \otimes M_n \to L$ the map $\varphi = \Phi \circ \iota$ is $n$-multilinear from $M_1 \times \cdots \times M_n$ to $L$, and

2. If $\varphi : M_1 \times \cdots \times M_n \to L$ is an $n$-multilinear map then there is a unique $R$-module homomorphism $\Phi : M_1 \otimes \cdots \otimes M_n \to L$ such that $\varphi = \Phi \circ \iota$.

Hence there is a bijection

$$\left\{ \text{n-multilinear maps } \varphi : M_1 \times \cdots \times M_n \to L \right\} \leftrightarrow \left\{ \text{R-module homomorphisms } \Phi : M_1 \otimes \cdots \otimes M_n \to L \right\}$$

with respect to which the following diagram commutes:

$$\begin{array}{ccc}
M_1 \times \cdots \times M_n & \xrightarrow{\iota} & M_1 \otimes \cdots \otimes M_n \\
\varphi \downarrow & & \Phi \\
& L & \\
\end{array}$$
We have already seen examples where $M_1 \otimes_R N$ is not contained in $M \otimes_R N$ even when $M_1$ is an $R$-submodule of $M$. The next result shows in particular that (an isomorphic copy of) $M_1 \otimes_R N$ is contained in $M \otimes_R N$ if $M_1$ is an $R$-module direct summand of $M$.

**Theorem 17. (Tensor Products of Direct Sums)** Let $M, M'$ be right $R$-modules and let $N, N'$ be left $R$-modules. Then there are unique group isomorphisms

\[
(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N)
\]

\[
M \otimes_R (N \oplus N') \cong (M \otimes_R N) \oplus (M \otimes_R N')
\]

such that $(m, m') \otimes n \mapsto (m \otimes n, m' \otimes n)$ and $m \otimes (n, n') \mapsto (m \otimes n, m \otimes n')$ respectively. If $M, M'$ are also $(S, R)$-bimodules, then these are isomorphisms of left $S$-modules. In particular, if $R$ is commutative, these are isomorphisms of $R$-modules.

**Proof:** The map $(M \oplus M') \times N \rightarrow (M \otimes_R N) \oplus (M' \otimes_R N)$ defined by $((m, m'), n) \mapsto (m \otimes n, m' \otimes n)$ is well defined since $m$ and $m'$ in $M \oplus M'$ are uniquely defined in the direct sum. The map is clearly $R$-balanced, so induces a homomorphism $f$ from $(M \oplus M') \otimes N$ to $(M \otimes_R N) \oplus (M' \otimes_R N)$ with

\[f((m, m') \otimes n) = (m \otimes n, m' \otimes n).
\]

In the other direction, the $R$-balanced maps $M \times N \rightarrow (M \otimes M') \otimes_R N$ and $M' \times N \rightarrow (M \otimes M') \otimes_R N$ given by $(m, n) \mapsto (m, 0) \otimes n$ and $(m', n) \mapsto (0, m') \otimes n$, respectively, define homomorphisms from $M \otimes_R N$ and $M' \otimes_R N$ to $(M \oplus M') \otimes_R N$. These in turn give a homomorphism $g$ from the direct sum $(M \otimes_R N) \oplus (M' \otimes_R N)$ to $(M \oplus M') \otimes_R N$ with

\[g((m \otimes n_1, m' \otimes n_2)) = (m, 0) \otimes n_1 + (0, m') \otimes n_2.
\]

An easy check shows that $f$ and $g$ are inverse homomorphisms and are $S$-module isomorphisms when $M$ and $M'$ are $(S, R)$-bimodules. This completes the proof.

The previous theorem clearly extends by induction to any finite direct sum of $R$-modules. The corresponding result is also true for arbitrary direct sums. For example

\[M \otimes (\bigoplus_{i \in I} N_i) \cong \bigoplus_{i \in I} (M \otimes N_i),\]

where $I$ is any index set (cf. the exercises). This result is referred to by saying that tensor products commute with direct sums.

**Corollary 18. (Extension of Scalars for Free Modules)** The module obtained from the free $R$-module $N \cong R^n$ by extension of scalars from $R$ to $S$ is the free $S$-module $S^n$, i.e.,

\[S \otimes_R R^n \cong S^n\]

as left $S$-modules.

**Proof:** This follows immediately from Theorem 17 and the isomorphism $S \otimes_R S \cong S$ proved in Example 7 previously.

