MATH 465/565: More dimension problems

These problems are provided for your edification and enjoyment only; do not write them up and turn them in. There are at least two "dimension problems" on your closed-book, closed-notes, five-hour final exam, which is due at 5:00pm on May 4, 2011 in my office.

We work over an algebraically closed field k.

1. Let $\mathbb{P}(R_d) = \mathbb{P}^{\binom{n+d}{d}-1}$ be the space of nonzero homogeneous polynomials of degree d in $R = k[X_0, \ldots, X_n]$, mod scalars. Let $X \subseteq \mathbb{P}(R_d)$ be the set of polynomials that have a square factor, i.e. $F = G^2 H$ for some $G, H \in R$ with G non-constant. Show that X is closed in $\mathbb{P}(R_d)$ and compute its dimension.

2. (a) Show that the locus of symmetric $n \times n$ matrices of rank at most r is irreducible and compute its dimension.

   (b) Show that the locus of skew-symmetric $n \times n$ matrices of rank at most $2m$ is irreducible and compute its dimension.

3. Let $G(r, n)$ be the Grassmannian of r-planes in k^n, and let W be a fixed d-plane in k^n with $r + d \geq n$. Let S_i be the subset of $G(r, n)$ consisting of r-planes V for which $\dim(V + W) \leq n - i$. Show that S_i is an irreducible closed subset of $G(r, n)$ and compute its dimension.

4. Let $\mathbb{G}(1, 3)$ be the Grassmannian of lines in $\mathbb{P}^3$.

   (a) Show that the subset $X \subseteq \mathbb{G}(1,3) \times \mathbb{G}(1,3)$

   $$X = \{(\ell_1, \ell_2) : \ \ell_1 \cap \ell_2 \neq \varnothing\}$$

   is closed and irreducible. Compute its dimension.

   (b) Show that the subset $Y \subseteq \mathbb{G}(1,3) \times \mathbb{G}(1,3) \times \mathbb{G}(1,3)$

   $$Y = \{(\ell_1, \ell_2, \ell_3) : \ \ell_1 \cap \ell_2 \neq \varnothing, \ \ell_1 \cap \ell_3 \neq \varnothing, \ \ell_2 \cap \ell_3 \neq \varnothing\}$$

   is reducible. Find its irreducible components and compute their dimensions.

5. Let V and W be vector spaces of dimensions m and n respectively, and let $A \subseteq V$ be a subspace of dimension l. Let $\mathbb{P}(\mathrm{Hom}(V, W)) \cong \mathbb{P}^{mn-1}$ be the projective space of non-zero linear maps $\phi \colon V \to W$ mod scalars, and for any integer $j \leq l$, let

   $$\Psi_j = \left\{\phi \in \mathbb{P}^{nm-1} : \ \mathrm{rank}(\phi|_A) \leq j\right\}.$$

   Show that Ψ_j is an irreducible closed subset of $\mathbb{P}^{nm-1}$ and compute its dimension.

6. Let $X \subseteq \mathbb{A}^{2n^2}$ be the variety of pairs (A, B), where A and B are $n \times n$ matrices such that $AB = BA = 0$. Find the irreducible components of X and show that each irreducible component of X has dimension n^2.

7. Let X be a n-dimensional closed subset of $\mathbb{P}^m$. In the Grassmannian $\mathbb{G}(1,m)$ of lines in $\mathbb{P}^m$, let $S(X)$ be the set of lines which are secant to X, i.e. which meet X in at least two distinct points.

   (a) Show that if X is not a linear subspace of $\mathbb{P}^m$, then the closure of $S(X)$ in $\mathbb{G}(1,m)$ is irreducible of dimension $2n$.

   (b) Let $C(X) = \bigcup_{\ell \in S(X)} \ell \subseteq \mathbb{P}^m$. Show that the closure of $C(X)$ is irreducible of dimension at most $2n+1$.

   (c) Find an example of an X for which $\dim C(X) < \min\{m, 2n+1\}$

8. Let $X \subseteq \mathbb{P}^m$ be an irreducible projective variety of dimension n, and let $F \subseteq \mathbb{G}(1,m)$ be the set of lines contained in X.

   (a) Show that F is a closed subset of $\mathbb{G}(1,m)$.

   (b) Show that $\dim F \leq 2n-2$, with equality holding if and only if $X \subset \mathbb{P}^m$ is an n-plane.