MATH 465/565: Homework 5

Due Friday, February 18, 2011

You should try to do as many of these problems as possible, but **write up and turn in solutions to exactly <u>six</u> of them**. We work over an algebraically closed field k. Problems 3 and 6 will appear again on the next homework (turn in each at most once).

1. Show that the quasiprojective variety $\mathbb{A}^2 \backslash \{(0,0)\}$ is not isomorphic to a closed subset of $\mathbb{A}^n$. [Hint: use the Nullstellensatz for affine varieties.]

2. (a) Show that the Zariski topology on $\mathbb{A}^n_i \subset \mathbb{P}^n$ agrees with the subspace topology induced by the Zariski topology on $\mathbb{P}^n$.
   (b) Show that an affine variety $U \subseteq \mathbb{A}^n_i \subset \mathbb{P}^n$ is irreducible if and only if its projective closure $\overline{U} \subseteq \mathbb{P}^n$ is irreducible.
   (c) Show that the operation of projective closure defines a one-to-one correspondence between the closed subsets of $\mathbb{A}^n_i$ and the closed subsets of $\mathbb{P}^n$ no irreducible component of which is contained in the hyperplane "at infinity" defined by $X_i = 0$.

3. Prove that every rational map from $\mathbb{P}^1$ to $\mathbb{P}^n$ is regular.

4. Prove that any regular map $f\colon \mathbb{P}^1 \to \mathbb{A}^n$ maps $\mathbb{P}^1$ to a point.

5. The *twisted cubic* in $\mathbb{P}^3$ is the image of $\mathbb{P}^1$ under the degree 3 Veronese map, $\nu_3(\mathbb{P}^1)$, where in homogeneous coordinates we write:

$$\nu_3([X_0, X_1]) = [X_0^3, X_0^2 X_1, X_0 X_1^2, X_1^3].$$

   (a) Show that $\nu_3(\mathbb{P}^1)$ is the locus of common zeros of the three polynomials $F_1 = Z_0 Z_2 - Z_1^2$, $F_2 = Z_0 Z_3 - Z_1 Z_2$, and $F_3 = Z_1 Z_3 - Z_2^2$, where $[Z_0, Z_1, Z_2, Z_3]$ are the homogeneous coordinates on $\mathbb{P}^3$.
   (b) Find the irreducible components of the common zero locus of each of the pairs $\{F_1, F_2\}$, $\{F_2, F_3\}$, and $\{F_1, F_3\}$.

6. Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the rational map defined by

$$f([X_0, X_1, X_2]) = [X_1 X_2, X_0 X_2, X_0 X_1].$$

   (a) Show that f defines a birational map from $\mathbb{P}^2$ to itself.
   (b) At which points is f not regular? At which points is f^{-1} not regular?
   (c) Find non-empty open sets $U, V \subseteq \mathbb{P}^2$ so that f restricts to an isomorphism $U \to V$.

7. Show that the Veronese image $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^N$ is not contained in any hyperplane of $\mathbb{P}^N$.

8. Show that any $d+1$ points of the rational normal curve $\nu_d(\mathbb{P}^1) \subset \mathbb{P}^d$ span all of $\mathbb{P}^d$.

9. Show that if $X \subseteq \mathbb{P}^2$ is a plane conic, then $\mathbb{P}^2 \backslash X$ is isomorphic to an affine variety.

[Hint: use the Veronese embedding.]

10. (Extra Credit) A finite set $S \subset \mathbb{P}^n$ is in *(linear) general position* if for any $0 \leq \ell \leq n$, the linear span of any $\ell + 1$ points $p_0, \ldots, p_\ell \in S$ is a $\mathbb{P}^\ell$ (i.e. no three points on a line, no four points in a plane, etc.).

    (a) Show that if $S \subset \mathbb{P}^n$ is any collection of $d \leq 2n$ points in general position, then S is the set of common zeros of a collection homogeneous polynomials of degree 2.

    (b) Show that if $p_1, \ldots, p_{2n+1}$ are any $2n+1$ points on the rational normal curve $\nu_n(\mathbb{P}^1) \subset \mathbb{P}^n$, then $p_1, \ldots, p_{2n+1}$ are in general position, but any homogeneous polynomial of degree 2 that vanishes on all the p_i must vanish on all of $\nu_n(\mathbb{P}^1)$.

    (c) Generalize parts (a) and (b) to polynomials of degree m.

11. (Extra Credit)

    (a) Show that the subring $k[xy, xy^2, xy^3, \ldots] \subseteq k[x, y]$ is not a finitely generated k-algebra.

    (b) Show that any sub-k-algebra of $k[x]$ is finitely generated as a k-algebra.