Sec. 10.4 Tensor Products of Modules 373
Corollary 19. Let R be a commutative ring and let \( M \cong R^s \) and \( N \cong R^t \) be free \( R \)-modules with bases \( m_1, \ldots, m_s \) and \( n_1, \ldots, n_t \), respectively. Then \( M \otimes_R N \) is a free \( R \)-module of rank \( st \), with basis \( m_i \otimes n_j \), \( 1 \leq i \leq s \) and \( 1 \leq j \leq t \), i.e.,
\[
R^s \otimes_R R^t \cong R^{st}.
\]

Remark: More generally, the tensor product of two free modules of arbitrary rank over a commutative ring is free (cf. the exercises).

Proof: This follows easily from Theorem 17 and the first example following Corollary 9.

Proposition 20. Suppose \( R \) is a commutative ring and \( M, N \) are left \( R \)-modules, considered with the standard \( R \)-module structures. Then there is a unique \( R \)-module isomorphism
\[
M \otimes_R N \cong N \otimes_R M
\]

mapping \( m \otimes n \) to \( n \otimes m \).

Proof: The map \( M \times N \rightarrow N \otimes M \) defined by \( (m, n) \mapsto n \otimes m \) is \( R \)-balanced. Hence it induces a unique homomorphism \( f \) from \( M \otimes N \) to \( N \otimes M \) with \( f(m \otimes n) = n \otimes m \). Similarly, we have a unique homomorphism \( g \) from \( N \otimes M \) to \( M \otimes N \) with \( g(n \otimes m) = m \otimes n \) giving the inverse of \( f \), and both maps are easily seen to be \( R \)-module isomorphisms.

Remark: When \( M = N \) it is not in general true that \( a \otimes b = b \otimes a \) for \( a, b \in M \). We shall study "symmetric tensors" in Section 11.6.

We end this section by showing that the tensor product of \( R \)-algebras is again an \( R \)-algebra.

Proposition 21. Let \( R \) be a commutative ring and let \( A \) and \( B \) be \( R \)-algebras. Then the multiplication \( (a \otimes b)(a' \otimes b') = aa' \otimes bb' \) is well defined and makes \( A \otimes_R B \) into an \( R \)-algebra.

Proof: Note first that the definition of an \( R \)-algebra shows that
\[
r(a \otimes b) = ra \otimes b = ar \otimes b = a \otimes rb = a \otimes br = (a \otimes b)r
\]
for every \( r \in R \), \( a \in A \) and \( b \in B \). To show that \( A \otimes B \) is an \( R \)-algebra the main task is, as usual, showing that the specified multiplication is well defined. One way to proceed is to use two applications of Corollary 16, as follows. The map \( \varphi : A \times B \times A \times B \rightarrow A \otimes B \) defined by \( f(a, b, a', b') = aa' \otimes bb' \) is multilinear over \( R \). For example,
\[
f(a, r_1 b_1 + r_2 b_2, a', b') = aa' \otimes (r_1 b_1 + r_2 b_2)b'
\]
\[
= aa' \otimes r_1 b_1 b' + aa' \otimes r_2 b_2 b'
\]
\[
= r_1 f(a, b_1, a', b') + r_2 f(a, b_2, a', b').
\]
By Corollary 16, there is a corresponding $R$-module homomorphism $\Phi$ from $A \otimes B \otimes A \otimes B$ to $A \otimes B$ with $\Phi(a \otimes b \otimes a' \otimes b') = aa' \otimes bb'$. Viewing $A \otimes B \otimes A \otimes B$ as $(A \otimes B) \otimes (A \otimes B)$, we can apply Corollary 16 once more to obtain a well defined $R$-bilinear mapping $\varphi'$ from $(A \otimes B) \times (A \otimes B)$ to $A \otimes B$ with $\varphi'(a \otimes b, a' \otimes b') = aa' \otimes bb'$. This shows that the multiplication is indeed well defined (and also that it satisfies the distributive laws). It is now a simple matter (left to the exercises) to check that with this multiplication $A \otimes B$ is an $R$-algebra.

**Example**

The tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is free of rank 4 as a module over $\mathbb{R}$ with basis given by $e_1 = 1 \otimes 1$, $e_2 = 1 \otimes i$, $e_3 = i \otimes 1$, and $e_4 = i \otimes i$ (by Corollary 19). By Proposition 21, this tensor product is also a (commutative) ring with $e_1 = 1$, and, for example,

$$e_4^2 = (i \otimes i)(i \otimes i) = i^2 \otimes i^2 = (-1) \otimes (-1) = (-1)(-1) \otimes 1 = 1.$$ 

Then $(e_4 - 1)(e_4 + 1) = 0$, so $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is not an integral domain.

