MATH 465/565: Homework 1

Due Friday, January 21, 2011

1. Suppose that $f, g \in k[x, y]$ where f is irreducible, $f \nmid g$, and $f \notin k[x]$. Show that if we regard f and g as elements of $k(x)[y]$, it is still the case that f is irreducible and $f \nmid g$. [Hint: use Gauss's Lemma.]

2. Suppose that $C = V(f)$ is an irreducible affine plane curve of degree $n \geq 2$ with infinitely many points, where $f = f_{n-1} + f_n$, with f_j homogeneous of degree j. Show that C is rational. [Hint: show that if your parameterization were constant, then f would be reducible.]

3. Show that if k is an algebraically closed field and $f \in k[x, y]$ is a homogeneous polynomial of degree n, then f factors into a product of n homogeneous linear polynomials.

4. Suppose char $k \neq 2, 3$. Show that the affine plane curve $y^2 = x^3 + px + q$ over k is singular if and only if the polynomial $x^3 + px + q$ has a multiple root in k.

5. Suppose that C is an irreducible affine plane curve of degree 3 that has infinitely many points. Show that C can have at most one singular point and that a singular point of C must have multiplicity two. What if C is reducible?

6. Show that if P_1 and P_2 are distinct points of an affine plane curve C, then there exists a rational function $f \in k(C)$ so that $f(P_1) = 0$ and $f(P_2) = 1$.