MATH 465/565: Homework 3

Due Friday, February 4, 2011

You should try to do as many of these problems as possible, but **write up and turn in solutions to exactly <u>six</u> of them**. We work over an algebraically closed field k.

1. Let R be a ring (by which we mean commutative ring with unity). Prove the following are equivalent:

   (a) Every ideal of R has a finite generating set.

   (b) Given an ideal $I = \langle S \rangle$ of R generated by a subset $S \subset R$, there is a finite subset $G \subseteq S$ so that $I = \langle G \rangle$.

   (c) Every ascending chain $I_1 \subseteq I_2 \subseteq \cdots$ of ideals of R eventually stabilizes, i.e. for some N, $I_N = I_{N+1} = \cdots = I_n = \cdots$ for all $n \geq N$.

   (d) If $\{I_\alpha\}$ is any collection of ideal of R, then some ideal I_α is maximal with respect to inclusion (i.e. I_α is not properly contained in any $I_{\alpha'}$ in the collection).

2. Let $X \subseteq \mathbb{A}^2$ be the common zero locus of the two polynomials $f(x, y) = x^2 + y^2 - 1$ and $g(x, y) = x - 1$. Find $I(X)$. Is it true that $I(X) = \langle f, g \rangle$?

3. Let $X = V(x^2 + y^2 + z^2, x^2 - y^2 - z^2 + 1) \subseteq \mathbb{A}^3$. Find the irreducible components of X.

4. (a) Given closed sets $X, Y \subseteq \mathbb{A}^n$, show that $I(X \cup Y) = I(X) \cap I(Y)$.

   (b) Given ideals $I, J \subseteq k[x_1, \ldots, x_n]$, show that $V(I \cap J) = V(IJ) = V(I) \cup V(J)$.

   (c) Let $X \subset \mathbb{A}^n$ be closed and $S \subset X$ be a dense subset of X in the Zariski topology (i.e. $\overline{S} = X$ or equivalently $V(I(S)) = X$. Let $f, g \in k[X]$. Show that if f and g agree on S, then $f = g$.

   (d) Suppose X is an irreducible affine variety and $U \subset X$ is a non-empty open set, and let $f, g \in k[X]$. Show that if f and g agree on U, then $f = g$.

5. Suppose char $k = 0$. Prove that there are three points $P_1, P_2, P_3 \in \mathbb{A}^2$ such that

$$\sqrt{\langle x^2 - 2xy^4 + y^8, y^3 - y \rangle} = \mathfrak{m}_{P_1} \cap \mathfrak{m}_{P_2} \cap \mathfrak{m}_{P_3},$$

   where $\mathfrak{m}_P = I(\{P\})$ is the maximal ideal of $k[x, y]$ consisting of polynomials that vanish at P. To what extent is this true if char $k \neq 0$?

6. (a) Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be closed subsets. Show that $X \times Y \subseteq \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$ is closed in the Zariski topology.

   (b) Show that $k[X \times Y] \cong k[X] \otimes_k k[Y]$.

   (c) Show that the product topology on $\mathbb{A}^{n+m} = \mathbb{A}^n \times \mathbb{A}^m$ (using the Zariski topologies on $\mathbb{A}^n$ and $\mathbb{A}^m$) is not the same as the Zariski topology on $\mathbb{A}^{n+m}$.

7. A topological space X is *irreducible* if whenever $X = Z_1 \cup Z_2$ with Z_1 and Z_2 closed, either $X = Z_1$ or $X = Z_2$. A subset $S \subset X$ is irreducible if it is irreducible in the subspace topology. An *irreducible component* of X is an irreducible subset that is maximal with respect to inclusion.

   (a) If X is Hausdorff, show that the irreducible components of X are points.

   (b) Show that X is irreducible if and only if every non-empty open subset of X is dense.

   (c) Show that a subset $S \subseteq X$ is irreducible if and only if its closure $\overline{S}$ is irreducible.

   (d) Show that any irreducible subset of X is contained in an irreducible component. [Hint: use Zorn's lemma.]

8. A topological space X is *Noetherian* if any descending chain $Z_1 \supseteq Z_2 \supseteq \cdots$ of closed subsets of X eventually stabilizes.

   (a) Show that X is Noetherian if and only if in every collection $\{Z_\alpha\}$ of closed subsets of X, some Z_α is minimal with respect to inclusion.

   (b) Show that $\mathbb{A}^n$ in the Zariski topology is a Noetherian topological space.

   (c) Show that a topological space X is Noetherian if and only if every subset of X, including X itself, is compact[1] (in the subspace topology).

   (d) Show that a Noetherian topological space has finitely many irreducible components. [Hint: consider the collection of closed sets that have infinitely many irreducible components.]

   (e) Show that a Noetherian Hausdorff topological space is finite.

9. Let $P_1,\ldots,P_5$ be five distinct points in $\mathbb{P}^2$, with no four of the P_i in a line. Show that there is a unique conic in $\mathbb{P}^2$ passing through all five points. (A *conic* in $\mathbb{P}^2$ is the zero locus of a degree 2 homogeneous polynomial $F \in k[X, Y, Z]$, not necessarily irreducible.)

10. Prove the Hilbert Basis Theorem: if R is a noetherian ring, then $R[x]$ is also a noetherian ring. [Hint: given an ideal $I \subseteq R[x]$, consider ideals in R generated by the leading coefficients of suitable elements of I.]

11. Let $X \subset \mathbb{A}^3$ be the curve given parametrically by $x = t^3,\ y = t^4,\ z = t^5$. Show that X is closed and find (with proof) generators for $I(X)$. Show that $I(X)$ cannot be generated by two elements.

[1] Some reserve the term "compact" for compact Hausdorff spaces and use the word *quasicompact* to refer to a topological space with the property that every open cover has a finite subcover.