The ring $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is an $\mathbb{R}$-algebra and the left and right $\mathbb{R}$-actions are the same: $xr = rx$ for every $r \in \mathbb{R}$ and $x \in \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. The ring $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ has a structure of a left $\mathbb{C}$-module because the first $\mathbb{C}$ is a $(\mathbb{C}, \mathbb{R})$-bimodule. It also has a right $\mathbb{C}$-module structure because the second $\mathbb{C}$ is an $(\mathbb{R}, \mathbb{C})$-bimodule. For example,

$$i \cdot e_1 = i \cdot (1 \otimes 1) = (i \cdot 1) \otimes 1 = i \otimes 1 = e_3$$

and

$$e_1 \cdot i = (1 \otimes 1) \cdot i = 1 \otimes (1 \cdot i) = 1 \otimes i = e_2.$$ 

This example also shows that even when the rings involved are commutative there may be natural left and right module structures (over some ring) that are not the same.

**EXERCISES**

Let $R$ be a ring with 1.

1. Let $f : R \rightarrow S$ be a ring homomorphism from the ring $R$ to the ring $S$ with $f(1_R) = 1_S$. Verify the details that $sr = sf(r)$ defines a right $R$-action on $S$ under which $S$ is an $(S, R)$-bimodule.

2. Show that the element "$2 \otimes 1$" is 0 in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ but is nonzero in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$.

3. Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are both left $\mathbb{R}$-modules but are not isomorphic as $\mathbb{R}$-modules.

4. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ are isomorphic left $\mathbb{Q}$-modules. [Show they are both 1-dimensional vector spaces over $\mathbb{Q}$.]

5. Let $A$ be a finite abelian group of order $n$ and let $p^k$ be the largest power of the prime $p$ dividing $n$. Prove that $\mathbb{Z}/p^k \mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow $p$-subgroup of $A$.

6. If $R$ is any integral domain with quotient field $Q$, prove that $(Q/R) \otimes_R (Q/R) = 0$.

7. If $R$ is any integral domain with quotient field $Q$ and $N$ is a left $R$-module, prove that every element of the tensor product $Q \otimes_R N$ can be written as a simple tensor of the form $(1/d) \otimes n$ for some nonzero $d \in R$ and some $n \in N$.

8. Suppose $R$ is an integral domain with quotient field $Q$ and let $N$ be any $R$-module. Let $U = R^x$ be the set of nonzero elements in $R$ and define $U^{-1}N$ to be the set of equivalence classes of ordered pairs of elements $(u, n)$ with $u \in U$ and $n \in N$ under the equivalence relation $(u, n) \sim (u', n)$ if and only if $u'n = un'$ in $N$. 

Sec. 10.4 Tensor Products of Modules 375
Prove that $U^{-1}N$ is an abelian group under the addition defined by $(u_1, n_1) + (u_2, n_2) = (u_1u_2, u_2n_1 + u_1n_2)$. Prove that $r(u, n) = (ru, rn)$ defines an action of $R$ on $U^{-1}N$ making it into an $R$-module. [This is an example of localization considered in general in Section 4 of Chapter 15, cf. also Section 5 in Chapter 7.]

(b) Show that the map from $Q \times N$ to $U^{-1}N$ defined by sending $(a/b, n)$ to $(b, an)$ for $a \in R, b \in U, n \in N$, is an $R$-balanced map, so induces a homomorphism $f$ from $Q \otimes_R N$ to $U^{-1}N$. Show that the map $g$ from $U^{-1}N$ to $Q \otimes_R N$ defined by $g((u, n)) = (1/u) \otimes n$ is well defined and is an inverse homomorphism to $f$. Conclude that $Q \otimes_R N \cong U^{-1}N$ as $R$-modules.

(c) Conclude from (b) that $(1/d) \otimes n = 0$ in $Q \otimes_R N$ if and only if $rn = 0$ for some nonzero $r \in R$.

(d) If $A$ is an abelian group, show that $Q \otimes_Z A = 0$ if and only if $A$ is a torsion abelian group (i.e., every element of $A$ has finite order).

