MATH 465/565: Homework 8

Due Friday, April 1, 2011

You should try to do as many of these problems as possible, but **write up and turn in solutions to exactly <u>six</u> of them**. We work over an algebraically closed field k.

1. Let S be the set of hypersurfaces of degree d in $\mathbb{A}^n$. (For the purpose of this problem, a *hypersurface of degree* d on $\mathbb{A}^n$ will be a polynomial f of degree d, where two polynomials define the same hypersurface if they differ by multiplication by a non-zero scalar.[1])
   (a) Show that S has a natural structure of a quasiprojective variety.
   (b) Let $U \subset S$ be the set of hypersurfaces $[f]$ of degree d where f is irreducible. Show that U is open in S.
   (c) Let $V \subset S$ be the set of hypersurfaces $[f]$ of degree d where f is square-free. Show that V is open in S.

2. Show that each of the following quasiprojective varieties is neither affine nor projective:
   (a) $\mathbb{A}^2 \setminus \{p\}$, where $p \in \mathbb{A}^2$ is any point,
   (b) $\mathbb{P}^2 \setminus \{p\}$, where $p \in \mathbb{P}^2$ is any point, and
   (c) $\mathbb{P}^1 \times \mathbb{A}^1$.

3. Determine whether the following maps are finite:
   (a) $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ defined by $f(x, y) = (xy, y)$, and
   (b) $g\colon \mathbb{P}^2 \to \mathbb{P}^2$ defined by $g([X_0, X_1, X_2]) = [X_0^2, X_1^2, X_2^2]$.

4. Find (with proof) a finite map from $\mathbb{A}^1 \setminus \{0\}$ onto the affine line $\mathbb{A}^1$.

5. Let $X \subseteq \mathbb{P}^n$ be a projective variety and $F_0, \dots, F_m$ be homogeneous polynomials of the same degree such that $X \cap V(F_0, \dots, F_m) = \varnothing$. Prove that the regular map $f\colon X \to \mathbb{P}^m$ defined by $f(x) = [F_0(x), \dots, F_m(x)]$ restricts to a finite map $X \xrightarrow{f} f(X)$.

6. Find an example of a surjective regular map with finite fibers that is not a finite map.

7. Let Y be a quasiprojective variety and $X \subseteq Y \times \mathbb{P}^1$ be a closed subset. Let $\pi\colon X \to Y$ be the restriction to X of projection onto the first factor. Assume that π is surjective and has finite fibers. Show that π is finite. [Hint: first reduce to the case where Y is affine.]

[1] This definition is slightly "wrong" in that, for example, if $d = ab$ and $f = \tilde{f}^a$ for some polynomial $\tilde{f}$ of degree b, then $V(f) = V(\tilde{f})$ should really have degree b. We can "fix" this "problem" replacing S with the variety V from part (c).

8. (Extra Credit) Let A be a subring of B. Suppose there exist $b_1, \ldots, b_m \in B$ so that

$$\langle b_1, \ldots, b_m \rangle = \langle 1 \rangle$$

and $B[b_i^{-1}]$ is a finitely generated A-algebra for $i = 1, \ldots, m$. Prove that B is a finitely generated A-algebra.

9. (Extra Credit) Let X be a quasiprojective variety. Given $f \in k[X]$, we may consider

$$X_f = \{x \in X \colon\ f(x) \neq 0\},$$

an open subset of X.

   (a) Show that $k[X_f] \cong k[X][f^{-1}]$.

   (b) Suppose there exist $f_1, \ldots, f_m \in k[X]$ such that $\langle f_1, \ldots, f_n \rangle = \langle 1 \rangle$ and each X_{f_i} is affine. Show that X is affine.

10. (Extra Credit) Let $f \colon X \to Y$ be a finite map of quasiprojective varieties. Show that if $V \subseteq Y$ is any affine open, then $U = f^{-1}(V)$ is affine and the restriction $U \xrightarrow{f} V$ is finite.