MATH 465/565: Homework 4

Due Friday, February 11, 2011

You should try to do as many of these problems as possible, but **write up and turn in solutions to exactly <u>five</u> of them**. We work over an algebraically closed field k.

1. Assume char $k \neq 2$. Let X be the affine plane curve $y^2 = x^3 + x^2$. Let $f\colon \mathbb{A}^1 \to X$ be the regular map defined by $f(t) = (t^2 - 1, t(t^2 - 1))$.
   (a) Show that $f^*\colon k[X] \to k[\mathbb{A}^1]$ is injective.
   (b) Show that the image of f^* is the subring of $k[\mathbb{A}^1] \cong k[t]$ consisting of polynomials $g(t)$ such that $g(-1) = g(1)$.
   (c) Show that every element of the function field $k(X)$ has a unique representation in the form $u(x) + v(x)y$, where u and v are rational functions of x.

2. Assume char $k \neq 2$.
   (a) Find (with proof) the set of points on $Y = V(x^2+y^2-1) \subset \mathbb{A}^2$ where the rational function $\frac{1-y}{x}$ is regular.
   (b) Find (with proof) the set of points on $X = V(y^2 - x^3 - x^2) \subset \mathbb{A}^2$ where the rational function $\frac{y}{x}$ is regular.

3. Show that if a rational function on $\mathbb{A}^2$ is regular at all points in $\mathbb{A}^2 - \{(0,0)\}$, then it is regular on all of $\mathbb{A}^2$.

4. Show that the quasiprojective variety $\mathbb{A}^2 - \{(0,0)\}$ is not isomorphic to an affine variety. [Hint: use the Nullstellensatz for affine varieties.]

5. (a) Consider the regular map $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ defined by $f(x,y) = (x, xy)$. Find the image $f(\mathbb{A}^2)$; is it open in $\mathbb{A}^2$? Is it closed? Is it dense?
   (b) Consider the regular map $g\colon \mathbb{A}^3 \to \mathbb{A}^3$ defined by $g(x,y,z) = (x, xy, xyz)$. Find the image $f(\mathbb{A}^3)$. Show that the image is a finite union of finite intersections of open and closed sets.

6. Find a birational map from the cubic surface $x^3 + x^2y + z^3 = 1$ to $\mathbb{A}^2$.

7. (a) Show that the Zariski topology on $\mathbb{A}^n_i \subset \mathbb{P}^n$ agrees with the subspace topology induced by the Zariski topology on $\mathbb{P}^n$.
   (b) Show that an affine variety $U \subseteq \mathbb{A}^n_i \subset \mathbb{P}^n$ is irreducible if and only if its projective closure $\overline{U} \subseteq \mathbb{P}^n$ is irreducible.
   (c) Show that the operation of projective closure defines a one-to-one correspondence between the closed subsets of $\mathbb{A}^n_i$ and the closed subsets of $\mathbb{P}^n$ no irreducible component of which is contained in the hyperplane "at infinity" defined by $X_i = 0$.

8. Let char $k = p \neq 0$. Let $\varphi\colon \mathbb{A}^n \to \mathbb{A}^n$ be the regular map

$$\varphi(x_1, \ldots, x_n) = (x_1^p, \ldots, x_n^p).$$

We call φ the *Frobenius map.*

   (a) Show that φ is a bijection but not an isomorphism.

   (b) Show that $\mathbb{F}_p^n$ is the set of fixed points of φ.

   (c) Show that if $f_1, \ldots, f_k \in \mathbb{F}_p[x_1, \ldots, x_n]$, and $X = V(f_1, \ldots, f_k)$, then $\varphi(X) = X$.

   (d) (Extra Credit) Show that conversely, if $X \subseteq \mathbb{A}^n$ is closed and $\varphi(X) = X$, then $I(X)$ can be generated by polynomials with coefficients in $\mathbb{F}_p$.

9. (Extra Credit) This exercise generalizes the example in Problem 1, showing that it is always possible to "glue together" two points of an affine variety and get another affine variety.

   (a) Let R be a finitely generated k-algebra and $S \subset R$ be a sub-k-algebra. Assume R is finitely generated as an S-module. Prove that S is a finitely generated k-algebra.[1]

   (b) Let X be an affine variety and $p, q \in X$ distinct points. Show that there exists an affine variety Y and a dominant regular map $f\colon X \to Y$ so that $f(p) = f(q)$ but f is otherwise injective.

   (c) Show that your map from the previous part is surjective.

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[1]This commutative algebra lemma also allows one to prove the Nullstellensatz using a somewhat weaker theorem from field theory than the proof in Shafarevich. We must show that if R is a finitely generated k-algebra, then R contains as a sub-k-algebra no transcedential field extension of k.

Replacing R by its quotient by a maximal ideal, we may assume R is itself a field. Let $t_1, \ldots, t_l \in R$ be a transcendence basis for R over k, and set $S = k(t_1, \ldots, t_l)$. Then the extension R/S is algebraic and finitely generated, hence R is a finite dimensional vector space over S, and we can apply the exercise to conclude that S is a finitely generated k-algebra.

We can thus reduce the Nullstellensatz to showing that $S = k(t_1, \ldots, t_l)$ is not a finitely generated k-algebra for any $l \geq 1$, which we can do directly using unique factorization in $k[t_1, \ldots, t_l]$ and the fact that there are infinitely many irreducibles in this ring.