9. Suppose $R$ is an integral domain with quotient field $Q$ and let $N$ be any $R$-module. Let $Q \otimes_R N$ be the module obtained from $N$ by extension of scalars from $R$ to $Q$. Prove that the kernel of the $R$-module homomorphism $i : N \rightarrow Q \otimes_R N$ is the torsion submodule of $N$ (cf. Exercise 8 in Section 1). [Use the previous exercise.]

10. Suppose $R$ is commutative and $N \cong R^n$ is a free $R$-module of rank $n$ with $R$-module basis $e_1, \ldots, e_n$.

(a) For any nonzero $R$-module $M$ show that every element of $M \otimes N$ can be written uniquely in the form $\sum_{j=1}^n m_i \otimes e_i$ where $m_i \in M$. Deduce that if $\sum_{i=1}^n m_i \otimes e_i = 0$ in $M \otimes N$ then $m_i = 0$ for $i = 1, \ldots, n$.

(b) Show that if $\sum m_i \otimes n_i = 0$ in $M \otimes N$ where the $n_i$ are merely assumed to be $R$-linearly independent then it is not necessarily true that all the $m_i$ are 0. [Consider $R = \mathbb{Z}, n = 1, M = \mathbb{Z}/2\mathbb{Z}$, and the element $1 \otimes 2$.]

11. Let $(e_1, e_2)$ be a basis of $V = \mathbb{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_R V$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in \mathbb{R}^2$.

12. Let $V$ be a vector space over the field $F$ and let $u, v'$ be nonzero elements of $V$. Prove that $u \otimes v' = v' \otimes v$ in $V \otimes_F V$ if and only if $v = au'$ for some $a \in F$.

13. Prove that the usual dot product of vectors defined by letting $(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n)$ be $a_1b_1 + \cdots + a_nb_n$ is a bilinear map from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$.

14. Let $I$ be an arbitrary nonempty index set and for each $i \in I$ let $N_i$ be a left $R$-module. Let $M$ be a right $R$-module. Prove the group isomorphism: $M \otimes (\oplus_{i \in I} N_i) \cong \oplus_{i \in I} (M \otimes N_i)$, where the direct sum of an arbitrary collection of modules is defined in Exercise 20, Section 3. [Use the same argument as for the direct sum of two modules, taking care to note where the direct sum hypothesis is needed — cf. the next exercise.]

15. Show that tensor products do not commute with direct products in general. [Consider the extension of scalars from $\mathbb{Z}$ to $\mathbb{Q}$ of the direct product of the modules $M_i = \mathbb{Z}/2^i\mathbb{Z}$, $i = 1, 2, \ldots$]

16. Suppose $R$ is commutative and let $I$ and $J$ be ideals of $R$, so $R/I$ and $R/J$ are naturally $R$-modules.

(a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 \mod I) \otimes (r \mod J)$.

(b) Prove that there is an $R$-module isomorphism $R/I \otimes_R R/J \cong R/(I + J)$ mapping $(r \mod I) \otimes (r' \mod J)$ to $rr' \mod (I + J)$.

17. Let $I = (2, x)$ be the ideal generated by 2 and $x$ in the ring $R = \mathbb{Z}[x]$. The ring $\mathbb{Z}/2\mathbb{Z} = R/I$ is naturally an $R$-module annihilated by both 2 and $x$. 

376
(a) Show that the map \( \varphi : I \times I \to \mathbb{Z}/2\mathbb{Z} \) defined by
\[
\varphi(a_0 + a_1 x + \cdots + a_n x^n, b_0 + b_1 x + \cdots + b_m x^m) = \frac{a_0}{2} b_1 \mod 2
\]
is \( R \)-bilinear.
(b) Show that there is an \( R \)-module homomorphism from \( I \otimes_R I \to \mathbb{Z}/2\mathbb{Z} \) mapping
\[
p(x) \otimes q(x) \text{ to } \frac{p(0)}{2} q'(0) \text{ where } q' \text{ denotes the usual polynomial derivative of } q.
\]
(c) Show that \( 2 \otimes x \neq x \otimes 2 \) in \( I \otimes_R I \).

18. Suppose \( I \) is a principal ideal in the integral domain \( R \). Prove that the \( R \)-module \( I \otimes_R I \) has no nonzero torsion elements (i.e., \( rm = 0 \) with \( 0 \neq r \in R \) and \( m \in I \otimes_R I \) implies that \( m = 0 \)).

