MATH 465/565: Homework 2

Due Friday, January 28, 2011

1. Let $C = V(f)$ be an irreducible affine curve with infinitely many points, let P be a singular point of C and let L be a line passing through P. Show that the intersection multiplicity of C and L at P is at least the multiplicity of the singularity. Are these multiplicities always equal?

2. Prove that if $\alpha, \beta \in k^\times$ are non-zero elements of k and $g \in k[x]$, then the map

$$f(x, y) = (\alpha x, \beta y + g(x))$$

   is an automorphism of $\mathbb{A}^2$ (i.e. show that it is a bijection and that its inverse can also be expressed as a pair of polynomials).

3. Find two different local parameters at the origin for the affine plane curve $C = V(y - x^2)$, and verify directly that each satisfies the defining properties of a local parameter.

4. Find a local parameter[1] at $P = [0, 0, 1]$ on the elliptic curve $Y^2Z = X^3 + XZ^2$ in $\mathbb{P}^2$.

5. Let $F \in k[X, Y, Z]$ be a homogeneous polynomial of degree n over an arbitrary field k. Prove that

$$X\frac{\partial F}{\partial X} + Y\frac{\partial F}{\partial Y} + Z\frac{\partial F}{\partial Z} = nF.$$

6. (a) Show that if k has characteristic p, then the curve $y = x^{p+1}$ is non-singular, and every tangent line to the curve passes through the origin.

   (b) Suppose that k has characteristic 0 and $C = V(f)$ is an irreducible affine plane curve over k. Show that if $P \in \mathbb{A}^2$ is not a singular point of C, then there are there are at most a finite number of lines that pass through P and are tangent to C.

7. Let $P_1, \ldots, P_5$ be five distinct points in the affine plane $\mathbb{A}^2$. Prove that if no four of the points P_i are collinear, then there exists a polynomial $f \in k[x, y]$ of degree two such that $f(P_1) = 1$ and $f(P_i) = 0$ for $2 \leq i \leq 5$. What if four of the points are collinear?

8. (Extra Credit) We showed in class that if char $k \neq 2, 3$ and $x^3 + px + q$ does not have a multiple root, then the fields $k\left(x, \sqrt{x^3 + px + q}\right)$ and $k(t)$ are not isomorphic *over* k (i.e. there exists no isomorphism of fields between them that preserves every element of k).

   Find a field k with char $k \neq 2, 3$ so that $k\left(x, \sqrt{x^3 + px + q}\right) \cong k(t)$ as fields.

[1] For a projective curve, it won't be possible to find a local parameter that is regular on the entire curve, but it is possible to find a rational function, regular at P, that satisfies the defining condition of a local parameter: it will simply be a local parameter at P of the intersection of the curve with one of our standard affine open sets U_i.