MATH 465/565: Homework 6

Due Friday, March 11, 2011

You should try to do as many of these problems as possible, but **write up and turn in solutions to exactly <u>six</u> of them**. We work over an algebraically closed field k.

1. Prove that every rational map from $\mathbb{P}^1$ to $\mathbb{P}^n$ is regular. [You should show this directly in this case, without using Theorem 1.1 as in the proof of Theorem 1.2.]

2. Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the rational map defined by

$$f([X_0, X_1, X_2]) = [X_1X_2, X_0X_2, X_0X_1].$$

   (a) Show that f defines a birational map from $\mathbb{P}^2$ to itself.

   (b) At which points is f not regular? At which points is f^{-1} not regular?

   (c) Find non-empty open sets $U, V \subseteq \mathbb{P}^2$ so that f restricts to an isomorphism $U \to V$.

3. Prove that the Segre variety $\sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m) \subset \mathbb{P}^N$, where $N = (n+1)(m+1)-1$ is not contained in any hyperplane in $\mathbb{P}^N$.

4. Let $\Sigma_{n,m} = \sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m)$ be the Segre image and let $p_1 \colon \Sigma_{n,m} \to \mathbb{P}^n$ and $p_2 \colon \Sigma_{n,m} \to \mathbb{P}^m$ be the functions obtained by composing the inverse function $\sigma_{n,m}^{-1} \colon \Sigma_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m$ with the projections $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n$ and $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$, respectively.

   (a) Show that p_1 and p_2 are regular maps.

   (b) Show that if W is any quasiprojective variety and $f_1 \colon W \to \mathbb{P}^n$ and $f_2 \colon W \to \mathbb{P}^m$ are regular maps, then there is a unique regular map $f \colon W \to \Sigma_{n,m}$ such that $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$.

5. Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be locally closed subsets (i.e. quasiprojective varieties).

   (a) Show that the image under the Segre map $\sigma_{n,m}(X \times Y) \subseteq \sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m) \subseteq \mathbb{P}^N$ is locally closed (i.e. $\sigma_{n,m}(X \times Y)$ is a quasiprojective variety).

   (b) Show that the functions $p_1 \colon \sigma_{n,m}(X \times Y) \to X$ and $p_2 \colon \sigma_{n,m}(X \times Y) \to Y$, defined as in the previous problem, are regular maps, and that if W is any quasiprojective variety and $f_1 \colon W \to X$ and $f_2 \colon W \to Y$ are regular maps, then there is a unique regular map $f \colon W \to \sigma_{n,m}(X \times Y)$ such that $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$.[1]

   (c) Suppose that $X' \subset \mathbb{P}^{n'}$ and $Y' \subset \mathbb{P}^{m'}$ are locally closed subsets so that $X \cong X'$ and $Y \cong Y'$. Show that the quasiprojective varieties $\sigma_{n,m}(X \times Y) \subset \mathbb{P}^N$ and $\sigma_{n',m'}(X' \times Y') \subset \mathbb{P}^{N'}$ are isomorphic.

[1] The is the *universal property* of the product; we say that $\sigma_{n,m}(X \times Y)$ is the product of X and Y in the category of quasiprojective varieties.

6. Let L, M, and $N \subseteq \mathbb{P}^3$ be any pairwise skew (i.e. disjoint) lines. Show that, after a linear change of variables on $\mathbb{P}^3$, the union of the lines that meet L, M, and N is the Segre surface $\Sigma_{1,1} \subseteq \mathbb{P}^3$.

7. Let X be a quasi-projective variety and $U, V \subseteq X$ be affine open subsets. Show that $U \cap V$ is affine. [Hint: Consider $U \times V \subseteq X \times X$.]

8. Show that the twisted cubic curve $\nu_3(\mathbb{P}^1) \subset \mathbb{P}^3$ can be realized as the intersection of $\Sigma_{1,2} \subseteq \mathbb{P}^5$ with a three-plane $\mathbb{P}^3 \subseteq \mathbb{P}^5$.

9. For this problem, let $k = \mathbb{C}$, the field of complex numbers. In this case, $\mathbb{A}^n$ has another standard topology, generated by open balls, which we will call the *analytic topology* or the *classical topology.*

   (a) Show that the classical topology on $\mathbb{A}^n$ is finer than the Zariski topology (i.e. show that every Zariski closed subset is also closed in the classical topology).

   (b) Let $X \subsetneq \mathbb{A}^n$ be closed in the Zariski topology. Prove that $\mathbb{A}^n \backslash X$ is dense in the classical topology.

   (c) Describe what the classical topology should be on $\mathbb{P}^n$. Show that $\mathbb{P}^n$ is compact in the classical topology.

   (d) Show that every regular function on an irreducible projective variety (over $k = \mathbb{C}$) is constant.

10. (Extra Credit) Let $\ell \subseteq \Sigma_{n,m} \subset \mathbb{P}^N$ be a line contained in the Segre image. Show that either ℓ is the image under $\sigma_{n,m}$ of $\ell_1 \times \{y_2\}$ for some line $\ell_1 \subseteq \mathbb{P}^n$ and point $y_2 \in \mathbb{P}^m$, or ℓ is the image of $\{x_1\} \times \ell_2$ for some point $x_1 \in \mathbb{P}^n$ and line $\ell_2 \subseteq \mathbb{P}^m$.