MATH 465/565: Midterm Exam - solutions to selected exercises

We work over an algebraically closed field k.

2. Let $X \subseteq \mathbb{A}^n$ be an irreducible hypersurface defined by vanishing of the polynomial $f = f_{d-1} + f_d$, where $f_{d-1}, f_d \in k[x_1, \ldots, x_n]$ are non-zero homogeneous polynomials of degrees $d-1$ and d. Show that X is birational to $\mathbb{A}^{n-1}$.

Define $\phi \colon A^n \dashrightarrow \mathbb{A}^{n-1}$ by

$$(t_1, \ldots, t_{n-1}) = \phi(x_1, \ldots, x_n) = \left(\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}\right),$$

which is regular for $x_n \neq 0$; it is intuitively obvious[1] that X is not contained in the hyperplane $x_n = 0$, since $X = V(f)$ is itself an irreducible hypersurface and f is not a scalar multiple of x_n, so ϕ ought to restrict to a rational map on X.

Let ϕ_X be the restriction of ϕ to $X \cap \{x_n \neq 0\}$. For $t = (t_1, \ldots, t_{n-1}) \in A^{n-1}$, we consider the preimage $\phi_X^{-1}(t)$. If $\phi(x_1, \ldots, x_n) = t$, then $t_i = \frac{x_i}{x_n}$, hence $x_i = t_i x_n$ for $1 \leq i \leq n-1$. For such a point $x = (x_1, \ldots, x_n)$ that lies in X, we have

$$\begin{aligned} 0 = f(x_1, \ldots, x_n) &= f_{d-1}(t_1 x_n, \ldots, t_{n-1} x_n, x_n) + f_d(t_1 x_n, \ldots, t_{n-1} x_n, x_n) \\ &= x_n^{d-1} f_{d-1}(t_1, \ldots, t_{n-1}, 1) + x_n^d f_d(t_1, \ldots, t_{n-1}, 1) \\ &= x_n^{d-1} \left(f_{d-1}\left(t_1, \ldots, t_{n-1}\right) + x_n f_d\left(t_1, \ldots, t_{n-1}, 1\right)\right), \end{aligned}$$

so for $x_n \neq 0$ and $f_d(t_1, \ldots, t_{n-1}, 1) \neq 0$, we have a unique such x, namely the point defined by $x_n = -\dfrac{f_{d-1}(t_1, \ldots, t_{n-1})}{f_d(t_1, \ldots, t_{n-1}, 1)}$ and $x_i = t_i x_n$ for $1 \leq i \leq n-1$.

However, since f_d is a non-zero homogeneous polynomial, its dehomogenization $f_d(t_1, \ldots, t_{n-1}, 1)$ is also a non-zero polynomial, so the set $U = \{(t_1, \ldots, t_{n-1}) \in A^{n-1} : \ f_d(t_1, \ldots, t_{n-1}, 1) \neq 0\}$ is a nonempty (hence dense) open subset of $\mathbb{A}^{n-1}$, and letting $V = \phi_X^{-1}(U)$, we see that for $t \in V$, there is a unique $x \in U$ with $\phi_X(x) = t$, and in particular $V \subset X$ is nonempty since U is nonempty, and ϕ_X is a well-defined rational map on X that induces a bijection from V to U. The above computation of x given t shows that ϕ_X^{-1} is regular on U, so ϕ_X is a birational map between X and $\mathbb{A}^{n-1}$, as desired.

[1] Unfortunately, making this intuition more precise in general is based on the notion of dimension, which we haven't discussed yet (but note that in the special case of $\mathbb{A}^2$, we have shown that for $f \in k[x, y]$ irreducible and $g \in k[x, y]$ non-constant, we have either $V(f)$ is an irreducible component of $V(g)$ in the case that f divides g, or else $V(f)$ and $V(g)$ meet in only finitely many points). Fortunately, in this case we're going to construct plenty of points of X outside of the hyperplane $x_n \neq 0$ shortly.

6. Assume char $k \neq 2, 3$. Let $S \subset \mathbb{P}^3$ be the cubic surface $X^3 + Y^3 + Z^3 + W^3 = 0$. Let $f\colon S \dashrightarrow \mathbb{P}^1$ be the rational map defined by

$$f([X, Y, Z, W]) = [X^2 - XY + Y^2, WZ + W^2].$$

Determine where f is regular.

