MATH 465/565: Homework 7

Due Wednesday, March 23, 2011

You should try to do as many of these problems as possible, but **write up and turn in solutions to exactly <u>six</u> of them**. We work over an algebraically closed field k.

1. (a) Show that $G(r, n) \cong G(n-r, n)$.
   (b) For V a finite dimensional vector space of dimension n, show that there is a natural isomorphism $G(r, V) \cong G(n-r, V^*)$.

2. Assume that char $k = 0$, with V a finite-dimensional k-vector space of dimension n.
   (a) Show that for any $x \in \bigwedge^2 V$, there is a basis $v_1, \ldots, v_n$ of V so that
   $$x = v_1 \wedge v_2 + v_3 \wedge v_4 + \cdots + v_{2r-1} \wedge v_{2r}$$
   for some $0 \leq r \leq n/2$.
   (b) Show that $x^r = \underbrace{x \wedge x \wedge \ldots \wedge x}_{r \text{ times}}$ is non-zero, but $x^{r+1} = 0$.
   (c) Show that $x \in \bigwedge^2 V$ is totally decomposable if and only if $x \wedge x = 0$. Is this always true for $x \in \bigwedge^3 V$?

3. Let $\ell_1, \ell_2 \subseteq \mathbb{P}^3$ be skew lines. Show that the set $Q \subseteq \mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$ that meet both ℓ_1 and ℓ_2 is the intersection of $\mathbb{G}(1,3)$ with a 3-plane $\mathbb{P}^3 \subset \mathbb{P}^5$ (and hence is isomorphic to a quadric surface). Explain what the two rulings (families of lines) on this quadric surface correspond to in terms of ℓ_1 and ℓ_2.

4. Let $0 < r_1 < r_2 < n$. Show that
   $$F(r_1, r_2; n) := \{(\Lambda_1, \Lambda_2) \in G(r_1, n) \times G(r_2, n) : \quad \Lambda_1 \subseteq \Lambda_2\}$$
   is a closed subset of $G(r_1, n) \times G(r_2, n)$.[1] [Hint: closedness is a local property.]

5. Let $\mathbb{G}(r, n) = G(r+1, n+1)$ be the set of r-dimensional linear subspaces $\mathbb{P}^r \subseteq \mathbb{P}^n$. Given a point $p \in \mathbb{P}^3$ and a plane $H \subset \mathbb{P}^3$ containing p, let $\Sigma_{p,H} \subseteq \mathbb{G}(1,3)$ be the set of lines in $\mathbb{P}^3$ lying in H and passing through p. Show that under the Plücker embedding $\mathbb{G}(1,3) \hookrightarrow \mathbb{P}^5$, each $\Sigma_{p,H}$ is mapped to a line and that conversely, every line in $\mathbb{P}^5$ contained in image of $\mathbb{G}(1,3)$ under the Plücker embedding arises in this way.

[1] This is an example of a *flag variety*. More generally, an ordered m-tuple $(\Lambda_1, \Lambda_2, \ldots, \Lambda_m)$ of linear subspaces of V with $\{0\} \subsetneq \Lambda_1 \subsetneq \Lambda_2 \subsetneq \cdots \subsetneq \Lambda_m \subsetneq V$ is called a *flag* and given integers $0 < r_1 < \cdots < r_m < n$, we may form a flag variety $G(r_1, r_2, \ldots, r_m; n)$ consisting of all such flags with $\dim \Lambda_i = r_i$.

6. Let $\dim V = 4$ and $\{0\} \subsetneq \Lambda_1 \subsetneq \Lambda_2 \subsetneq \Lambda_3 \subsetneq V$ be subspaces with $\dim \Lambda_i = i$. To any $\Lambda \in G(2, V)$, we may associate an ordered triple of nonnegative integers

$$f(\Lambda) = (\dim(\Lambda \cap \Lambda_1), \dim(\Lambda \cap \Lambda_2), \dim(\Lambda \cap \Lambda_3)).$$

   (a) Which ordered triples can arise in this way?

   (b) Show that the function f is *upper semi-continuous* on $G(2, V)$, in the sense that if (a, b, c) is an ordered triple of integers, then the set

$$\{\Lambda \in G(2, V) \colon \ f(\Lambda) \geq^2 (a, b, c)\}$$

   is a closed subset of $G(2, V)$.[3]

   (c) Let $V = k^4$ with standard basis e_1, e_2, e_3, e_4, and assume $\Lambda_1 = \text{span}\,\{e_4\}$, $\Lambda_2 = \text{span}\,\{e_3, e_4\}$, and $\Lambda_3 = \text{span}\,\{e_2, e_3, e_4\}$. Let $\Lambda \in G(2, 4)$ be the row space of a 2×4 matrix M of rank 2. Show that computing $f(\Lambda)$ is equivalent to finding the locations of the leading 1's in the reduced row echelon form for M.

   (d) Show that for each (a, b, c) from part (a), the subset

$$\{\Lambda \in G(2, V) \colon \ f(\Lambda) = (a, b, c)\}$$

   is a quasi-projective variety, isomorphic to $\mathbb{A}^m$ for some m (depending on a,b,c).

7. Let $k = \mathbb{C}$; we consider $G(2, 4)$ as a topological space in the analytic topology.

   (a) Show that $G(2, 4)$ is a manifold and compute its dimension.

   (b) Show that $G(2, 4)$ is homeomorphic to a finite CW complex.

   (c) Compute the fundamental group and homology groups of $G(2, 4)$.

   (d) (Extra Credit) Repeat parts (a)-(c) for the real Grassmannian $G_{\mathbb{R}}(2, 4)$.[4]

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[2] Given two ordered triples of integers (a_1, b_1, c_1) and (a_2, b_2, c_2), we say $(a_1, b_1, c_1) \geq (a_2, b_2, c_2)$ if $a_1 \geq b_1$, $a_2 \geq b_2$, and $a_3 \geq b_3$.

[3] These are called *Schubert varieties*. Note that the sets $\Sigma_{p,H}$ from the previous problem are a special case.

[4] A point of $G_{\mathbb{R}}(2, 4)$ is a 2-dimensional real subspace of the real vector space $\mathbb{R}^4$; $G_{\mathbb{R}}(2, 4)$ can be embedded in $\mathbb{R}P^5$ by the Plücker embedding just as in the complex case, and the image is the zero locus of the same polynomial (which has integer coefficients). Moreover, $G_{\mathbb{R}}(2, 4)$ has local coordinate charts of the same form as in the complex case.