MATH 465/565: Homework 10

Due Friday, April 15, 2011

You should try to do as many of these problems as possible, but **write up and turn in solutions to exactly <u>six</u> of them**. We work over an algebraically closed field k.

1. Let $X \subseteq \mathbb{P}^n$ be finite.
   (a) Show that for all sufficiently large d, for every $x \in X$ there exists a homogeneous polynomial F of degree d such that $F(x) \neq 0$ but $F(x') = 0$ for $x' \in X \backslash \{x\}$.
   (b) Show that the Hilbert function $h_X(d)$ is constant for $d \gg 0$.

2. Let $\mathbb{P}^9 = \mathbb{P}(k[X_0, X_1, X_2, X_3]_2)$ be the space of quadric surfaces in $\mathbb{P}^3$, and let Γ be the incidence variety[1]

$$\Gamma_2 = \{(S, \ell) \in \mathbb{P}^9 \times \mathbb{G}(1,3) : \ \ell \subseteq S\},$$

   and let $p_1 \colon \Gamma_2 \to \mathbb{P}^9$ and $p_2 \colon \Gamma \to \mathbb{G}(1,3)$ be the projections.
   (a) Show that Γ_2 is a closed subset of $\mathbb{P}^9 \times \mathbb{G}(1,3)$.
   (b) Using the projection p_2, show that Γ_2 is irreducible and compute its dimension.
   (c) Describe, up to isomorphism, all the possible fibers $p_1^{-1}(S)$ for $S \in \mathbb{P}^9$.

3. Let $\mathbb{P}^N$ be the space of all quartic surfaces, i.e. hypersurfaces in $\mathbb{P}^3$ of degree 4, and let $\Gamma_4 \subseteq \mathbb{P}^N \times \mathbb{G}(1,3)$ be the incidence variety

$$\Gamma_4 = \{(S, \ell) \in \mathbb{P}^N \times \mathbb{G}(1,3) : \ \ell \subseteq S\}.$$

   (a) What is N?
   (b) Show that Γ_4 is an irreducible projective variety and compute its dimension.
   (c) Let $V \subseteq \mathbb{P}^N$ be the set of quartic surfaces in $\mathbb{P}^3$ that contain a line. Show that V is closed and has codimension at least 1 in $\mathbb{P}^N$.
   (d) Find a quartic surface in V that contains only finitely many lines. Explain why this implies that V has codimension exactly 1 in $\mathbb{P}^N$.
   (e) Show that there is a polynomial Φ in the coefficients of an arbitrary quartic $F(X_0, X_1, X_2, X_3)$, such that the quartic surface $F = 0$ contains a line if and only if $\Phi(F) = 0$.

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[1] In the notation of problem 6 on the previous homework, where $R = k[X_0, \ldots, X_3]$, we could write instead:

$$\Gamma_2 = \{([F], \ell) \in \mathbb{P}(R_2) \times \mathbb{G}(1,3) : \ F|_\ell \equiv 0\}.$$

4. Let $\mathbb{P}^9$ be the space of all cubic curves in $\mathbb{P}^2$. Let $S \subseteq \mathbb{P}^9$ be the locus of singular cubic curves. Show that S is closed and irreducible and compute its dimension. Does there exist a single polynomial Δ in the coefficients of an arbitrary plane cubic so that the cubic is singular if and only if Δ vanishes?

5. Let $X \subseteq \mathbb{P}^n$ be an irreducible projective variety. For $m \leq n - \dim X$, we consider the set

$$\mathscr{C}_m(X) = \{\Lambda \in \mathbb{G}(m,n) : \ \Lambda \cap X \neq \varnothing\}.$$

Show that $\mathscr{C}_m(X)$ is an irreducible closed subset of $\mathbb{G}(m,n)$ and compute its dimension.

6. (a) Compute the Hilbert function of the twisted cubic in $\mathbb{P}^3$.
   (b) Compute the Hilbert function of the Veronese image $\nu_m(\mathbb{P}^n) \subseteq \mathbb{P}^{\binom{n+m}{m}-1}$.
   [Hint: what does a degree d homogeneous polynomial on $\mathbb{P}^{\binom{n+m}{m}-1}$ pull back to on $\mathbb{P}^n$?]

7. Assume $\operatorname{char} k \neq 2$. Show that the orthogonal group $O(n)$ over k is a quasiprojective variety and compute its dimension. Is $O(n)$ irreducible?

8. (Extra Credit) A field F is *quasi-algebraically-closed*[2] if for every homogeneous polynomial $g \in F[x_1, \ldots x_n]$ of degree d with $d < n$, there are elements $a_1, \ldots, a_n \in F$, not all zero, so that $g(a_1, \ldots, a_n) = 0$.[3]
   (a) Show that $\mathbb{F}_p$ is quasi-algebraically-closed. [Hint: consider $\sum_{a \in \mathbb{F}_p^n} \left(1 - g(a)^{p-1}\right)$.]
   (b) Show that if F is quasi-algebraically-closed and $F \subseteq K$ is an algebraic extension, then K is quasi-algebraically-closed. [Hint: use the norm.]

[2] Such fields are also called C_1. More generally, a field is C_i if every homogeneous polynomial of degree d in $n > d^i$ variables has a non-trivial zero.

[3] Note that we can restate Tsen's Theorem (p. 73) in this language: if k is algebraically closed, then $k(t)$ is quasi-algebraically-closed.