Let ζ and $\bar{\zeta}$ be the primitive sixth roots of unity.[2] Then we have

$$f([X, Y, Z, W]) = [(X - \zeta Y)(X - \bar{\zeta} Y), W(Z + W)],$$

which is defined (and is certainly regular where defined) as long as the two components aren't both zero. Thus, this expression for f is undefined at the points of S where either $W = 0$ or $W = -Z$ and either $Y = \zeta X$ or $Y = \bar{\zeta} X$. Since

$$X^3 + Y^3 = (X - \zeta Y)(X - \bar{\zeta} Y),$$

any point of S satisfying these conditions must also have $W^3 + Z^3 = 0$, so in the $W = 0$ case we must also have $Z = 0$. This then yields two points: $[1, \zeta, 0, 0]$ and $[1, \bar{\zeta}, 0, 0]$.

In the $W = -Z$ case, we still have $Y = \zeta X$ or $Y = \bar{\zeta} X$, and the equation $X^3 + Y^3 + Z^3 + W^3 = 0$ is automatically satisfied. We thus get two lines, defined parametrically as those points of the forms

$$[A, \zeta A, B, -B] \quad \text{and} \quad [A, \bar{\zeta} A, B, -B],$$

for $[A, B] \in \mathbb{P}^1$. The two points we found in the $W = 0$ case are actually just points on these lines.

Now, while this shows that the given expression for f is undefined on these lines, it does *not* show that f is not regular on these lines: f may have other expressions as two homogeneous polynomials of the same degree that are defined at some of those points. To find another expression in this case, we use the equation for S, together with the fact that $X^2 - XY + Y^2$ divides $X^3 + Y^3$ and $Z + W$ divides $Z^3 + W^3$:

$$\begin{aligned} f = [X^2 - XY + Y^2, WZ + W^2] &= [X^3 + Y^3, (X + Y)(Z + W)W] \\ &= [-Z^3 - W^3, (X + Y)(Z + W)W] \\ &= [-Z^2 + ZW - W^2, (X + Y)W]. \end{aligned}$$

To find where this second expression for f is undefined, we must solve the system: $X^3 + Y^3 + Z^3 + W^3 = 0$, $Z^2 - ZW + W^2 = 0$, and $(X + Y)W = 0$. This goes more or less the same as above. Now, the case $X + Y = 0$ yields the two lines

$$[A, -A, B, \zeta B] \quad \text{and} \quad [A, -A, D, \bar{\zeta} D],$$

whereas in the case $W = 0$, we must have $Z = 0$ as well, and we get three points $[1, \zeta, 0, 0]$, $[1, \bar{\zeta}, 0, 0]$, and $[1, -1, 0, 0]$, the last of which is the point of intersection of the two lines.

Now, the rational function is certainly regular wherever either expression is defined, so we find that f is regular except at the points $[1, \zeta, 0, 0]$ and $[1, \bar{\zeta}, 0, 0]$. We now show that f is indeed not

[2] Here we're already using the char $k \neq 3$ hypothesis. The char $k \neq 2$ hypothesis won't actually be needed (but it does make it correct to use the quadratic formula and say $\zeta = \frac{1 \pm \sqrt{-3}}{2}$; note though that $e^{\pm 2\pi i/6}$ doesn't technically make sense here).

regular at these points. Assume that f is regular at $[1, \zeta, 0, 0]$. Then f is still regular when restricted to the irreducible curve $C \subset S$ defined by $X^3 + Y^3 + Z^3 = 0$ and $W = 0$. But, except at the points $[1, \zeta, 0, 0]$ and $[1, \bar{\zeta}, 0, 0]$, the restriction of f to C is the constant function $[1, 0]$. Thus, since f is regular and C is irreducible, we must have $f([1, \zeta, 0, 0]) = [1, 0]$.

On the other hand, the restriction of f to the line $[A, \zeta A, B, -B]$ is also a regular function. Let $g\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the composition of f with the parameterization of the line. Then

$$\begin{aligned} g([A, B]) &= \left[-(-B)^2 + (-B)B - B^2, (A + \zeta A)(-B)\right] \\ &= \left[-3B^2, -(1 + \zeta)AB\right] \\ &= \left[3B, (1 + \zeta)A\right], \end{aligned}$$

so we find that $f([1, \zeta, 0, 0]) = g([1, 0]) = [0, 1 + \zeta] = [0, 1]$. But, restricting to C, we had found that $f([1, \zeta, 0, 0]) = [1, 0]$, a contradiction. Similarly, we can show that f is not regular at $[1, \bar{\zeta}, 0, 0]$. Thus the set of points where f is regular is $S \backslash \{[1, \zeta, 0, 0], [1, \bar{\zeta}, 0, 0]\}$.