19. Let \( I = (2, x) \) be the ideal generated by 2 and \( x \) in the ring \( R = \mathbb{Z}[x] \) as in Exercise 17. Show that the nonzero element \( 2 \otimes x - x \otimes 2 \) in \( I \otimes_R I \) is a torsion element. Show in fact that \( 2 \otimes x - x \otimes 2 \) is annihilated by both 2 and \( x \) and that the submodule of \( I \otimes_R I \) generated by \( 2 \otimes x - x \otimes 2 \) is isomorphic to \( R/I \).

20. Let \( I = (2, x) \) be the ideal generated by 2 and \( x \) in the ring \( R = \mathbb{Z}[x] \). Show that the element \( 2 \otimes 2 + x \otimes x \) in \( I \otimes_R I \) is not a simple tensor, i.e., cannot be written as \( a \otimes b \) for some \( a, b \in I \).

21. Suppose \( R \) is commutative and let \( I \) and \( J \) be ideals of \( R \).
   (a) Show there is a surjective \( R \)-module homomorphism from \( I \otimes_R J \) to the product ideal \( IJ \) mapping \( i \otimes j \) to the element \( ij \).
   (b) Give an example to show that the map in (a) need not be injective (cf. Exercise 17).

22. Suppose that \( M \) is a left and a right \( R \)-module such that \( rm = mr \) for all \( r \in R \) and \( m \in M \). Show that the elements \( r_1 r_2 \) and \( r_2 r_1 \) act the same on \( M \) for every \( r_1, r_2 \in R \). (This explains why the assumption that \( R \) is commutative in the definition of an \( R \)-algebra is a fairly natural one.)

23. Verify the details that the multiplication in Proposition 19 makes \( A \otimes_R B \) into an \( R \)-algebra.

24. Prove that the extension of scalars from \( Z \) to the Gaussian integers \( \mathbb{Z}[i] \) of the ring \( \mathbb{R} \) is isomorphic to \( \mathbb{C} \) as a ring: \( \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C} \) as rings.

25. Let \( R \) be a subring of the commutative ring \( S \) and let \( x \) be an indeterminate over \( S \). Prove that \( S[x] \) and \( S \otimes_R R[x] \) are isomorphic as \( S \)-algebras.

26. Let \( S \) be a commutative ring containing \( R \) (with \( 1_S = 1_R \)) and let \( x_1, \ldots, x_n \) be independent indeterminates over the ring \( S \). Show that for every ideal \( I \) in the polynomial ring \( R[x_1, \ldots, x_n] \) that \( S \otimes_R (R[x_1, \ldots, x_n]/I) \cong S[x_1, \ldots, x_n]/IS[x_1, \ldots, x_n] \) as \( S \)-algebras.

The next exercise shows the ring \( C \otimes_R C \) introduced at the end of this section is isomorphic to \( C \times C \). One may also prove this via Exercise 26 and Proposition 16 in Section 9.5, since \( C \cong \mathbb{R}[x]/(x^2 + 1) \). The ring \( C \times C \) is also discussed in Exercise 23 of Section 1.

27. (a) Write down a formula for the multiplication of two elements \( a \cdot 1 + b \cdot e_2 + c \cdot e_3 + d \cdot e_4 \) and \( a' \cdot 1 + b' \cdot e_2 + c' \cdot e_3 + d' \cdot e_4 \) in the example \( A = C \otimes_R C \) following Proposition 21 (where \( 1 = 1 \otimes 1 \) is the identity of \( A \)).
   (b) Let \( e_1 = \frac{1}{2} (1 \otimes 1 + i \otimes i) \) and \( e_2 = \frac{1}{2} (1 \otimes 1 - i \otimes i) \). Show that \( e_1 e_2 = 0, e_1 + e_2 = 1, \) and \( e_1^2 = e_1 \) for \( j = 1, 2 \). \((e_1 \text{ and } e_2 \text{ are called orthogonal idempotents in } A)\). Deduce that \( A \) is isomorphic as a ring to the direct product of two principal ideals: \( A \cong A e_1 \times A e_2 \) (cf. Exercise 1, Section 7.6).
   (c) Prove that the map \( \varphi : C \times C \to C \times C \) by \( \varphi(z_1, z_2) = (z_1 z_2, z_1 \overline{z_2}) \), where \( \overline{z_2} \) denotes the complex conjugate of \( z_2 \), is an \( R \)-bilinear map.