MATH 465/565: Midterm Exam

This is an open-book, open-notes exam. You may not discuss these problems with anyone other than the instructor. There is no time limit and you don't need to take the exam in one sitting, but the exam is due in class on Friday, February 25, 2011.

We work over an algebraically closed field k.

1. Assume char $k = 0$. Let $X = V(x^2 - yz, xz - x, y^3 z - y^3 - 2yz + 2y) \subseteq \mathbb{A}^3$. Find the irreducible components of X.

2. Let $X \subseteq \mathbb{A}^n$ be an irreducible hypersurface defined by vanishing of the polynomial $f = f_{d-1} + f_d$, where $f_{d-1}, f_d \in k[x_1, \ldots, x_n]$ are non-zero homogeneous polynomials of degrees $d-1$ and d. Show that X is birational to $\mathbb{A}^{n-1}$.

3. Assume char $k = 0$. Let $C \subseteq \mathbb{A}^2$ be the affine plane curve $y^2 = x^5 + x^3$.

   (a) Find all the singular points of C and compute their multiplicities.

   (b) Find an equation for the projective closure $\overline{C}$ of C in $\mathbb{P}^2 \supseteq \mathbb{A}^2$.

   (c) Find all the singular points of $\overline{C}$ and compute their multiplicities.

4. Show that is $X \subseteq \mathbb{P}^n$ is a quasiprojective variety, then X is an open subset of its projective closure $\overline{X} \subseteq \mathbb{P}^n$.

5. Given two ideals $I, J \subseteq k[x_1, \ldots, x_n]$, the *ideal quotient* of I by J is

$$(I : J) = \{g \in k[x_1, \ldots, x_n] \colon \; gJ \subseteq I\}$$

   Let $X, Y \subseteq \mathbb{A}^n$ be closed. Show that $I(X \backslash Y) = (I(X) : I(Y))$.[1]

6. Assume char $k \neq 2, 3$. Let $S \subset \mathbb{P}^3$ be the cubic surface $X^3 + Y^3 + Z^3 + W^3 = 0$. Let $f \colon S \dashrightarrow \mathbb{P}^1$ be the rational map defined by

$$f([X, Y, Z, W]) = [X^2 - XY + Y^2, WZ + W^2].$$

   Determine where f is regular.

[1] By $X \backslash Y$, we mean $\{x \in X \colon \; x \notin Y\}$. This is in fact a quasi-projective variety, but here we're just thinking of it as a subset of $\mathbb{A}^n$.