MATH 465/565: Homework 11

Due Friday, April 22, 2011

We work over an algebraically closed field k.

1. Let $F, G \in k[X_0, \dots, X_n]$ be homogeneous of degrees a and b with no common factor, and set $I = \langle F, G \rangle$. Let $R_d = k[X_0, \dots, X_n]_d$ be the vector space of degree d homogeneous polynomials in $n+1$ variables[1], and let $I_d = I \cap R_d$.
   (a) Show that for each degree d, there is an exact sequence of k-vector spaces
   $$0 \longrightarrow R_{d-a-b} \longrightarrow R_{d-a} \oplus R_{d-b} \longrightarrow I_d \longrightarrow 0.$$
   (b) Compute the function $h_I(d) = \dim(R_d/I_d)$.[2]
   (c) Show that for all sufficiently large d, the function $h_I(d)$ is equal to a polynomial in d. Find the leading term of that polynomial.
   (d) Suppose that $X \subseteq \mathbb{P}^2$ is a dimension zero complete intersection, i.e. $I(X) = \langle F, G \rangle$ for $F, G \in k[X_0, X_1, X_2]$ homogeneous of degrees a and b with no common factor. Compute the Hilbert function $h_X(d)$. How many points are there in X?

2. (a) Let $I \subseteq k[X_0, \dots, X_n]$ be an ideal and $X \subset \mathbb{P}^n$ its zero locus. Show that $h_I(d) \geq h_X(d)$ for all d.
   (b) Prove the "weak Bézout theorem in the plane": let $F, G \in k[X_0, X_1, X_2]$ be homogeneous of degrees a and b with no common factor. Then F and G have at most ab common zeros in $\mathbb{P}^2$.

3. Let $C_n \subseteq \mathbb{A}^2$ be the curve given by $y^2 = x^{2n+1}$. Show that the singularity of C_n can be resolved by n successive blow-ups.

4. Let $V \subseteq \mathbb{A}^3$ be the quadric cone $z^2 = xy$. Prove that the blowup $\pi \colon Bl_O V \to V$ with center at the origin resolves the singularity of V. Also, show that the exceptional divisor $\pi^{-1}(O)$ is a nonsingular rational curve.

5. Prove that a quadric (degree 2) hypersurface in $\mathbb{P}^n$ with a singular point is a cone. [Hint: the tangent space of a hypersurface $X = V(F)$ at p is the union of the lines L that are tangent to X at p in the sense that the restriction of $F|_L$ has a multiple root at p.]

6. Show that if a hypersurface $X \subseteq \mathbb{P}^n$ of degree $d > 1$ contains a linear subspace $\Lambda \cong \mathbb{P}^r$ with $r \geq n/2$, then X is singular. [Hint: write the equation for X in a suitable coordinate system.]

[1] If $d < 0$, let R_d be the zero vector space. Recall that for $d \geq 0$, $\dim R_d = \binom{n+d}{d} = \binom{n+d}{n}$.

[2] This is called the *Hilbert function* of the ideal I.

7. (Extra Credit) If $U \subseteq \mathbb{C}^n$ is an open subset (in the analytic topology) containing the origin 0, then the *blow-up* of U at 0 is

$$Bl_0(U) = \left\{(((x_1, \ldots, x_n), [Y_1, \ldots, Y_n]) \in U \times \mathbb{P}^{n-1}_{\mathbb{C}} : \quad x_i Y_j - x_j Y_i = 0 \text{ for } 1 \le i, j \le n\right\}.$$

More generally, if M is an n-dimensional complex and p be a point of M, the *blow-up* $Bl_p(M)$ of M at p is obtained by taking a local holomorphic coordinate chart $U \to M$ sending 0 to p and replacing it with $Bl_0(U)$. Then $Bl_p(M)$ is an n-dimensional complex manifold (which does not depend, up to isomorphism, on the choice of local coordinate near p). Compute the homology groups of $Bl_p(M)$ in terms of those of M.